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Epistemic Game Theory without Types Structures: An Application to Psychological Games*

Pierpaolo Battigalli† Roberto Corrao‡ Federico Sanna§

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Abstract

We consider multi-stage games with incomplete information and observable actions, and we analyze strategic reasoning by means of epistemic events within a "total" state space made of all the profiles of behaviors (paths of play) and possibly incoherent infinite hierarchies of conditional beliefs. Thus, we do not rely on types structures, or similar epistemic models. Subjective rationality is defined by the conjunction of coherence of belief hierarchies, rational planning, and consistency between plan and on-path behavior. Since consistent hierarchies uniquely induce beliefs about behavior and belief hierarchies of others, we can define rationality and common strong belief in rationality, and analyze their behavioral and low-order beliefs implications, which are characterized by strong rationalizability. Our approach allows to extend known techniques to the epistemic analysis of psychological games where the utilities of outcomes depend on beliefs of order k or lower. This covers almost all applications of psychological game theory.

JEL Classification Numbers: C72, C73, D82.

Keywords: Epistemic game theory, belief hierarchies, consistency, subjective rationality, strong rationalizability, psychological games.

1 Introduction

Epistemic game theory (EGT) is the formal analysis of players' interactive strategic reasoning in games.[1] Such analysis posits, or obtains by construction, a set **states of the world** Ω so that each $\omega \in \Omega$ is an all-encompassing implicit or explicit specification of all the relevant aspects of the strategic situation, including what players do and how they think about each others' behavior and beliefs. This permits the definition of events (measurable subsets of Ω) such as "players are rational" [viz. $R \subseteq \Omega$] and "it is common belief that

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[1]See, for example, the survey by Dekel & Siniscalchi (2015), or the texbook by Perea (2012), and the relevant references therein.

players are rational" [viz. $\mathrm{CB}(R) \subseteq \Omega$]. These events are the **epistemic assumptions** of interest and relate behavior to beliefs. The typical theorem of epistemic game theory provides a characterization of the interesting implications of such epistemic assumptions, such as the behavioral implications. For example, given appropriate definitions of "rationality," "common belief," and "rationalizability," we have the following theorem:[2] *Players' behavior is consistent with* $R \cap \mathrm{CB}(R)$ *if and only if it is rationalizable.*"

Standard approach The standard approach of EGT is to posit or construct Ω so that it is true at *every* state ω in Ω that every player is **cognitively rational**, that is, that his system of beliefs satisfies appropriate coherence requirements. Specifically, each state ω determines for each player i a hierarchy of beliefs such that the "first-order" belief of i about the behavior of other players $-i$ is the marginal of "higher-order" joint beliefs about the behavior *and* beliefs of $-i$, and similar coherence restrictions hold for higher-order beliefs. Since cognitive rationality holds at every state, it has to be commonly believed at every state that players are cognitively rational. Say that an event is **transparent** if is true and commonly believed. With this, *the standard approach assumes that cognitive rationality is transparent.*

Yet, rationality has also a behavioral aspect: roughly, a player is **rational** at ω if he is cognitively rational at ω *and* his behavior at ω is a best reply to his beliefs at ω. Most contributions to EGT do *not* assume that players are rational at every state. In particular, allowing for states where players are irrational is important in the epistemic analysis of multi-stage games that contain histories inconsistent with either mere rationality, or with rationality and common belief in rationality. Indeed, assuming that rationality holds at *every* state would imply "by *fiat*" that the actions in such histories cannot be chosen even if they comply with the rules of the game. Several authors (including ourselves) find this feature conceptually problematic. Thus, standard EGT postulates transparency of cognitive rationality, but instead treats rationality and common belief in rationality as a property that holds only in some states. We accept the latter, but question the former: *Why should cognitive rationality be transparent* while rationality holds only at some states?

In our view, the reason is more technical than conceptual: the standard approach allows to work with epistemic structures. In particular, much of the EGT literature on multi-stage games works with type structures, whereby each player is uncertain about the coplayers' behavior and a type of a player corresponds to a hierarchy of beliefs satisfying coherence and common belief in coherence. It is argued that this is without loss of generality, because one can show that the space of profiles of behaviors and belief hierarchies satisfying transparency of coherence gives the "largest" type structure.[3]

Our approach Working with type structures is traditional and in many ways convenient, but we show in this paper that eschewing them is both possible and fruitful. *We regard cognitive rationality simply as an aspect of rationality that—like rationality itself—holds only in some states.* Hence, we consider the "total" space of all profiles of behaviors and beliefs, including incoherent beliefs. Rationality of player i corresponds to the set of

states where i has coherent beliefs and his behavior is a sequential best reply to such beliefs. Since the belief-hierarchy of a rational player is coherent, it induces a belief (a

[2] One can give versions of this result for different decision criteria (forms of rationality), different kinds of game (e.g., simultaneous, or sequential, with complete or incomplete information), and—correspondingly—different definitions of rationalizability.

[3] See Battigalli & Siniscalchi (1999).

conditional probability system) about behaviors and beliefs hierarchies of the opponents (Battigalli & Siniscalchi 1999). Leveraging on this, we prove that, in multi-stage games with possibly incomplete information, strong rationalizability characterizes the behavioral implications of rationality and common strong beliefs in rationality, which represents forward-induction reasoning.

There are other novelties in our approach. We take advantage of our "fresh start" to introduce other innovations compared to standard epistemic game theory. Indeed, we take as primitive players' *uncertainty about the path of play*, rather than uncertainty about contingent behavior. With this, the only mathematical objects that look like "strategies" are players' (marginal) systems of beliefs about their own actions conditional of reaching non-terminal histories. These are mere plans. Rationality is modeled as coherence of beliefs, rational planning (incentive compatibility of plans), and on-path consistency of plan and actions.

Application to psychological games Besides its conceptual appeal, our approach has also a technical advantage: it allows us to extend to so called "psychological games" the techniques used by Battigalli & Tebaldi (2018) in the epistemic analysis of a class of games with compact strategy spaces. In a **psychological game** utilities of outcomes depend on beliefs. This allows to capture a wide range of emotional or otherwise psychological aspects of choice. In applications, the utility of outcomes is assumed to depend only on beliefs of the first k order, e.g., only on the first-order beliefs of everybody. Dependence on beliefs up to order k allows for a tractable definition of rationalizability by means of iterated elimination of non-best replies to beliefs of order $k+1$. The technical problem is to show that a $k+1$-order belief that justifies a player's behavior as rationalizable can be extended to an infinite hierarchy of beliefs that makes such behavior consistent with rationality and common strong belief in rationality. Battigalli & Tebaldi (2018) rely on the possibility to *factorize* the uncertainty space of any player i as follows $\Omega_{-i} = Y_{-i} \times M_{-i}$, where Y_{-i} is a set of utility-relevant states and M_{-i} is a space of opponents' beliefs (probability measures). In a psychological game, Y_{-i} should be the set of possible behaviors and opponents' beliefs up to order k, and M_{-i} should be the set of opponents' beliefs of order $k+1$ or higher. Yet, when working with type structures, Ω_{-i} cannot be factorized in this way, because low-order beliefs must be the marginal of higher-order beliefs; thus, $\Omega_{-i} \subsetneq Y_{-i} \times M_{-i}$, which prevents the application of the aforementioned methods. We instead work with the space of *all* belief hierarchies, including the incoherent ones, which implies that the relevant factorization holds. With this, we prove the results stated above for multi-stage games where psychological utility depends on beliefs of some given order k, or lower. To be clear, we are not claiming that our approach is necessary for an epistemic analysis of psychological games,[4] we only argue that our approach has an independent conceptual motivation and it also has the advantage of allowing the extension of known methods from standard games to psychological games.

Related literature To the best of our knowledge, this is the first paper on epistemic game theory that does not assume transparency of coherence and hence does not rely on standard epistemic structures. As we explained above, our epistemic justification of solution concepts adapts techniques borrowed from Battigalli & Tebaldi (2018). Our analysis of hierarchies of conditional beliefs builds on Battigalli & Siniscalchi (1999), our analysis of common strong belief in rationality (forward-induction reasoning) builds on Battigalli & Siniscalchi

[4] See our comments on the literature.

(2002) and Battigalli *et al.* (2013). In the latter paper, as in ours, primitive uncertainty concerns the path of play. Psychological games were first analyzed by Geanakoplos *et al.* (1989). Here we borrow from and modify the extended framework of Battigalli & Dufwenberg (2009), which also puts forward the first epistemic analysis of strong rationalizability in psychological games. The most important difference with Battigalli & Dufwenberg (2009) is that they do not assume a finite upper bound on the order of beliefs that affect psychological utility; hence, they cannot give a tractable definition of (strong) rationalizability as iterated elimination of non-best replies. Also, Battigalli & Dufwenberg (2009) assume transparency of consistency between plan and behavior, and that psychological utility does not depend on one's own plan, while we dispense with both assumptions. This allows for an explicit analysis of players' inferences about coplayers' intentions and it introduces the possibility of dynamically inconsistent preferences. Both features are crucial in some important applications of psychological game theory.[5] Finally, Jagau & Perea (2017, 2018) analyze rationality and common belief in rationality in simultaneous-move games where psychological utility depends only on *initial* beliefs. Our analysis differs from theirs in several aspects. First, we consider T-stage (rather than one-stage) games where psychological utility may depend on initial as well as updated beliefs, including *terminal* beliefs. Therefore, we are able to model—both in one-stage and in multistage games—backward and forward-induction reasoning, as well as concerns for the opinions of others (see Battigalli & Dufwenberg 2009). Second, Jagau & Perea (2017) rely on the standard type-structure approach; hence, they use different methods.

Outline The rest of the paper is organized as follows. Section 2 reviews some mathematical preliminaries. Section 3 presents multistage game forms. Section 4 defines hierarchies of beliefs and the total state space of the game. Section 5 defines k-th order psychological games and subjective rationality. Sections 6 defines strong rationalizability. Section 7 states and proves our main result, Theorem 1, which provides an epistemic justification of strong rationalizability (all the proofs not present in the article are available upon request). Finally, Section 8 discusses possible extensions of our work.

2 Mathematical preliminaries

For every set X and for every $t \in \mathbb{N}$, we denote with X^t the t-fold product of the set X with generic element denoted by x^t. By convention, we let $X^0 = \{\varnothing\}$, where $\varnothing$ denotes the empty sequence. Fix a (non-ordered) index set I and a profile of sets $(X_i)_{i\in I}$, we let $\prod_{i\in I} X_i$ denote the set of all selections from correspondence $i \mapsto X_i$, that is, the set of all functions $f : I \to \bigcup_{i\in I} X_i$ such that $f(i) \in X_i$. In other words, the Cartesian product $\prod_{i\in I} X_i$ is regarded as a set of functions and the order of "factors" is irrelevant.

Given any compact metrizable topological space (Ω, τ), we denote with $\mathcal{B}(\Omega)$ its Borel sigma-algebra and with $\Delta(\Omega)$ the space of all the Borel probability measures over $\mathcal{B}(\Omega)$. We always endow $\Delta(\Omega)$ with the topology of weak convergence τ^w. It follows that $(\Delta(\Omega), \tau^w)$ is a compact metrizable topological space (see for example Aliprantis & Border 2006, Chapter 15). We always endow finite spaces with the discrete topology, product of topological spaces

[5] For a comprehensive analysis of the main applications in psychological game theory see Battigalli & Dufwenberg (2009) and Battigalli *et al.* (2018).

with the product topology and subsets of topological spaces with the relative topology. Given a measurable subset F of Ω, we denote with $\mathcal{B}(\Omega) \cap F$ the relative Borel sigma-algebra of F.

Definition 1 *Let Ω be* compact metrizable *and let $\mathcal{F}$ be a countable collection of* clopen *subsets of Ω, then $(\Omega, \mathcal{F})$ is called* ***conditional measurable space****.*

The following definition is key in our analysis.

Definition 2 *Let $(\Omega, \mathcal{F})$ be a conditional measurable space. A* ***conditional probability system (CPS)*** *over $(\Omega, \mathcal{F})$ is an array $\mu \in [\Delta(\Omega)]^{\mathcal{F}}$ that satisfies:*

- *(*Knowledge implies belief) *For all $F \in \mathcal{F}$ and for all $E \in \mathcal{B}(\Omega)$*

$$F \subseteq E \Rightarrow \mu(E|F) = 1.$$

- (Chain Rule) *For all $F_1, F_2 \in \mathcal{F}$ with $F_2 \subseteq F_1$ and for all $E \in \mathcal{B}(\Omega) \cap F_2$*

$$\mu(E|F_1) = \mu(E|F_2)\,\mu(F_2|F_1).$$

We denote with $\Delta^{\mathcal{F}}(\Omega) \subseteq [\Delta(\Omega)]^{\mathcal{F}}$ the set of CPSs on $(\Omega, \mathcal{F})$.

The defining properties of CPSs essentially say that the rules of conditional probability apply whenever possible. In particular, they imply that, for each $F \in \mathcal{F}$, each measurable partition $\{F_1, ..., F_k\}$ of F and each measurable subset E of F,

$$\mu(E|F) = \sum_{\ell=1}^{k} \mu(E|F_\ell)\,\mu(F_\ell|F).$$

Battigalli & Siniscalchi (1999) proved the following result.

Lemma 1 *Set $\Delta^{\mathcal{F}}(\Omega)$ is compact metrizable.*

Next, consider two compact metrizable spaces Ω_1, Ω_2, their product $\Omega = \Omega_1 \times \Omega_2$ and a collection $\mathcal{F} \subseteq \mathcal{B}(\Omega_1)$ of clopen subsets of Ω_1. The "cylinders"

$$\mathcal{F}' = \{F \times \Omega_2 \subseteq \Omega : F \in \mathcal{F}\} \subseteq \mathcal{B}(\Omega)$$

form a family of clopen subsets of Ω; therefore, $(\Omega, \mathcal{F}')$ is a conditional measurable space. We can marginalize CPSs defined over product spaces through the map $\mathrm{marg}_{\Omega_1} : \Delta^{\mathcal{F}'}(\Omega) \to \Delta^{\mathcal{F}}(\Omega_1)$ defined by

$$\mathrm{marg}_{\Omega_1}(\mu)(E|F) = \mu(E \times \Omega_2 | F \times \Omega_2)$$

for all $E \in \mathcal{B}(\Omega_1)$ and $F \in \mathcal{F}$, which is clearly continuous.

3 Multistage game form

A multistage game form is a mathematical object that encodes the rules of interaction in a game: the set of players (roles), and, for each player, his information and feasible actions at each stage. Here we consider a simplified version with *observable actions*. The game proceeds through stages. In each stage, the set of feasible actions of each player may depend on the history of past actions, which is public information. A player is inactive when his set of feasible actions is a singleton. In each stage, all active players move simultaneously.

Let $\mathbb{T} = \{1, ..., T\}$ be a finite set of successive natural numbers, where T represents the height of the game, that is, its maximum length. A **multistage game form** is a mathematical structure

$$\Gamma = \left\langle I, (\Theta_i, A_i, \mathcal{A}_i)_{i \in I} \right\rangle$$

comprising the following elements:

- I is the finite set of players; for each $i \in I$,
- Θ_i is a *compact* metrizable space of possible personal traits of player i; we assume that each player i *knows* his personal traits $\theta_i \in \Theta_i$, but is uncertain about the personal traits of coplayers, $\theta_{-i} \in \Theta_{-i} = \prod_{j \in I \setminus \{i\}} \Theta_j$;
- A_i is a *finite* set of **actions** of player i;
- let $A = \prod_{i \in I} A_i$ denote the set of action profiles; with this, $\mathcal{A}_i = \left(\mathcal{A}_{i,t} : A^{t-1} \rightrightarrows A_i\right)_{t=1}^{T}$ is a T-tuple of **feasibility correspondences**, where $\mathcal{A}_{i,t}$ describes the feasible actions of i given any sequence of action profiles of length $t-1$.

We assume that, for all $i \in I$, $\mathcal{A}_{i,1}(\varnothing)$ is nonempty and, for all $t \in \mathbb{T}$ and $a^{t-1} \in A^{t-1}$, if $\mathcal{A}_{i,t}\left(a^{t-1}\right)$ is empty, then for every $j \in I$, $\mathcal{A}_{j,t}\left(a^{t-1}\right)$ is empty as well, that is, $\mathcal{A}_{i,t}\left(a^{t-1}\right) = \emptyset$ means "game over." The set of feasible action profiles given a^{t-1} is denoted by $\mathcal{A}_t\left(a^{t-1}\right) = \prod_{i \in I} \mathcal{A}_{i,t}\left(a^{t-1}\right)$.

Let $A^{\leq \mathbb{T}} = \{\varnothing\} \cup \bigcup_{t \in \mathbb{T}} A^t$ be the set of all the sequences of action profiles with length smaller or equal to the height of the game T. The *finite* set $H \subseteq A^{\leq \mathbb{T}}$ of (feasible) **histories** is

$$H = \left\{a^t \in A^{\leq \mathbb{T}} : \forall k \leq t,\ a_k \in \mathcal{A}_k\left(a^{k-1}\right)\right\}.$$

Let $\ell(h)$ denote the length of history $h \in H$. For each history h and stage $t \leq \ell(h)$, the action profile played in h at stage t is denoted by $a_t(h) \in A$, and the sequence of profiles played up to t is denoted by $a^t(h) \in A^t$. For all $i \in I$, $t \in \mathbb{T}$, and $h \in H$, $a_{i,t}(h) \in A_i$ and $a_i^t(h) \in A_i^t$ are similarly defined. With a slight abuse of notation, we write $\mathcal{A}_i(h) = \mathcal{A}_{i,\ell(h)+1}(h)$ and $\mathcal{A}(h) = \mathcal{A}_{\ell(h)+1}(h)$.

The set of **personal histories** of player i is

$$H_i = H \cup \{(h, a_i) \in H \times A_i : a_i \in \mathcal{A}_i(h)\}.$$

In words, (h, a_i) represents the interim information of player i as soon as he has chosen action a_i given h and before he obtains information about the actions simultaneously chosen

by the coplayers. Thus, the set H_i of personal histories of i contains both standard histories and such "interim histories."

The natural precedence relation on H is denoted by $\preceq$, that is, for all $h, h' \in H$, we write $h \preceq h'$ if and only if h is a prefix of h' (the irreflexive part of $\preceq$ is denoted by $\prec$). For each $i \in I$, $\preceq$ can be extended to a corresponding precedence relation $\preceq$ on H_i in an obvious way.[6]

The set of **terminal histories** or **paths** is

$$Z = \{h \in H : \mathcal{A}(h) = \emptyset\}.$$

Each $z \in Z$ is a complete description of the *actual* behavior of players until the end of the game. For every player $i \in I$ and personal history $h_i \in H_i$,

$$Z(h_i) = \{z \in Z : h_i \preceq z\}$$

denotes the set of paths consistent with h_i. For each $h \in H$, $a_{-i} \in \mathcal{A}_{-i}(h)$, $Z(h, a_{-i}) \subseteq Z$ is similarly defined.

4 Beliefs

In this section we first analyze systems of conditional beliefs about paths and personal features of coplayers, focusing on a key independence property. Next we analyze the hierarchies of conditional beliefs.

4.1 Conditional beliefs and Own-action independence of beliefs

For a fixed player [viz. $i \in I$] we posit an abstract compact metrizable space $\mathcal{T}_{-i}$ that we interpret as the set of profiles of possible personal features of the coplayers. Specifically, each $\tau_{-i} \in \mathcal{T}_{-i}$ is interpreted as a description the personal traits of the coplayers and of how they think, that is, what they think at the beginning of the game and how they update their beliefs upon receiving information about past play. Later we will provide a constructive definition of this abstract space, which right now is not necessary. With this, the (abstract) uncertainty space of player i is the product space $\Omega_{-i} = Z \times \mathcal{T}_{-i}$, and the corresponding collection of conditioning events is

$$\mathcal{F} = \{Z(h_i) \times \mathcal{T}_{-i}\}_{h_i \in H_i} \subseteq \mathcal{B}(\Omega_{-i}).$$

Since Z is finite, each $\Omega_{-i}(h_i) = Z(h_i) \times \mathcal{T}_{-i}$ is clopen. Given that $\mathcal{F}$ is isomorphic to H_i, we denote with (Ω_{-i}, H_i) the corresponding conditional measurable space. The space of CPSs on (Ω_{-i}, H_i) is denoted by $\Delta^{H_i}(\Omega_{-i})$. For simplicity, we write conditional beliefs as $\mu_i(\cdot|h_i)$ instead of $\mu_i(\cdot|\Omega_{-i}(h_i))$.

The definition of conditional belief as expressed in Definition 2 may represent the system of beliefs of an external observer that obtains the same information as player i. But, arguably, reasonable beliefs of player i should satisfy a further condition: *what i believes about his coplayers' features and simultaneous actions is independent of his own actions.* We express this property by means of probabilities conditional on histories, and marginal probabilities

[6] In particular, for $h \in H$, $a_i \in \mathcal{A}_i(h)$, and $h' \in H$, $(h, a_i) \preceq h'$ if and only if $h \prec h'$ and $(h, (a_i, a_{-i})) \preceq h'$ for some $a_{-i} \in \mathcal{A}_{-i}(h)$.

of actions, action profiles and events in $\mathcal{T}_{-i}$. Therefore, for any CPS μ_i on (Ω_{-i}, H_i), we introduce a simplified notation for marginal conditional probabilities summarized by the following table, where $h \in H$, $h_i, h_i' \in H_i$ such that $h_i \prec h_i'$, $E_{-i} \subseteq \mathcal{T}_{-i}$ is measurable, and $(a_i, a_{-i}) \in \mathcal{A}(h)$:

Notation	**Definition**
$\mu_i(h_i'\|h_i)$	$\mu_i(Z(h_i) \times \mathcal{T}_{-i}\|h_i)$
$\mu_i(E_{-i}\|h_i)$	$\mu_i(Z \times E_{-i}\|h_i)$
$\mu_i(a_i, a_{-i}\|h)$	$\mu_i(Z(h, (a_i, a_{-i})) \times \mathcal{T}_{-i}\|h)$
$\mu_i(a_i\|h)$	$\mu_i(Z(h, a_i) \times \mathcal{T}_{-i}\|h) = \sum_{a'_{-i} \in \mathcal{A}_{-i}(h)} \mu_i(a_i, a'_{-i}\|h)$
$\mu_i(a_{-i}\|h)$	$\mu_i(Z(h, a_{-i}) \times \mathcal{T}_{-i}\|h) = \sum_{a'_i \in A_i(h)} \mu_i(a'_i, a_{-i}\|h)$

Table 1. Marginal conditional probabilities

Definition 3 *A CPS μ_i on (Ω_{-i}, H_i) satisfies* own-action independence *(OAI) if*

$$\mu_i\left(Z\left(h, (a_i, a_{-i})\right) \times E_{-i}|h, a_i\right) = \mu_i\left(Z\left(h, \left(a_i', a_{-i}\right)\right) \times E_{-i}|h, a_i'\right)$$

for every $h \in H$, $a_i, a_i' \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$ and measurable $E_{-i} \subseteq \mathcal{T}_{-i}$.[7]
The set of CPSs of i satisfying OAI is denoted by $\Delta_i^{H_i}(\Omega_{-i})$.

Proposition 1 *Consider a CPS μ_i on (Ω_{-i}, H_i). The following are equivalent:*

i) μ_i satisfies OAI;

ii) For every $h \in H$, $(a_i, a_{-i}) \in \mathcal{A}(h)$, and measurable $E_{-i} \subseteq \mathcal{T}_{-i}$,

$$\mu_i\left(Z\left(h, a_{-i}\right) \times E_{-i}|h\right) = \mu_i\left(Z\left(h, (a_i, a_{-i})\right) \times E_{-i}|h, a_i\right);$$

OAI implies that a CPS of player i is made of two independent parts, i's beliefs about his own behavior and his beliefs about the coplayers' behavior and personal features.

Remark 1 *In two-person game forms with observable actions, a CPS μ_i on (Ω_{-i}, H_i) satisfies OAI if and only if there is a unique pair of behavior strategies (σ_i, σ_{-i}) and ν_i as in (iii) of the previous proposition.*

Given a CPS $\mu_i \in \Delta^{H_i}(\Omega_{-i})$, we interpret the corresponding behavior strategy σ_i as the **plan** of player i, and σ_{-i} as the conjecture of i about the coplayers' behavior.

Relying on Lemma 1, we can prove the following result.

Lemma 2 *Set $\Delta_i^{H_i}(\Omega_{-i})$ is compact metrizable.*

To ease the exposition, we refer to CPSs that satisfy the OAI independence property with the acronym ICPS.

[7] The previous condition holds vacuously for terminal information sets.

Definition 4 *We say that an ICPS $\mu_i \in \Delta_i^{H_i}(\Omega_{-i})$ strongly believes event $E_{-i} \in \mathcal{B}(\Omega_{-i})$ if, for every $h \in H$,*

$$\Omega_{-i}(h) \cap E_{-i} \neq \emptyset \Rightarrow \mu_i(E_{-i}|h) = 1.$$

We say that $\mu_i \in \Delta_i^{H_i}(\Omega_{-i})$ strongly believes $\left(E_{-i}^1, ..., E_{-i}^n\right) \in \mathcal{B}(\Omega_{-i})^n$ if μ_i strongly believes E_{-i}^k for every $k \in \{1, ..., n\}$.[8]

The following result adapts Lemma 3 of Battigalli & Tebaldi (2018).[9] It will be crucial in the proof of the main Theorem of this article.

Lemma 3 *Let $\Omega_{-i} = Z \times \mathcal{T}_{-i}$ as above and let X_{-i} be compact metrizable. Fix a decreasing chain $\left(E_{-i}^1, ..., E_{-i}^n\right)$ in $\Omega_{-i} \times X_{-i}$ and let $\mathrm{proj}_{\Omega_{-i}} E_{-i}^m$ be measurable for each $m \in \{1, ..., n\}$. For each $\mu_i \in \Delta_i^{H_i}(\Omega_{-i})$ that strongly believes $\left(\mathrm{proj}_{\Omega_{-i}} E_{-i}^1, \ldots, \mathrm{proj}_{\Omega_{-i}} E_{-i}^n\right)$ there exists $\eta_i \in \Delta_i^{H_i}(\Omega_{-i} \times X_{-i})$ that strongly believes $\left(E_{-i}^1, ..., E_{-i}^n\right)$ and satisfies $marg_{\Omega-i}\eta_i = \mu_i$.*

4.2 Hierarchies of conditional beliefs

For each player $i \in I$, the space of **primitive uncertainty** of i is $\Omega_{-i}^0 = Z \times \Theta_{-i}$, that is, player i is first of all uncertain about how the game is going to be played ($z \in Z$) and of the personal traits of the coplayers ($\theta_{-i} \in \Theta_{-i}$). The set Ω_{-i}^0 is compact metrizable because Z is finite and Θ_{-i} is compact metrizable. It is immediate to see that Ω_{-i}^0 is a particular case of the (abstract) uncertainty space Ω_{-i} introduced in Subsection 4.1: let $\mathcal{T}_{-i} = \Theta_{-i}$. With this, the conditional measurable space $\left(\Omega_{-i}^0, H_i\right)$ is well defined and we can consider ICPSs in $\Delta_i^{H_i}\left(\Omega_{-i}^0\right)$.

For all $i \in I$, hierarchies of ICPSs are recursively defined as follows:

- $\Omega_{-i}^0 = Z \times \Theta_{-i}$; $\quad M_{i,1} = M_i^1 = \Delta_i^{H_i}\left(\Omega_{-i}^0\right)$,
- $\Omega_{-i}^k = \Omega_{-i}^{k-1} \times \prod_{j \in I \setminus \{i\}} M_{j,k}$; $\quad M_{i,k+1} = \Delta_i^{H_i}\left(\Omega_{-i}^k\right)$; $\quad M_i^{k+1} = \prod_{m=1}^{k+1} M_{i,m} \quad (k \in \mathbb{N})$;

where, for all $k \in \mathbb{N}$, Ω_{-i}^k is the **k-th order uncertainty space** of i, $M_{i,k}$ is the space of **k-th order** ICPSs of i and M_i^k is the space of **k-th order hierarchies** of ICPSs of i. A generic element $\mu_i^k = \left(\mu_{i,m}\right)_{m=1}^k$ in M_i^k is a sequence of length k of ICPSs defined on uncertainty spaces of increasing orders. Moreover, for all $k \in \mathbb{N}$, it can be checked that

$$\Omega_{-i}^k = Z \times \Theta_{-i} \times \prod_{m=1}^{k} \prod_{j \in I \setminus \{i\}} M_{j,m}.$$

For each $i \in I$ and $k \in \mathbb{N}$, let

$$M_{-i}^k = \prod_{m=1}^{k} \prod_{j \in I \setminus \{i\}} M_{j,m} \text{ and } \mathcal{T}_{-i}^k = \Theta_{-i} \times M_{-i}^k$$

[8] We can show that OAI implies that if the condition displayed above holds for all $h \in H$, then it also holds for all $h_i \in H_i$.

[9] In particular, we rely on a generalization of their result proved in the working paper version of the article (IGIER w.p. 609).

respectively denote the spaces of k-order hierarchies and personal features of coplayers; with this, each k-order uncertainty space $\Omega_{-i}^k = Z \times \mathcal{T}_{-i}^k$ has the same structure of the abstract space Ω_{-i} introduced in Subsection 4.1. By repeated applications of Lemma 2 and Tychonoff's Theorem, one can show that each M_i^k is compact metrizable. Thus, for every $k \in \mathbb{N}$, $\mathcal{T}_{-i}^k$ is compact metrizable and $\left(\Omega_{-i}^k, H_i\right)$ is a well defined conditional measurable space.[10]

An **infinite hierarchy of ICPSs** of player i is a denumerable sequence $\mu_i^\infty = \left(\mu_{i,k}\right)_{k\in\mathbb{N}} \in M_i^\infty$, where $M_i^\infty = \prod_{k\in\mathbb{N}} M_{i,k}$. For every $i \in I$, M_i^∞ is compact metrizable, and so are $M_{-i}^\infty = \prod_{j\in I\setminus\{i\}} M_j^\infty$ and $M^\infty = \prod_{i\in I} M_i^\infty$. The **total state space** of game Γ is

$$\Omega^\infty = Z \times \Theta \times M^\infty.$$

Each state $\omega^\infty = (z, \theta, \mu^\infty)$ in Ω^∞ is a complete description of the actual behavior of players, of their personal traits, and of their systems of conditional beliefs of all orders. Since player i is assumed to know both his personal traits θ_i and his infinite hierarchy μ_i^∞, the **total uncertainty space** of i is

$$\Omega_{-i}^\infty = Z \times \Theta_{-i} \times M_{-i}^\infty.$$

The **spaces of personal features** of player i and coplayers $-i$ are, respectively, $\mathcal{T}_i^\infty = \Theta_i \times M_i^\infty$ and $\mathcal{T}_{-i}^\infty = \Theta_{-i} \times M_{-i}^\infty$. With this, we can write the total uncertainty space of i as $\Omega_{-i}^\infty = Z \times \mathcal{T}_{-i}^\infty$, and $\left(\Omega_{-i}^\infty, H_i\right)$ is a well defined conditional measurable space.

4.3 Belief Coherence

So far, we did not impose the requirement that beliefs of different orders in a hierarchy are mutually consistent, i.e., that they assign the same probabilities to events of lower order of uncertainty, such as events about behavior. Next we consider hierarchies satisfying this requirement, that we interpret as a cognitive rationality condition.[11]

Definition 5 *Fix an infinite belief hierarchy $\mu_i^\infty \in M_i^\infty$. We say that μ_i^∞ is* ***coherent*** *if, for all $k \in \mathbb{N}$, and $h_i \in H_i$,*

$$marg_{\Omega_{-i}^{k-1}} \mu_{i,k+1}\left(\cdot|h_i\right) = \mu_{i,k}\left(\cdot|h_i\right). \tag{1}$$

For each $i \in I$, we let C_i^∞ denote the subset of coherent hierarchies in M_i^∞ and let $C^\infty = \prod_{i\in I} C_i^\infty$. For each $n \in \mathbb{N}$, C_i^n and C^n are similarly defined, i.e., condition (1) must hold for all $k \leq n$. The following technical result implies that C_i^∞ has the same topological properties of M_i^∞.

Lemma 4 *Set C_i^∞ is compact metrizable.*

[10] Of course, the countable collection of clopen subsets of Ω_{-i}^k is given by

$$\left\{Z\left(h_i\right) \times \mathcal{T}_{-i}^k\right\}_{h_i\in H_i} \subseteq \mathcal{B}\left(\Omega_{-i}^k\right),$$

which is isomorphic to H_i for all $k \in \mathbb{N}$.

[11] Note, we did impose some cognitive rationality requirements: the chain rule and OAI. We did this only for brevity. We could allow for more deviations from cognitive rationality at arbitrary states of the world, requiring all the relevant consistency conditions only at states where players are rational.

This easily follows from the fact that, for all $k \in \mathbb{N}$ and $h_i \in H_i$, the map

$$\mu_{i,k+1}(\cdot|h_i) \mapsto \text{marg}_{\Omega_{-i}^{k-1}} \mu_{i,k+1}(\cdot|h_i)$$

is continuous (see for example Aliprantis & Border 2006, Chapter 15).

The following result adapts Proposition 1 of Battigalli & Siniscalchi (1999) to the current setup.[12]

Proposition 2 *There exists a* ***canonical*** *homeomorphism* $g_i : C_i^\infty \to \Delta_i^{H_i}\left(\Omega_{-i}^\infty\right)$ *such that for all* $\mu_i^\infty \in C_i^\infty$ *and* $k \in \mathbb{N}$, $marg_{\Omega_{-i}^{k-1}} g_i\left(\mu_i^\infty\right) = \mu_{i,k}$.

Finally, It is straightforward to define the space of **coherent k-th order hierarchies** of ICPSs which is denoted by C_i^k.

5 Psychological Games and Rationality

Next we introduce belief-dependent utilities to obtain a psychological game from the game form. Fix a multistage game form Γ, a **k-th order psychological game** based on Γ is a structure

$$PG_k = \left\langle \Gamma, (v_i)_{i \in I} \right\rangle$$

where, for every $i \in I$, the **psychological utility function** $v_i : Z \times \Theta \times M^k \to \mathbb{R}$ is *continuous*.[13] According to the definition, the **utility-relevant state space** of each player is $\Omega^k = Z \times \Theta \times M^k$. Note that the total state space can be factorized as follows:

$$\Omega^\infty = \Omega^k \times \prod_{i \in I} \prod_{m \geq k} \Delta_i^{H_i}\left(\Omega_{-i}^m\right) = Z \times \prod_{i \in I} \left(\mathcal{T}_i^k \times \prod_{m > k} M_i^m \right).$$

Since player i knows his own **utility-relevant personal features** $\left(\theta_i, \mu_i^k\right) \in \Theta_i \times M_i^k = \mathcal{T}_i^k$, his **utility-relevant uncertainty space** is $\Omega_{-i}^k = Z \times \mathcal{T}_{-i}^k$.

Clearly, a rational player i has to consult his $(k+1)$-order ICPS $\mu_{i,k+1}$ in order to take expectations of his psychological utility. Specifically, if i has observed $h \in H$, for each $a_i \in \mathcal{A}_i(h)$, he computes the corresponding expected utility given θ_i and $\mu_{i,k+1}$, that is,

$$u_{i,h}\left(a_i, \theta_i, \mu_{i,k+1}\right) = \int_{\Omega_{-i}^k} v_i\left(\theta_i, \mu_i^k, \omega_{-i}^k\right) \mu_{i,k+1}\left(\mathrm{d}\omega_{-i}^k | h, a_i\right). \tag{2}$$

Hence, for all $i \in I$, we have a vector

$$\left(u_{i,h} : \mathcal{A}_i(h) \times \Theta_i \times M_{i,k+1} \to \mathbb{R}\right)_{h \in H}$$

of **psychological decision utilities** defined as in (2).

[12] Proposition 1 of Battigalli & Siniscalchi (1999) considers an abstract conditional measurable space, not one based on a game; therefore, the game-based OAI condition used in the definition of C_i^∞ and $\Delta_i^{H_i}\left(\Omega_{-i}^\infty\right)$ cannot even be expressed. The game-theoretic applications of the second part of that article instead impose a requirement in the spirit of OAI as part of the definition of rationality.

[13] As in Battigalli & Dufwenberg (2009), we take continuity of psychological utilities as a maintained assumption. In their analysis of static psychological games, Jagau & Perea (2017) relax the continuity.

Remark 2 *For every $i \in I$ and $h \in H$, $u_{i,h}$ is continuous.*

In the rest of the paper we only refer to such decision utilities to define players' rationality and derive our results. Note that we could have considered vector $(u_{i,h})_{(i,h)\in I\times H}$ as the primitive element of the analysis, focusing on the reduced-form continuous utilities $(u_{i,h})_{(i,h)\in I\times H}$.[14]

Given $h \in H$, player i with personal traits θ_i and beliefs $\mu_{i,k+1}$ has to solve the following problem:

$$\max_{a_i\in\mathcal{A}_i(h)} u_{i,h}\left(a_i, \theta_i, \mu_{i,k+1}\right).$$

With this, for every $h \in H$, we define the **local best-reply correspondence** of i at h as

$$\begin{array}{rccl} r_{i,h}: & \Theta_i \times M_{i,k+1} & \rightrightarrows & \mathcal{A}_i(h), \\ & \left(\theta_i, \mu_{i,k+1}\right) & \mapsto & \arg\max_{a_i\in\mathcal{A}_i(h)} u_{i,h}\left(a_i, \theta_i, \mu_{i,k+1}\right). \end{array}$$

Remark 3 *For every $i \in I$ and $h \in H$, $r_{i,h}$ is non-empty valued and upper hemicontinuous.*

5.1 Belief and plans

Conditional on each history $h \in H$ and given his first-order ICPS $\mu_{i,1} \in M_{i,1} = \Delta_i^{H_i}\left(\Omega_{-i}^0\right)$, player i assigns probability[15]

$$\text{marg}_Z\mu_{i,1}\left(Z\left(h,(a_i,a_{-i})\right)|h\right) = \text{marg}_Z\mu_{i,1}\left(Z\left(h,a_i\right)|h\right) \times \text{marg}_Z\mu_{i,1}\left(Z\left(h,a_{-i}\right)|h\right)$$

to each action profile (a_i, a_{-i}), where $Z(h, a_{-i}) = \bigcup_{j\in I\setminus\{i\}} Z(h, a_j)$ and the factorization follows from OAI and the chain rule. We interpret the system of conditional probabilities of *own* actions as the subjective plan of i (given $\mu_{i,1}$). Consistently with the notation of Section 4.1, we define the plan function

$$\begin{array}{rccl} \hat{\sigma}_i: & M_{i,1} & \to & \prod_{h\in H}\Delta\left(\mathcal{A}_i(h)\right), \\ & \mu_{i,1} & \mapsto & \left(\left(\text{marg}_Z\mu_{i,1}\left(Z\left(h,a_i\right)|h\right)\right)_{a_i\in\mathcal{A}_i(h)}\right)_{h\in H}, \end{array}$$

which gives the subjective plan of player i implied by $\mu_{i,1}$.[16] Note that, we take the space of first-order beliefs as domain of $\hat{\sigma}_i$. This choice—although natural—is somewhat arbitrary, because one can derive a plan for i from a belief of any order. Yet, we will focus on plans of rational players, whose belief hierarchy is coherent and hence yields the same plan starting from beliefs of any order.

The following example illustrates the concepts introduced so far.

Example 1 *(Guilt aversion in the Trust Minigame)* Following Dufwenberg (2002) and Battigalli & Dufwenberg (2009), we model simple guilt aversion in a two-person game with monetary payoffs as the desire—other things being equal—not to disappoint the other player. The

[14] Such reduced form approach is more general because it does not require that decision utilities are derived from an overall utility over terminal nodes. As explained in Battigalli et al. (2018) this allows to model some belief-dependent action tendencies like being aggressive when frustrated.

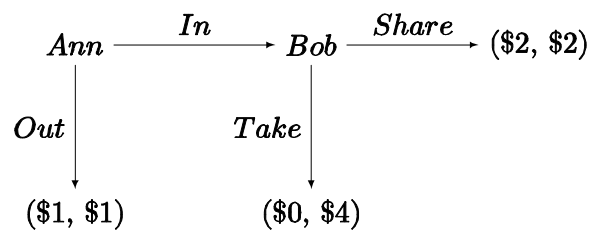
[15] Note, we have to consider marginal probabilities because player i is also uncertain about the co-players' personal traits θ_{-i}.

[16] In the standard language of game theory such plans are called "behavior strategies."

disappointment of player j at terminal history z given first-order ICPS $\mu_{j,1}$ is the difference, if positive, between her or his (initially) expected monetary playoff and his realized monetary payoff at z. Let $D_j\left(z, \mu_{j,1}\right)$ denote this difference. The psychological utility function of player $i \neq j$ is

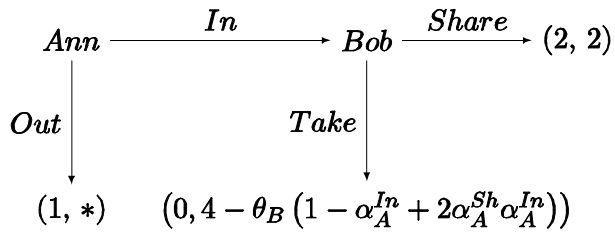
$$v_i\left(z, \theta, \mu_1\right) = m_i\left(z\right) - \theta_i D_j\left(z, \mu_{j,1}\right),$$

where $m_i\left(z\right)$ denotes the monetary payoff of i at z, and $\theta_i \in \Theta_i = \left[\underline{\theta}_i, \bar{\theta}_i\right] \subseteq \mathbb{R}_+$. Suppose that the game with monetary payoffs is the Trust Minigame in Figure 1:



Trust Minigame with monetary payoffs

Furthermore suppose for simplicity that it is common knowledge that Ann is selfish and risk neutral, that is, $\Theta_A = \{0\}$. To ease notation, let $\alpha_A^{In} = \hat{\sigma}_A\left(\mu_{A,1}\right)(In|\varnothing)$ and $\alpha_A^{Sh} = \hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In)$ denote Ann's plan to play In and her conjecture about Bob choosing $Share$. With this, the resulting psychological game is represented in Figure 1



Trust Minigame with (asymmetric) guilt aversion

The decision utility of $Take$ for Bob given In is

$$u_{B,In}\left(Tk, \theta_B, \mu_{B,2}\right) = 4 - \theta_B \mathbb{E}_{\mu_{B,2}}\left[D_A\left((In, Tk), \mu_{A,1}\right) | (In, Tk)\right] \tag{3}$$

$$= 4 - \theta_B \int_{M_{A,1}} \left[1 - \hat{\sigma}_A\left(\mu_{A,1}\right)(In|\varnothing) + 2\hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In)\,\hat{\sigma}_A\left(\mu_{A,1}\right)(In|\varnothing)\right] \mu_{B,2}\left(\mathrm{d}\mu_{A,1}|In\right),$$

where in the second equality we rely on OAI, which implies $\mu_{B,2}\left(\mathrm{d}\mu_{A,1}|In\right) = \mu_{B,2}\left(\mathrm{d}\mu_{A,1}|\left(In, Tk\right)\right)$.
▲

5.2 Rational planning and material consistency

Recall that player i has a coherent belief at $\omega_i^\infty = (z, \theta_i, \mu_i^\infty) \in \Omega_i^\infty$ whenever $\mu_i^\infty \in C_i^\infty$, that is, whenever the prevailing state of the world ω_i^∞ belongs to

$$\Omega_i^{\infty,*} = \left\{ \left(\bar{z}, \bar{\theta}_i, \bar{\mu}_i^\infty \right) \in \Omega_i^\infty : \bar{\mu}_i^\infty \in C_i^\infty \right\}$$

which is clearly measurable. Event $\Omega_i^{\infty,*}$ represents the statement "Player i has coherent beliefs."

Remark 4 *The set $\Omega_i^{\infty,*}$ is compact.*[17]

Our notion of rationality is given by the conjunction of several consistency conditions. We begin with the condition requiring optimality of the subjective plan of player i:

Definition 6 *Player i is a **rational planner (RP)** at $(z, \theta_i, \mu_i^\infty) \in \Omega_i^\infty$ if $\mu_i^\infty \in C_i^\infty$ and*

$$\hat{\sigma}_i \left(\mu_{i,1} \right) \left(r_{i,h} \left(\theta_i, \mu_{i,k+1} \right) | h \right) = 1$$

for every $h \in H$.

Thus, the event that player i is a rational planner is:

$$RP_i = \left\{ (z, \theta_i, \mu_i^\infty) \in \Omega_i^{\infty,*} : \forall h \in H, \hat{\sigma}_i \left(\mu_{i,1} \right) \left(r_{i,h} \left(\theta_i, \mu_{i,k+1} \right) | h \right) = 1 \right\}.$$

Lemma 5 *Event RP_i is non-empty and compact.*

Another aspect of rationality is the consistency between planned behavior and actual behavior:

Definition 7 *Player i is **materially consistent (MC)** at $(z, \theta_i, \mu_i^\infty) \in \Omega_i^\infty$ if $\mu_i^\infty \in C_i^\infty$ and, for all $h \in H$,*

$$h \prec z \implies \hat{\sigma}_i \left(\mu_{i,1} \right) \left(a_{i,h} (z) | h \right) > 0,$$

where $a_{i,h}(z) \in \mathcal{A}_i(h)$ is the unique feasible action of i at h implied by z.

The corresponding event is

$$MC_i = \left\{ (z, \theta_i, \mu_i^\infty) \in \Omega_i^{\infty,*} : \forall h \in H, h \prec z \implies \hat{\sigma}_i \left(\mu_{i,1} \right) \left(a_{i,h} (z) | h \right) > 0 \right\}.$$

Note that we can similarly define the event that player i is **strictly materially consistent** as

$$sMC_i = \left\{ (z, \theta_i, \mu_i^\infty) \in \Omega_i^{\infty,*} : \forall h \in H, h \prec z \implies \hat{\sigma}_i \left(\mu_{i,1} \right) \left(a_{i,h} (z) | h \right) = 1 \right\}$$

where we require that player i's subjective plan assigns probability 1 to his actual behavior.

Lemma 6 *Set MC_i is nonempty and measurable, sMC_i is nonempty and compact.*

Rationality is therefore defined as follows.

[17] To see this, just notice that $\Omega_i^{\infty,*} = Z \times \Theta_i \times C_i^\infty$, where C_i^∞ is compact (Lemma 4).

Definition 8 *Player i is* ***rational*** *at $(z, \theta_i, \mu_i^\infty) \in \Omega_i^\infty$ if i is a rational planner and is materially consistent at $(z, \theta_i, \mu_i^\infty)$.*

The event that player i is rational is denoted by $R_i = RP_i \cap MC_i$. Finally, we define the events "Every player is rational" and "Every coplayer of player i is rational," respectively in Ω^∞ and Ω_{-i}^∞, as:

$$R = \bigcap_{i \in I} \left(R_i \times \mathcal{T}_{-i}^\infty \right) \text{ and } R_{-i} = \bigcap_{j \in I \setminus \{i\}} \left(R_j \times \prod_{\iota \in I \setminus \{i,j\}} \mathcal{T}_\iota^\infty \right),$$

with the convention that, whenever we analyze 2-player games, the set $\prod_{\iota \in I \setminus \{i,j\}} \mathcal{T}_\iota^\infty$ is a singleton and R_{-i} is equal to R_j. Moreover, we define the sets sR_i, sR and sR_{-i} by replacing MC_i with sMC_i in all the previous definitions.

Remark 5 *Sets R_i, R_{-i} and R are non-empty and measurable; sets sR_i, sR_{-i} and sR are compact.*

Note that sR_i is the event that i is rational and has a deterministic plan. This set can be empty. Indeed, there are cases in which only non-deterministic plans are consistent with rational planning.

6 Strong Belief and Strong Rationalizability

In this section, we adapt Battigalli & Siniscalchi's (2002) notion of strong belief to our framework and then provide the definition of strong rationalizability. We first focus on events within the utility-relevant space of uncertainty of player i. We say that $(k+1)$-th order belief $\mu_{i,k+1} \in M_{i,k+1}$ **strongly believes** an event $E_{-i} \subseteq \Omega_{-i}^k$ if

$$\Omega_{-i}^k(h) \cap E_{-i} \neq \emptyset \implies \mu_{i,k+1}(E_{-i} | h) = 1,$$

for all $h \in H$.

We are now ready to present the algorithm defining **strong rationalizability**, which is meant to capture the predictions for behavior and low-order beliefs implied by rationality and forward-induction reasoning (see Theorem 1). Recall that, for all $i \in I$, $\mathcal{T}_i^k = \Theta_i \times M_i^k$ and $\mathcal{T}_{-i}^k = \Theta_{-i} \times M_{-i}^k$ are the spaces of (k-order) personal features of i and $-i$ respectively.

(Step 0) For all $i \in I$, let $\mathbf{P}_i(0) = \Omega_i^k$, $\mathbf{P}_{-i}(0) = \Omega_{-i}^k$ and $\mathbf{P}(0) = \Omega^k$.

(Step $n > 0$) Assume that $\mathbf{P}_i(m)$, $\mathbf{P}_{-i}(m)$ and $\mathbf{P}(m)$ have been defined for all $m \in \{0, ..., n-1\}$. For each $i \in I$, let $\left(z, \theta_i, \mu_i^k\right) \in \mathbf{P}_i(n)$ if there exists $\mu_{i,k+1} \in M_{i,k+1}$ such that:

- *(Coherence)* $\left(\mu_i^k, \mu_{i,k+1}\right) \in C_i^{k+1}$;
- *(RP)* for all $h \in H$, $\hat{\sigma}_i\left(\mu_{i,1}\right)\left(r_{i,h}\left(\theta_i, \mu_{i,k+1}\right) | h\right) = 1$;
- *(MC)* for all $h \in H$, if $h \prec z$, then $\hat{\sigma}_i\left(\mu_{i,1}\right)\left(a_{i,h}(z) | h\right) > 0$;
- *(Strong belief)* for every $m \in \{1, ..., n-1\}$, $\mu_{i,k+1}$ strongly believes $\mathbf{P}_{-i}(m)$.

Finally, for all $i \in I$, let

$$\mathbf{P}_{-i}(n) = \bigcap_{j \in I \setminus \{i\}} \left(\mathbf{P}_j(n) \times \prod_{\iota \in I \setminus \{i,j\}} \mathcal{T}_\iota^k \right) \text{ and } \mathbf{P}(n) = \bigcap_{i \in I} \left(\mathbf{P}_i(n) \times \mathcal{T}_{-i}^k \right).$$

Proposition 3 *For every $n \in \mathbb{N}$, the following are true*

i) For every $i \in I$, $\mathbf{P}_i(n)$ is measurable;

ii) For every $i \in I$, $\mathbf{P}_i(n) \subseteq \mathbf{P}_i(n-1)$, $\mathbf{P}_{-i}(n) \subseteq \mathbf{P}_{-i}(n-1)$ and $\mathbf{P}(n) \subseteq \mathbf{P}(n-1)$.

Therefore, the sequence of prediction sets $(\mathbf{P}(n))_{n \in \mathbb{N}_0}$ is *decreasing* and it is standard to define $\mathbf{P}(\infty) = \bigcap_{n \in \mathbb{N}_0} \mathbf{P}(n)$. If $\omega^k \in \mathbf{P}(\infty)$ then we say that ω^k is **strongly rationalizable.**

If we replace condition MC above with *strict* material consistency (sMC), we obtain the decreasing chain of events $(\mathbf{sP}(n))_{n \in \mathbb{N}_0}$ and its limit set $\mathbf{sP}(\infty)$. Note that, in general, we cannot prove the non-emptiness of $\mathbf{P}(\infty)$ or $\mathbf{sP}(\infty)$. However, we can provide general and relevant sufficient conditions to obtain these results. For example, assume that v_i *does not depend* on the k-th order hierarchy of i.[18] In this case, v_i is just like a state-dependent utility function and we can rely on the standard results about subjective expected utility maximization, such as the one-shot deviation principle and the existence of deterministic optimal plans. This yields the behavioral equivalence and non-emptiness of $\mathbf{P}(\infty)$ or $\mathbf{sP}(\infty)$.[19]

Proposition 4 *If, for every $i \in I$, v_i does not depend on the k-th order hierarchy of i, then:*

i) For every $n \in \mathbb{N} \cup \{\infty\}$, $proj_Z \mathbf{P}(n) = proj_Z \mathbf{sP}(n)$.

ii) For every $n \in \mathbb{N} \cup \{\infty\}$, $\mathbf{sP}(n)$ is non-empty and compact.

Example 2 *(Strong rationalizability in the Trust Minigame)* Consider the Trust Minigame of Figure 1. Focusing only on the key steps and omitting θ_A—which is commonly known to be 0—it can be checked that

$$\mathbf{P}_A(1) = \left\{ \left(Out, \mu_{A,1}\right) : \hat{\sigma}_A\left(\mu_{A,1}\right)(Out|\varnothing) > 0, \hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In) \leq 1/2 \right\} \cup$$
$$\cup \left\{ \left((In, a_B), \mu_{A,1}\right) : a_B \in \{Sh, Tk\}, \hat{\sigma}_A\left(\mu_{A,1}\right)(In|\varnothing) > 0, \hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In) \geq 1/2 \right\},$$

$$\begin{aligned} \mathbf{P}_B(2) \;=\; & \left\{ \left(z, \theta_B, \mu_{B,1}\right) : z \in \{Out, (In, Tk)\}, \theta_B < 1, \hat{\sigma}_B\left(\mu_{B,1}\right)(Tk|In) = 1 \right\} \cup \\ & \cup \left\{ \left(z, \theta_B, \mu_{B,1}\right) : 1 \leq \theta_B \leq 2 \right\} \cup \\ & \cup \left\{ \left(z, \theta_B, \mu_{B,1}\right) : z \in \{Out, (In, Sh)\}, \theta_B > 2, \hat{\sigma}_B\left(\mu_{B,1}\right)(Sh|In) = 1 \right\}. \end{aligned}$$

[18] Actually, it is sufficient to assume that i's utility does not depend on i's plan, which is just a (first-order) feature of i's belief hierarchy. We consider the stronger assumption in the text to simplify the exposition.

[19] The proof of Proposition 4 is available upon request. The first part follows from the fact that a behavior strategy is a sequential best reply to a conditional probability system (about the co-players) if and only if every pure strategy in its "support" is also a sequential best reply. The proof of the second part adapts the proof of Theorem 13 in Battigalli & Dufwenberg (2009).

The first step for Ann says that she plans to go In with positive probability only she assigns at least 50% probability that Bob would $Share$ given In.[20] Note that material consistency implies that when Ann plans to go Out (respectively, In) her action on the actual path must be consistent with such plan. Similar considerations apply to the following steps. To understand the second step for Bob, note that his first-order ICPS $\mu_{B,1}$ must assign probability 1 to event

$$\left\{\left(\left(In, a_B\right), \mu_{A,1}\right) : a_B \in \{Sh, Tk\}, \hat{\sigma}_A\left(\mu_{A,1}\right)(In|\varnothing) > 0, \hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In) \geq 1/2\right\}$$

upon observing In, because he strongly believes in Ann's rationality (forward induction). This implies that, upon observing In, Bob is certain that Ann initially expected a payoff of at least 1; thus, $\mathbb{E}_{\mu_{B,2}}\left[D_A\left(\left(In, Tk\right), \mu_{A,1}\right)|In\right] \geq 1$. With this, the guilt-aversion formula (3) implies that, if $\theta_B > 2$, Bob would $Share$ if given the opportunity, that is, he plans to $Share$ with probability 1 given In. For each $\theta_B \in [1, 2]$ and plan σ_B we can find a second-order ICPS $\mu_{B,2}$ that satisfies the aforementioned forward-induction requirement and yields $\sigma_B = \hat{\sigma}_B\left(marg_Z \mu_{B,2}\right)$. If instead $\theta_B < 1$, despite forward-induction reasoning, Bob plans to $Take$ because he is insufficiently guilt averse. The third step (for Ann) depends on the bounds on θ_B. In particular,

$$\bar{\theta}_B < 1 \Rightarrow \mathbf{P}_A(3) = \left\{\left(Out, \mu_{A,1}\right) : \hat{\sigma}_A\left(\mu_{A,1}\right)(Out|\varnothing) = 1, \hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In) = 0\right\}$$

and

$$\underline{\theta}_B > 2 \Rightarrow \mathbf{P}_A(3) =$$
$$= \left\{\left(\left(In, a_B\right), \mu_{A,1}\right) : a_B \in \{Sh, Tk\}, \hat{\sigma}_A\left(\mu_{A,1}\right)(In|\varnothing) = 1, \hat{\sigma}_A\left(\mu_{A,1}\right)(Sh|In) = 1\right\}.$$

Case $\underline{\theta}_B < 2 < \bar{\theta}_B$ is a bit harder to describe in detail, but the upshot is that Ann's behavior is unrestricted. ▲

7 Epistemic Justification of strong rationalizability

In this section we provide an epistemic justification for the algorithm of strong rationalizability defined above. Recall that, for all $h \in H$, $\Omega_{-i}^{\infty}(h) = Z(h) \times \mathcal{T}_{-i}^{\infty}$.

Definition 9 *The **strong belief operator** of i is a map $\mathrm{SB}_i : \mathcal{B}\left(\Omega_{-i}^{\infty}\right) \to 2^{\Omega_i^{\infty}}$ defined as*

$$\mathrm{SB}_i(E_{-i}) = \left\{(z, \theta_i, \mu_i^{\infty}) \in \Omega_i^{\infty,*} : \forall h \in H, \Omega_{-i}^{\infty}(h) \cap E_{-i} \neq \emptyset \Rightarrow g_i(\mu_i^{\infty})(E_{-i}|h) = 1\right\},$$

for all $E_{-i} \in \mathcal{B}\left(\Omega_{-i}^{\infty}\right)$.

The following lemma clarifies some of the properties of SB_i.

Remark 6 *For all $E_{-i} \in \mathcal{B}\left(\Omega_{-i}^{\infty}\right)$, $\mathrm{SB}_i(E_{-i})$ is measurable; if E_{-i} is closed then $\mathrm{SB}_i(E_{-i})$ is closed.*

[20] Since in this step we are not assuming that Ann believes in Bob's rationality and θ_B does not directly affect Ann's utility, her beliefs about θ_B are unrestricted.

We express our epistemic assumptions as events about behavior and personal features (personal traits and beliefs) of each player i, that is, measurable subsets of $\Omega_i^\infty = Z \times \mathcal{T}_i^\infty$. For instance, we say that player i is rational and strongly believes the rationality of his coplayers at $(z, \tau_i^\infty) \in \Omega_i^\infty$ if $(z, \tau_i^\infty) \in R_i \cap \mathrm{SB}_i(R_{-i}) \subseteq \Omega_i^\infty$. On the one hand, the elements of R_i satisfy cross restrictions concerning actual behavior and the personal features τ_i^∞ of player i, on the other hand, the elements of $\mathrm{SB}_i(R_{-i})$ are characterized only by restrictions concerning the personal features of i, that is, $\mathrm{proj}_Z \mathrm{SB}_i(R_{-i}) = Z$. Similarly, the event that the coplayers $-i$ are rational and strongly believe in their coplayers' rationality is

$$\bigcap_{j \in I \setminus \{i\}} \left((R_j \cap \mathrm{SB}_j(R_{-j})) \times \prod_{\iota \in I \setminus \{i,j\}} \mathcal{T}_\iota^\infty \right) \subseteq \Omega_{-i}^\infty.$$

Consider the following epistemic assumptions of increasing strength:

(Step 1) For every $i \in I$, let $R_i(1) = R_i \subseteq \Omega_i^\infty$ and $R_{-i}(1) = R_{-i} \subseteq \Omega_{-i}^\infty$.

(Step n) Assume that $R_i(m)$ and $R_{-i}(m)$ have been defined for every $i \in I$ and $m \in \{1, ..., n-1\}$, then define

$$R_i(n) = R_i(n-1) \cap \mathrm{SB}_i(R_{-i}(n-1))$$

and

$$R_{-i}(n) = \bigcap_{j \in I \setminus \{i\}} \left(R_j(n) \times \prod_{\iota \in I \setminus \{i,j\}} \mathcal{T}_\iota^\infty \right).$$

With this, for every $n \in \mathbb{N}$, event $R(n) = \bigcap_{i \in I} \left[R_i(n) \times \mathcal{T}_{-i}^\infty \right]$ represents the hypothesis of rationality (of all players) and n-mutual strong belief in rationality. By definition, it follows that, for every $i \in I$ and $n \in \mathbb{N}$, $R_i(n+1) \subseteq R_i(n)$ and $R(n+1) \subseteq R(n)$. Therefore, the event in Ω^∞ that represents *rationality and common strong belief in rationality* is $R(\infty) = \bigcap_{n \in \mathbb{N}_0} R(n)$. As explained in Battigalli & Siniscalchi (2002) these assumptions require, on top of rationality, that each player ascribes to his coplayers the highest degree of "strategic sophistication" consistent with their observed behavior, that is, given any $h \in H$ player i assigns probability 1 to $R_{-i}\left(n^*_{-i,h}\right)$, where

$$n^*_{-i,h} = \max \left\{ n \in \mathbb{N} : \Omega_{-i}^\infty(h) \cap R_{-i}(n) \neq \emptyset \right\}.$$

We can now state the main result of this paper.

Theorem 1 *For every $n \in \mathbb{N}$,*

$$\forall i \in I, \mathbf{P}_i(n) = \mathrm{proj}_{\Omega_i^k} R_i(n) \quad \textit{and} \quad \mathbf{P}(n) = \mathrm{proj}_{\Omega^k} R(n).$$

Define Relying on Theorem 1, under the assumption that the psychological utility v_i of every player i does not depend on his own beliefs, one can use standard compact-continuity arguments to prove the following result.

Proposition 5 *If, for every $i \in I$, v_i does not depend on the k-th order hierarchy of i, then*

$$\mathbf{sP}(\infty) = \mathrm{proj}_{\Omega^k} sR(\infty).$$

8 Discussion

To ease the exposition and simplify our notation, we made some simplifying assumptions and omitted possible generalizations. Here we hint at extensions of our analysis.

Infinite games We considered finite games forms, but our analysis extends to all multistage games where players' feasible action sets are finite at all histories of height 2 or more and their psychological utility functions do not depend on terminal beliefs, as in this case we can still adapt the techniques of Battigalli & Tebaldi (2018). This covers, for example, all compact-continuous games with simultaneous moves where utility depends only on initial beliefs.

Imperfectly observable actions We assumed that the actions of previous stages are perfectly observed because this allows to simplify the notation and streamline the analysis. But we can prove our results for all multistage games. Conceptually, such generalization relies on the fact that cognitive rationality implies perfect recall (we model perfect recall as a personal feature, not a property of the rules of the game).

Generalized psychological utilities In our analysis, the decision utility of action a_i given history h is the subjective expectation of psychological utility v_i conditional on (h, a_i) given personal traits θ_i and $k+1$-order belief $\mu_{i,k+1}$. With this, we obtain a continuous "local" utility function $u_{i,h}\left(a_i, \theta_i, \mu_{i,k+1}\right)$. We emphasized that only the local utility functions $(u_{i,h})_{(i,h)\in I\times H}$ matter for our epistemic analysis. Thus, the results in this article are valid for psychological games with more general forms of psychological preferences. For example, we may obtain $u_{i,h}$ as a local "distortion" of the conditional expectation of "experience utility" v_i (see Battigalli *et al.*, 2018).

Rationalizable self-confirming equilibrium The self-confirming equilibrium (SCE) concept—a generalization of Nash equilibrium—characterizes the steady states of learning dynamics in games played recurrently. According to SCE, agents are asymptotic empiricists who need not believe in the strategic sophistication of others. In a (strongly) rationalizable SCE (RSCE) agents (strongly) believe in the strategic sophistication of others. We can provide an algorithmic definition of RSCE for multistage (k-order psychological) games and an epistemic justification of RSCE.

Restrictions on low-order beliefs and Δ-rationalizability In applications, it is often natural to impose some restrictions on low-order beliefs and assume that such restrictions shape strategic reasoning (see, e.g., Battigalli & Tebaldi, 2018 and references therein). This yields a modified notion of strong rationalizability, called strong Δ-rationalizability (where Δ represents the restricted set of profiles of beliefs). We can define strong Δ-rationalizability for k-order psychological games and provide an epistemic justification of this solution concept.

9 Proofs

Proof of Proposition 1.

$(i) \implies (ii)$ Fix $h \in H$, $(a_i, a_{-i}) \in \mathcal{A}(h)$, and a measurable $E_{-i} \subseteq \mathcal{T}_{-i}$. It follows

$$\begin{aligned}
\mu_i\left(Z\left(h, a_{-i}\right) \times E_{-i}|h\right) &= \sum_{a_i' \in \mathcal{A}_i(h)} \mu_i\left(Z\left(h, \left(a_i', a_{-i}\right)\right) \times E_{-i}|h\right) \\
&= \sum_{a_i' \in \mathcal{A}_i(h)} \mu_i\left(Z\left(h, \left(a_i', a_{-i}\right)\right) \times E_{-i}|h, a_i'\right) \mu_i\left(a_i'|h\right) \\
&= \mu_i\left(Z\left(h, \left(a_i, a_{-i}\right)\right) \times E_{-i}|h, a_i\right) \left(\sum_{a_i' \in \mathcal{A}_i(h)} \mu_i\left(a_i'|h\right)\right) \\
&= \mu_i\left(Z\left(h, \left(a_i, a_{-i}\right)\right) \times E_{-i}|h, a_i\right)
\end{aligned}$$

where the third equality holds by OAI.

$(ii) \implies (iii)$ Let $\mu_i \in \Delta^{H_i}(\Omega_{-i})$ satisfy (ii). For every $h \in H$, $(a_i, a_{-i}) \in \mathcal{A}(h)$, $z \in Z$ and measurable $E_{-i} \subseteq \mathcal{T}_{-i}$, define

$$\begin{aligned}
\sigma_i\left(a_i|h\right) &= \mu_i\left(Z\left(h, a_i\right) \times \mathcal{T}_{-i}|h\right), \\
\sigma_{-i}\left(a_{-i}|h\right) &= \mu_i\left(Z\left(h, a_{-i}\right) \times \mathcal{T}_{-i}|h\right), \\
\nu_i\left(E_{-i}|h\right) &= \mu_i\left(Z\left(h\right) \times E_{-i}|h\right).
\end{aligned}$$

First,

$$\mu_i\left(Z\left(h'\right) \times E_{-i}|h\right) = \mu_i\left(Z\left(h'\right) \times E_{-i}|h'\right) \mu_i\left(Z\left(h'\right) \times \mathcal{T}_{-i}|h\right)$$

$$\begin{aligned}
&= \mu_i\left(Z\left(h'\right) \times E_{-i}|h'\right) \left(\prod_{h'' \in H: h \preceq h'' \preceq h'} \mu_i\left(Z\left(h'', a_{i,h''}\left(h'\right)\right) \times \mathcal{T}_{-i}|h\right) \mu_i\left(Z\left(h, a_{-i,h''}\left(h'\right)\right) \times \mathcal{T}_{-i}|h, a_i\right)\right) \\
&= \mu_i\left(Z\left(h'\right) \times E_{-i}|h'\right) \left(\prod_{h'' \in H: h \preceq h'' \preceq h'} \mu_i\left(Z\left(h'', a_{i,h''}\left(h'\right)\right) \times \mathcal{T}_{-i}|h\right) \mu_i\left(Z\left(h, a_{-i,h''}\left(h'\right)\right) \times \mathcal{T}_{-i}|h\right)\right) \\
&= \mu_i\left(Z\left(h'\right) \times E_{-i}|h'\right) \prod_{h'' \in H: h \preceq h'' \preceq h'} \sigma_i\left(a_{i,h''}\left(h'\right)|h''\right) \sigma_{-i}\left(a_{-i,h''}\left(h'\right)|h''\right) \\
&= \nu_i\left(E_{-i}|h'\right) \sigma\left(h'|h\right).
\end{aligned}$$

Second,

$$\begin{aligned}
\nu_i\left(E_{-i}|h\right) &= \mu_i\left(Z\left(h\right) \times E_{-i}|h\right) \\
&= \sum_{h'' \in H: h \preceq h''} \mu_i\left(Z\left(h''\right) \times E_{-i}|h\right) \mu_i\left(Z\left(h''\right) \times \mathcal{T}_{-i}|h\right) \\
&\quad \sum_{h'' \in H: h \preceq h''} \nu_i\left(E_{-i}|h''\right) \sigma\left(h''|h\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\nu_i\left(E_{-i}|h, (a_i, a_{-i})\right) &= \mu_i\left(Z\left(h, (a_i, a_{-i})\right) \times E_{-i}|h, (a_i, a_{-i})\right) \\
&= \mu_i\left(Z\left(h, a_{-i}\right) \times E_{-i}|h, a_{-i}\right) \\
&= \mu_i\left(Z\left(h, \left(a_i', a_{-i}\right)\right) \times E_{-i}|h, \left(a_i', a_{-i}\right)\right) \\
&= \nu_i\left(E_{-i}|h, \left(a_i', a_{-i}\right)\right).
\end{aligned}$$

$(iii) \implies (i)$ Consider $\mu_i \in \Delta^{H_i}(\Omega_{-i})$ such that all the conditions in (iii) are satisfied. Fix $h \in H$, $a_i, a_i' \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$ and a measurable $E_{-i} \subseteq \mathcal{T}_{-i}$.

$$\begin{aligned}
\mu_i(Z(h,(a_i,a_{-i})) \times E_{-i}|h,a_i) &= \mu_i(Z(h,(a_i,a_{-i})) \times E_{-i}|h,(a_i,a_{-i}))\,\mu_i(Z(h,(a_i,a_{-i})) \times \mathcal{T}_{-i}|h,a_i) \\
&= \mu_i(Z(h,(a_i,a_{-i})) \times E_{-i}|h,(a_i,a_{-i}))\,\mu_i(Z(h,a_{-i}) \times \mathcal{T}_{-i}|h) \\
&= \nu_i(E_{-i}|h,(a_i,a_{-i}))\,\sigma_{-i}(a_{-i}|h) \\
&= \nu_i\left(E_{-i}|h,\left(a_i',a_{-i}\right)\right)\sigma_{-i}(a_{-i}|h) \\
&= \mu_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}|h,\left(a_i',a_{-i}\right)\right)\mu_i(Z(h,a_{-i}) \times \mathcal{T}_{-i}|h) \\
&= \mu_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}|h,\left(a_i',a_{-i}\right)\right)\mu_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times \mathcal{T}_{-i}|h,a_i'\right) \\
&= \mu_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}|h,a_i'\right)
\end{aligned}$$

Showing that μ_i satisfies OAI. ■

Proof of Lemma 2. From Lemma 1 we know that $\Delta^{H_i}(\Omega_{-i})$ is compact metrizable. Consider a sequence $(\mu_i^n)_{n\in\mathbb{N}}$ of elements in $\Delta_i^{H_i}(\Omega_{-i})$ such that it converges to $\mu_i \in \Delta^{H_i}(\Omega_{-i})$. We need to show $\mu_i \in \Delta_i^{H_i}(\Omega_{-i})$. For simplicity, we write μ_{i,h_i}^n and μ_{i,h_i} to denote the conditional probabilities at $h_i \in H_i$. Fix $h \in H$, $a_i, a_i' \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$ and consider the following class of subsets of $\mathcal{B}(\mathcal{T}_{-i})$:

$$\mathcal{D} = \left\{E_{-i} \in \mathcal{B}(\mathcal{T}_{-i}) : \mu_{i,(h,a_i)}(Z(h,(a_i,a_{-i})) \times E_{-i}) = \mu_{i,\left(h,a_i'\right)}\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}\right)\right\}.$$

We show that all the open sets of $\mathcal{T}_{-i}$ belong to $\mathcal{D}$ and that $\mathcal{D}$ is a Dynkin class. Finally, by the Dynkin's lemma, we have $\mathcal{B}(\mathcal{T}_{-i}) = \mathcal{D}$. Also, note that, for each $b_i \in \{a_i, a_i'\}$, the map

$$E_{-i} \mapsto \mu_{i,\left(h,a_i'\right)}(Z(h,(b,a_{-i})) \times E_{-i})$$

is a finite measure over $\mathcal{B}(\mathcal{T}_{-i})$. In the following, we will consider integrals with respect to the measures just introduced. For the sake of simplicity, we denote such measures, for all $E_{-i} \in \mathcal{B}(\mathcal{T}_{-i})$, as

$$\begin{aligned}
\mu_n(E_{-i}) &= \mu_{i,(h,a_i)}^n(Z(h,(a_i,a_{-i})) \times E_{-i}), \\
\mu(E_{-i}) &= \mu_{i,(h,a_i)}(Z(h,(a_i,a_{-i})) \times E_{-i}), \\
\mu_n'(E_{-i}) &= \mu_{i,\left(h,a_i'\right)}^n\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}\right), \\
\mu'(E_{-i}) &= \mu_{i,\left(h,a_i'\right)}\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}\right),
\end{aligned}$$

We proceed by steps.

1. (For all $n \in \mathbb{N}$, and for all measurable functions $f : \mathcal{T}_{-i} \to \mathbb{R}$, $\int_{\mathcal{T}_{-i}} f \mathrm{d}\mu_n = \int_{\mathcal{T}_{-i}} f \mathrm{d}\mu_n'$)[21]. Fix $n \in \mathbb{N}$. If $f = \mathbf{I}_{E_{-i}}$ for some measurable set $E_{-i} \in \mathcal{B}(\mathcal{T}_{-i})$ then the thesis is true since $\mu_i^n \in \Delta_i^{H_i}(\Omega_{-i})$. If f is a simple function, then there exists a finite

[21] Here, with an abuse of notation, we denote with $\int_{\mathcal{T}_{-i}}(\cdot)\,\mathrm{d}\mu_{i,h}$ the integral of the marginal over $\mathcal{T}_{-i}$ conditional on h.

partition $\left\{E_{-i}^1, ..., E_{-i}^Q\right\}$ of $\mathcal{T}_{-i}$ and a collection of real numbers $\left\{d^1, ..., d^Q\right\}$ such that $f = \sum_{q=1}^Q d^q \mathbf{I}_{E_{-i}^q}$. Therefore,

$$\begin{aligned}
\int_{\mathcal{T}_{-i}} f \mathrm{d}\mu_n &= \int_{\mathcal{T}_{-i}} \left(\sum_{q=1}^Q d^q \mathbf{I}_{E_{-i}^q} \right) \mathrm{d}\mu_n \\
&= \sum_{q=1}^Q d^q \left(\int_{\mathcal{T}_{-i}} \mathbf{I}_{E_{-i}^q} \mathrm{d}\mu_n \right) \\
&= \sum_{q=1}^Q d^q \left(\mu_{i,(h,a_i)}^n \left(Z\left(h, (a_i, a_{-i})\right) \times E_{-i}^q \right) \right) \\
&= \sum_{q=1}^Q d^q \left(\mu_{i,(h,a_i')}^n \left(Z\left(h, (a_i', a_{-i})\right) \times E_{-i}^q \right) \right) \\
&= \sum_{q=1}^Q d^q \left(\int_{\mathcal{T}_{-i}} \mathbf{I}_{E_{-i}^q} \mathrm{d}\mu_n' \right) \\
&= \sum_{q=1}^Q d^q \left(\int_{\mathcal{T}_{-i}} \mathbf{I}_{E_{-i}^q} \mathrm{d}\mu_n' \right) = \int_{\mathcal{T}_{-i}} (f) \, \mathrm{d}\mu_n'.
\end{aligned}$$

If f is a generic measurable function, then there exists a sequence $(f^m)_{m \in \mathbb{M}}$ of simple measurable functions such that $f^m \uparrow f$. Therefore,

$$\begin{aligned}
\int_{\mathcal{T}_{-i}} f \mathrm{d}\mu_n &= \int_{\mathcal{T}_{-i}} \left(\lim_m f^m \right) \mathrm{d}\mu_n \\
&= \int_{\mathcal{T}_{-i}} \lim_m (f^m) \, \mathrm{d}\mu_n \\
&= \lim_m \int_{\mathcal{T}_{-i}} f^m \mathrm{d}\mu_n \\
&= \lim_m \int_{\mathcal{T}_{-i}} f^m \mathrm{d}\mu_n' \\
&= \int_{\mathcal{T}_{-i}} \lim_m (f^m) \, \mathrm{d}\mu_n' = \int_{\mathcal{T}_{-i}} f \mathrm{d}\mu_n'
\end{aligned}$$

where the third and fifth equalities follow from the Monotone Convergence Theorem. This proves the claim.

2. (Every open subset of $\mathcal{T}_{-i}$ is in $\mathcal{D}$). Consider an open set $E_{-i} \in \mathcal{D}$. By Urysohn's lemma, there exists a sequence $(f^m)_{m \in \mathbb{N}}$ of continuous real functions defined over $\mathcal{T}_{-i}$

such that $f^m(\tau_{-i}) \uparrow \mathbf{I}_{E_{-i}}(\tau_{-i})$ for all $\tau_{-i} \in \mathcal{T}_{-i}$. Then, we have

$$\begin{aligned}
\mu(E_{-i}) &= \int_{\mathcal{T}_{-i}} \mathbf{I}_{E_{-i}} \mathrm{d}\mu \\
&= \int_{\mathcal{T}_{-i}} \lim_m (f^m) \mathrm{d}\mu \\
&= \lim_m \int_{\mathcal{T}_{-i}} f^m \mathrm{d}\mu \\
&= \lim_m \lim_n \int_{\mathcal{T}_{-i}} f^m \mathrm{d}\mu_n \\
&= \lim_m \lim_n \int_{\mathcal{T}_{-i}} f^m \mathrm{d}\mu'_n \\
&= \lim_m \int_{\mathcal{T}_{-i}} f^m \mathrm{d}\mu' \\
&= \int_{\mathcal{T}_{-i}} \lim_m f^m \mathrm{d}\mu' = \mu'(E_{-i})
\end{aligned}$$

where, the third and seventh equalities follow from the Monotone Convergence Theorem, the fourth and sixth equality follow from the characterization of weak convergence of measures (see Portmanteau Theorem), and the fifth equality follows from point 1.

3. ($\mathcal{D}$ is a dynkin class) It is immediate to show that $\mathcal{T}_{-i} \in \mathcal{D}$. Let $E_{-i}, E'_{-i} \in \mathcal{D}$ such that $E_{-i} \subset E'_{-i}$. It follows

$$\begin{aligned}
\mu_{i,(h,a_i)}\left(Z(h,(a_i,a_{-i})) \times \left(E'_{-i} - E_{-i}\right)\right) &= \mu\left(E'_{-i} - E_{-i}\right) \\
&= \mu\left(E'_{-i}\right) - \mu\left(E_{-i}\right) \\
&= \mu'\left(E'_{-i}\right) - \mu'\left(E_{-i}\right) \\
&= \mu'\left(E'_{-i} - E_{-i}\right) \\
&= \mu_{i,(h,a'_i)}\left(Z\left(h,\left(a'_i,a_{-i}\right)\right) \times \left(E'_{-i} - E_{-i}\right)\right)
\end{aligned}$$

showing that $\left(E'_{-i} - E_{-i}\right) \in \mathcal{D}$. Finally, consider a sequence $\left\{E^n_{-i}\right\}$ of pairwise disjoint measurable subsets of $\mathcal{T}_{-i}$ in $\mathcal{D}$. It follows

$$\begin{aligned}
\mu_{i,(h,a_i)}\left(Z(h,(a_i,a_{-i})) \times \left(\bigcup E^n_{-i}\right)\right) &= \sum_n \mu\left(E^n_{-i}\right) \\
&= \sum_n \mu'\left(E^n_{-i}\right) \\
&= \mu_{i,(h,a'_i)}\left(Z\left(h,\left(a'_i,a_{-i}\right)\right) \times \left(\bigcup E^n_{-i}\right)\right)
\end{aligned}$$

showing that $\bigcup E^n_{-i} \in \mathcal{D}$. This finally shows that $\mathcal{D}$ is a Dynkin class and $\mathcal{D} = \mathcal{B}(\mathcal{T}_{-i})$.

Given that $h \in H$, $a_i, a'_i \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$ were arbitrarily chosen, μ_i satisfies OAI and belongs to $\Delta_i^{H_i}(\Omega_{-i})$ proving that the latter is closed, hence compact metrizable. ■

Proof of Lemma 3. By Theorem 9 in Battigalli *et al.* (2017), there exists $\eta_i \in \Delta^{H_i}(\Omega_{-i} \times X_{-i})$ that strongly believes $\left(E^1_{-i}, ..., E^n_{-i}\right)$ and satisfies $\text{marg}_{\Omega-i}\eta_i = \mu_i$. We need to show that $\eta_i \in \Delta_i^{H_i}(Z \times \mathcal{T}_{-i} \times X_{-i})$. By inspection of the proof of Theorem 9 in Battigalli *et al.* (2017), we know that

$$\eta_i(B|h_i) = \mu_i^*\left(f^{-1}(B)|h_i\right)$$

for each measurable $B \subseteq Z \times \mathcal{T}_{-i} \times X_{-i}$ and where $\mu_i^*(\cdot|h_i)$ is the completion of $\mu_i(\cdot|h_i)$ and $f : Z \times \mathcal{T}_{-i} \to Z \times \mathcal{T}_{-i} \times X_{-i}$ is analytically measurable and defined as

$$f(z, \tau_{-i}) = (z, \tau_{-i}, q(z, \tau_{-i}))$$

for some analytically measurable $q : Z \times \mathcal{T}_{-i} \to Z \times \mathcal{T}_{-i} \times X_{-i}$.

Fix $h \in H$, $a_i, a_i' \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$ and measurable $E_{-i} \subseteq \mathcal{T}_{-i} \times X_{-i}$.

$$\mu_i(Z(h,(a_i,a_{-i})) \times E_{-i} \times B_{-i}|h,a_i) = \mu_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}|h,a_i'\right)$$

$$\begin{aligned}
\eta_i(Z(h,(a_i,a_{-i})) \times E_{-i} \times B_{-i}|h,a_i) &= \mu_i^*\left(f^{-1}(Z(h,(a_i,a_{-i})) \times E_{-i} \times B_{-i})|h,a_i\right) \\
&= \mu_i(Z(h,(a_i,a_{-i})) \times E_{-i}|h,a_i) \\
&= \mu_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i}|h,a_i'\right) \\
&= \mu_i^*\left(f^{-1}\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i} \times B_{-i}\right)|h,a_i'\right) \\
&= \eta_i\left(Z\left(h,\left(a_i',a_{-i}\right)\right) \times E_{-i} \times B_{-i}|h,a_i'\right).
\end{aligned}$$

Showing that also η_i satisfies OAI. ■

Proof of Proposition 2. We need some preliminary definitions. Let Y_i^∞ the set of coherent infinite hierarchies that do not necessarily satisfy OAI. Clearly, we have $C_i^\infty \subseteq Y_i^\infty$. From Proposition 1 of Battigalli & Siniscalchi (1999) we know that there exists an homeomorphism $b_i : Y_i^\infty \to \Delta^{H_i}\left(\Omega_{-i}^\infty\right)$ such that

$$\mu_i^k = \text{marg}_{\Omega_{-i}^k} b_i(\mu_i^\infty) \tag{4}$$

for all $k \in \mathbb{N}$. We only need to check that $b_i(C_i^\infty) = \Delta_i^{H_i}\left(\Omega_{-i}^\infty\right)$ and define $g_i = b_{i|C_i^\infty}$. Consider $\nu_i \in b_i(C_i^\infty) \subseteq b_i(Y_i^\infty) = \Delta^{H_i}\left(\Omega_{-i}^\infty\right)$. It follows that there exists $\mu_i^\infty \in C_i^\infty$ such that $b_i(\mu_i^\infty) = \nu_i$. Fix $h \in H$, $a_i, a_i' \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$ and consider the families of subsets

$$\mathcal{D} = \left\{E_{-i} \in \mathcal{B}\left(\mathcal{T}_{-i}^\infty\right) : \nu_{i,(h,(a_i,a_{-i}))}(Z(h_i,(a_i,a_{-i})) \times E_{-i}) = \nu_{i,\left(h,\left(a_i',a_{-i}\right)\right)}\left(Z\left(h_i,\left(a_i',a_{-i}\right)\right) \times E_{-i}\right)\right\}$$

and the class of cylinder subsets of $\mathcal{T}_{-i}^\infty(h_i)$

$$\mathcal{C} = \left\{E_{-i}^k \times \mathcal{T}_{-i}^\infty \subseteq \mathcal{T}_{-i}^\infty : k \in \mathbb{N}, E_{-i}^k \in \mathcal{B}\left(\mathcal{T}_{-i}^k\right)\right\}.$$

Given that $\mu_i^\infty \in C_i^\infty$ and (4) holds, it is easy to verify that $\mathcal{C} \subseteq \mathcal{D}$. With essentially the same passages used in the proof of Lemma 2 one can show that $\mathcal{D}$ is a Dynkin class of subsets of $\mathcal{T}_{-i}^\infty$ and therefore, by Dynkin's Lemma and the well known fact $\mathcal{C}$ is a π-class, we

have $\mathcal{D} = \mathcal{B}\left(\mathcal{T}_{-i}^{\infty}\right)$. This finally shows that $\nu_i \in \Delta_i^{H_i}\left(\Omega_{-i}^{\infty}\right)$. Next, let $\nu_i \in \Delta_i^{H_i}\left(\Omega_{-i}^{\infty}\right)$. We want to show that there exists $\mu_i^{\infty} \in C_i^{\infty}$ such that $b_i\left(\mu_i^{\infty}\right) = \nu_i$. Given that $\Delta_i^{H_i}\left(\Omega_{-i}^{\infty}\right) \subseteq \Delta^{H_i}\left(\Omega_{-i}^{\infty}\right) = b_i\left(Y_i^{\infty}\right)$, there exists $\mu_i^{\infty} \in Y_i^{\infty}$ with $b_i\left(\mu_i^{\infty}\right) = \nu_i$. Finally, (4) and $\nu_i \in \Delta_i^{H_i}\left(\Omega_{-i}^{\infty}\right)$ necessarily implies that each μ_i^k satisfies OAI, showing that $\mu_i^{\infty} \in C_i^{\infty}$. ■

Proof of Lemma 5. Define the correspondence

$$\begin{array}{rccc} \Sigma_i^* : & \Omega_i^{\infty} & \rightrightarrows & \Sigma_i, \\ & (z, \theta_i, \mu_i^{\infty}) & \mapsto & \prod_{h \in H} \Delta\left(r_{i,h}\left(\theta_i, \mu_{i,k+1}\right)\right). \end{array}$$

Note that inherits all the properties of each $r_{i,h}$ (in particular, it is upper hemicontinuous) and that

$$RP_i = \left\{\omega_i^{\infty} \in \Omega_i^{\infty,*} : \omega_i^{\infty} \in \left(\hat{\sigma}_i\right)^{-1}\left(\Sigma_i^*\left(\omega_i^{\infty}\right)\right)\right\},$$

that is, RP_i coincides with the set of fixed point of the correspondence $\left(\hat{\sigma}_i\right)^{-1} \circ \Sigma_i^*$. By upper hemicontinuity of $\left(\hat{\sigma}_i\right)^{-1} \circ \Sigma_i^*$ and Kakutani fixed point theorem, we have that RP_i is non-empty and compact. ■

Proof of Lemma 6. Nonemptiness of both MC_i and sMC_i is trivially satisfied. Define the function $q_i : \Omega_i^{\infty,*} \to \mathbb{R}^{\mathbb{T}}$ as

$$q_i\left(z, \theta, \mu^{\infty}\right)_t = \begin{cases} g_i\left(\mu_i^{\infty}\right)\left(\left[h_i^{t-1}(z), a_{i,t}(z)\right] \middle| h_i^{t-1}(z)\right) & if \quad t \leq \ell(z) \\ 3 & else \end{cases}$$

for all $t \in \mathbb{T}$. In words, the function $q_i(\cdot)$ takes value in those states of the world at which player i is fully coherent and gives back the list of probability with which he planned to play the actions implied by z. It follows that we can write

$$\begin{array}{rcl} MC_i & = & \left\{\left(z, \theta, \mu^{\infty}\right) \in \Omega_i^{\infty,*} : q_i\left(z, \theta, \mu^{\infty}\right) > \mathbf{0}\right\}, \\ sMC_i & = & \left\{\left(z, \theta, \mu^{\infty}\right) \in \Omega_i^{\infty,*} : q_i\left(z, \theta, \mu^{\infty}\right) = \mathbf{1}\right\} \end{array}$$

where $\mathbf{0} = (0, ...0)', \mathbf{1} = (1, ..., 1) \in \mathbb{R}^{\mathbb{T}}$. We thus need to show that q is measurable with respect to the Borel sigma-algebra over $\Omega_i^{\infty,*}$. In particular, it is sufficient (and necessary) that $q_i(\cdot)_t : \Omega_i^{\infty,*} \to \mathbb{R}$ is measurable for each $t \in \mathbb{T}$. Recalling that g_i is an homeomorphism, that $h_i^{t-1}(\cdot)$ and $a_{i,t}(\cdot)$ are functions between finite spaces and that the Borel sigma-algebra of the weak* topology on the set of probability measures of a compact metrizable space is generated by all the bounded continuous functionals over that space, it follows that each $q_i(\cdot)_t$ is continuous hence measurable. Continuity of each $q_i(\cdot)_t$ also implies that sMC_i is closed, hence compact. ■

Proof of Proposition 3. We prove this by induction. Clearly,

$$\mathbf{P}_i(1) \subseteq \Omega_i^k = \mathbf{P}_i(0), \mathbf{P}_{-i}(1) \subseteq \Omega_{-i}^k = \mathbf{P}_{-i}(1) \text{ and } \mathbf{P}(1) \subseteq \Omega^k = \mathbf{P}(0).$$

Moreover, note that $\mathbf{P}_i(1)$ can be written as the intersection of the following two sets:

$$\left\{\left(z, \theta_i, \mu_i^k\right) \in \Omega_i^k : \left(\mu_i^k \in C_i^k\right) \wedge \left(\forall h \in H, h \prec z \implies \hat{\sigma}_i\left(\mu_{i,1}\right)\left(a_{i,h}(z) | h\right) > 0\right)\right\}$$

and

$$\text{proj}_{\Omega_i^k}\left\{\left(z,\theta_i,\mu_i^{k+1}\right)\in\Omega_i^{k+1}:\left(\mu_i^{k+1}\in C_i^{k+1}\right)\wedge\left(h\in H,\hat{\sigma}_i\left(\mu_{i,1}\right)\left(r_{i,h}\left(\theta_i,\mu_{i,k+1}\right)|h\right)=1\right)\right\}.$$

On the one hand, through the same passages used in the proof of Lemma 6, one can show that the former set is measurable. On the other hand, the latter set is the image through a continuous function (i.e., the projection) of a compact set, hence measurable. With this, $\mathbf{P}_i(1)$ is measurable as well. Next, assume that $(i)-(ii)$ hold for every $k\in\{0,1,...,n\}$. Let $\left(z,\theta_i,\mu_i^k\right)\in\mathbf{P}_i(n+1)$. It follows that there exists $\mu_{i,k+1}\in M_{i,k+1}$ such that $\left(z,\theta_i,\mu_i^k\right)$ and $\mu_{i,k+1}$ satisfy Coherence, RP, MC and Strong belief for each $m\in\{1,...,n\}$. Therefore, $\left(z,\theta_i,\mu_i^k\right)\in\mathbf{P}_i(n)$. We can similarly show that

$$\mathbf{P}_{-i}(n+1)\subseteq\mathbf{P}_{-i}(n)\text{ and }\mathbf{P}(n+1)\subseteq\mathbf{P}(n).$$

For (i), measurability of $\mathbf{P}_i(n)$ follows from the fact that $\mathbf{P}_i(n-1)$ is measurable and the measurability property of strong belief (see Lemma 6). ■

Proof of Theorem 1 We prove the thesis by induction on n.

(Basis step, $n=1$**)** Fix $i\in I$ and let $\left(z,\theta_i,\mu_i^k\right)\in\mathbf{P}_i(1)$. Then, there exists $\mu_{i,k+1}\in\Delta_i^{H_i}\left(\Omega_{-i}^k\right)$ such that $\left(z,\theta_i,\mu_i^k,\mu_{i,k+1}\right)$ satisfies coherence, RP and MC. Then, it is not hard to verify that, for all $\bar{\mu}_i^\infty\in C_i^\infty$ such that $\bar{\mu}_i^{k+1}=\left(\mu_i^k,\mu_{i,k+1}\right)$, $(z,\theta_i,\bar{\mu}_i^\infty)\in R_i(1)$ and therefore $\left(z,\theta_i,\mu_i^k\right)\in\text{proj}_{\Omega_i^k}R_i(1)$. Conversely, $\left(z,\theta_i,\mu_i^k\right)\in\text{proj}_{\Omega_i^k}R_i(1)$. Then, there exists $\bar{\mu}_i^\infty\in C_i^\infty$ such that $(z,\theta_i,\bar{\mu}_i^\infty)\in R_i(1)$ and $\bar{\mu}_i^k=\mu_i^k$. One can check that $\bar{\mu}_{i,k+1}\in\Delta_i^{H_i}\left(\Omega_{-i}^k\right)$ is such that $\left(z,\theta_i,\mu_i^k,\bar{\mu}_{i,k+1}\right)$ satisfies coherence, RP and MC, that is, $\left(z,\theta_i,\mu_i^k\right)\in\mathbf{P}_i(1)$. Since i was arbitrarily chosen, it follows that, for every $i\in I$, $\mathbf{P}_i(1)=\text{proj}_{\Omega_i^k}R_i(1)$.

(Inductive step) Assume that, for every $i\in I$ and $m\in\{1,...,n\}$, $\mathbf{P}_{-i}(m)=\text{proj}_{\Omega_i^k}R_i(m)$. First, we show that the inductive hypothesis (IH) implies $\mathbf{P}_i(m)=\text{proj}_{\Omega_{-i}^k}R_{-i}(m)$, for every $i\in I$, and $m\in\{1,...,n\}$. For the sake of simplicity, we write $\mathcal{T}_{-i,j}^k$ instead of $\prod_{\iota\in I\setminus\{i,j\}}\mathcal{T}_\iota^k$, for every $i,j\in I$ and $k\in\mathbb{N}\cup\{\infty\}$. Fix $i\in I$ and $m\in\{0,...,n\}$. Next, let $\left(z,\theta_{-i},\mu_{-i}^k\right)\in\mathbf{P}_{-i}(m)=\bigcap_{j\in I\setminus\{i\}}\left(\mathbf{P}_j(m)\times\mathcal{T}_{-i,j}^k\right)$. It follows that, for every $j\in I\setminus\{i\}$, $\left(z,\theta_j,\mu_j^k\right)\in\mathbf{P}_j(m)$. By IH, for every $j\in I\setminus\{i\}$, there exists $\bar{\mu}_j^\infty\in M_j^\infty$ such that $\bar{\mu}_j^k=\mu_j^k$ and $\left(z,\theta_j,\bar{\mu}_j^\infty\right)\in R_j(m)$. Given that, by definition we have $R_{-i}(m)=\bigcap_{j\in I\setminus\{i\}}\left(R_j(m)\times\mathcal{T}_{-i,j}^\infty\right)$, for every $j\in I\setminus\{i\}$, $\left(z,\theta_{-i},\bar{\mu}_{-i}^\infty\right)\in R_j(m)\times\mathcal{T}_{-i,j}^\infty$, where $\bar{\mu}_{-i}^\infty=\left(\bar{\mu}_j^\infty\right)_{j\in I\setminus\{i\}}$. Therefore, $\left(z,\theta_{-i},\bar{\mu}_{-i}^\infty\right)\in R_{-i}(m)$, proving that $\left(z,\theta_{-i},\mu_{-i}^k\right)\in\text{proj}_{\Omega_{-i}^k}R_{-i}(m)$. Conversely, assume that $\left(z,\theta_{-i},\mu_{-i}^k\right)\in\text{proj}_{\Omega_{-i}^k}R_{-i}(m)$. It follows that there exists $\bar{\mu}_{-i}^\infty$ such that $\bar{\mu}_{-i}^k=\mu_{-i}^k$ and, for every $j\in I\setminus\{i\}$, $\left(z,\theta_j,\bar{\mu}_j^\infty\right)\in R_j(m)$. By the inductive hypothesis, for every $j\in I\setminus\{i\}$, we have $\left(z,\theta_j,\mu_j^k\right)\in\mathbf{P}_j(m)$ and, as a consequence, $\left(z,\theta_{-i},\mu_{-i}^k\right)\in\mathbf{P}_j(m)\times\mathcal{T}_{-i,j}^k$. With this, $\left(z,\theta_{-i},\mu_{-i}^k\right)\in\mathbf{P}_{-i}(m)$. Since i and m were arbitrarily chosen, the claim holds.

Next, we show that, for every $i\in I$, $\mathbf{P}_i(n+1)=\text{proj}_{\Omega_i^k}R_i(n+1)$. Fix $i\in I$, and assume first that $\left(z,\theta_i,\mu_i^k\right)\in\mathbf{P}_i(n+1)$. Then, there exists $\mu_{i,k+1}\in\Delta_i^{H_i}\left(\Omega_{-i}^k\right)$ such that

$\left(z,\theta_i,\mu_i^k,\mu_{i,k+1}\right)$ satisfies coherence, RP and MC. Moreover, $\mu_{i,k+1}$ strongly believes the decreasing chain $(\mathbf{P}_{-i}(m))_{m=1}^n$ of events in Ω_{-i}^k. By the previous claim, we have that $\mu_{i,k+1}$ strongly believes $\left(\text{proj}_{\Omega_{-i}^k} R_{-i}(m)\right)_{m=1}^n$. Therefore, by Lemma 3, there exists $\nu_i \in \Delta_i^{H_i}\left(\Omega_{-i}^\infty\right)$ that strongly believes the decreasing chain $(R_{-i}(m))_{m=1}^n$ of events in Ω_{-i}^∞ and such that $\text{marg}_{\Omega_{-i}^k}\nu_i = \mu_{i,k+1}$. Next, let $\bar{\mu}_i^\infty \in C_i^\infty$ be defined as $g_i^{-1}(\nu_i)$. We claim that $(z,\theta_i,\bar{\mu}_i^\infty) \in R_i(n+1)$ and $\bar{\mu}_i^k = \mu_i^k$. The second part is immediate since, by Proposition 2, for all $q \leq k$,

$$\bar{\mu}_{i,q} = \text{marg}_{\Omega_{-i}^{q-1}} g_i\left(\bar{\mu}_i^\infty\right) = \text{marg}_{\Omega_{-i}^{q-1}} g_i\left(g_i^{-1}(\nu_i)\right) = \text{marg}_{\Omega_{-i}^{q-1}}\nu_i = \text{marg}_{\Omega_{-i}^{q-1}}\mu_{i,k+1} = \mu_{i,q},$$

where the last equality follows from the fact that $\left(\mu_i^k,\mu_{i,k+1}\right)$ is in C_i^{k+1} by hypothesis. As for the first part of the claim, note that

$$R_i(n+1) = R_i(n) \cap \text{SB}_i(R_{-i}(n)) = R_i \cap \bigcap_{m=1}^n \text{SB}_i(R_{-i}(m)).$$

Therefore, it is enough to show that $(z,\theta_i,\bar{\mu}_i^\infty) \in R_i$ and $(z,\theta_i,\bar{\mu}_i^\infty) \in \text{SB}_i(R_{-i}(m))$ for every $m \in \{1,...,n\}$. The fact that $(z,\theta_i,\bar{\mu}_i^\infty) \in R_i$ is trivial since $\bar{\mu}_i^\infty \in C_i^\infty$ and $\bar{\mu}_i^k = \mu_i^k$ satisfies rational planning and material consistency. Since $g_i(\bar{\mu}_i^\infty) = \nu_i$ strongly believes $R_{-i}(m)$ for every $m \in \{1,...,n\}$, it follows that $(z,\theta_i,\bar{\mu}_i^\infty) \in \text{SB}_i(R_{-i}(m))$ for every $m \in \{1,...,n\}$. With this, we proved that $(z,\theta_i,\bar{\mu}_i^\infty) \in R_i(n+1)$ and therefore that $\left(z,\theta_i,\mu_i^k\right) = \left(z,\theta_i,\bar{\mu}_i^k\right) \in \text{proj}_{\Omega_i^k} R_i(n+1)$. Conversely, assume that $\left(z,\theta_i,\mu_i^k\right) \in \text{proj}_{\Omega_i^k} R_i(n+1)$. It follows that there exists $\bar{\mu}_i^\infty \in M_i^\infty$ such that $(z,\theta_i,\bar{\mu}_i^\infty) \in R_i(n+1)$ and $\bar{\mu}_i^k = \mu_i^k$. Now, consider $\bar{\mu}_{i,k+1} \in \Delta_i^{H_i}\left(\Omega_{-i}^k\right)$. It is clear that $\left(\mu_i^k,\bar{\mu}_{i,k+1}\right) \in C_i^{k+1}$ and that $\left(z,\theta_i,\mu_i^k,\bar{\mu}_{i,k+1}\right)$ satisfies RP and MC since $(z,\theta_i,\bar{\mu}_i^\infty) \in R_i(n+1) \subseteq R_i$. We still need to show that $\bar{\mu}_{i,k+1}$ strongly believes the chain $(P_{-i}(m))_{m=1}^n$. By the first claim, this is equivalent to showing that $\bar{\mu}_{i,k+1}$ strongly believes the chain $\left(\text{proj}_{\Omega_{-i}^k} R_{-i}(m)\right)_{m=1}^n$. Fix $h \in H$, $m \in \{1,...,n\}$ and let $\Omega_{-i}^k(h) \cap \text{proj}_{\Omega_{-i}^k} R_{-i}(m) \neq \emptyset$. This implies that there exists $z \in Z(h)$ and $\tau_{-i}^\infty \in \mathcal{T}_{-i}^\infty$ such that $\left(z,\tau_{-i}^\infty\right) \in R_{-i}(m)$. Then, we have $\Omega_{-i}^\infty(h) \cap R_{-i}(m) \neq \emptyset$ which, by hypothesis, implies $g_i(\bar{\mu}_i^\infty)(R_{-i}(m)|h) = 1$. By Proposition 2,

$$\begin{aligned}
\bar{\mu}_{i,k+1}\left(\text{proj}_{\Omega_{-i}^k} R_{-i}(m)|h\right) &= \text{marg}_{\Omega_{-i}^k} g_i\left(\bar{\mu}_i^\infty\right)\left(\text{proj}_{\Omega_{-i}^k} R_{-i}(m)|h\right) \\
&= g_i\left(\bar{\mu}_i^\infty\right)\left(\left(\text{proj}_{\Omega_{-i}^k}\right)^{-1} \circ \left(\text{proj}_{\Omega_{-i}^k}\right)(R_{-i}(m))|h\right) \\
&= g_i\left(\bar{\mu}_i^\infty\right)(R_{-i}(m)|h) = 1.
\end{aligned}$$

Given that h and m were arbitrarily chosen, this implies that $\bar{\mu}_{i,k+1}$ strongly believes the chain $\left(\text{proj}_{\Omega_{-i}^k} R_{-i}(m)\right)_{m=1}^n$, i.e., $(P_{-i}(m))_{m=1}^n$, showing that $\left(z,\theta_i,\mu_i^k\right) \in \mathbf{P}_i(n+1)$. Since i was arbitrarily chosen, for every $i \in I$, $\mathbf{P}_i(n+1) = \text{proj}_{\Omega_i^k} R_i(n+1)$. Since the result holds for all n,

$$\mathbf{P}_i(\infty) = \bigcap_{n\in\mathbb{N}} \mathbf{P}_i(n) = \bigcap_{n\in\mathbb{N}} \mathbf{P}_i(n)$$

Finally, we need to show that, for every $n \in \mathbb{N}$, $\mathbf{P}(n) = \text{proj}_{\Omega^k} R(n)$. Fix $n \in \mathbb{N}$ and let $\left(z,\tau^k\right) \in \mathbf{P}(n)$. It follows that, for every $i \in I$, $\left(z,\tau_i^k\right) \in \mathbf{P}_i(n)$ and therefore that

there exists $\bar{\tau}_i^\infty \in \mathcal{T}_i^\infty$, such that $(z, \bar{\tau}_i^\infty) \in R_i(n)$ and $\bar{\tau}_i^k = \tau_i^k$. Thus, for every $i \in I$, $(z, \bar{\tau}^\infty) \in R_i(n) \times \mathcal{T}_{-i}^\infty$, hence $(z, \bar{\tau}^\infty) \in \bigcap_{i \in I} \left(R_i(n) \times \mathcal{T}_{-i}^\infty\right) = R(n)$. This shows that $\left(z, \tau^k\right) \in \mathrm{proj}_{\Omega^k} R(n)$. The proof of the converse is almost identical to the proof of the first claim and left to the reader. ■

Proof of Proposition 5 By repeating the same passages of the Proof of Theorem 1 we can show that, for every $n \in \mathbb{N}$,

$$\forall i \in I, \mathbf{sP}_i(n) = \mathrm{proj}_{\Omega_i^k} sR_i(n) \quad \text{and} \quad \mathbf{sP}(n) = \mathrm{proj}_{\Omega^k} sR(n).$$

This, together with Proposition 3, show that

$$\begin{aligned} \mathbf{sP}(\infty) &= \bigcap_{n \in \mathbb{N}} \mathbf{sP}(n) \\ &= \bigcap_{n \in \mathbb{N}} \mathrm{proj}_{\Omega^k} sR(n) \\ &\supseteq \mathrm{proj}_{\Omega^k} \bigcap_{n \in \mathbb{N}} sR(n) \\ &= \mathrm{proj}_{\Omega^k} sR(\infty). \end{aligned}$$

Conversely, let $\left(z, \theta, \mu^k\right) \in \mathbf{sP}(\infty) = \bigcap_{n \in \mathbb{N}} \mathrm{proj}_{\Omega^k} sR(n)$, so that, for every $n \in \mathbb{N}$, there exists $\bar{\mu}^\infty(n) \in M^\infty$ such that $(z, \theta, \bar{\mu}^\infty(n)) \in sR(n)$ and $\bar{\mu}^k(n) = \mu$. This implies that, for every $n \in \mathbb{N}$, the section $(sR(n))_{(z,\theta,\mu^k)}$ is nonempty. In particular, $\left((sR(n))_{(z,\theta,\mu^k)}\right)_{n \in \mathbb{N}}$ is a decreasing sequences of nonempty compact sets and, by the finite intersection property, $\bigcap_{n \in \mathbb{N}} (sR(n))_{(z,\theta,\mu^k)} \neq \emptyset$. If, $\left(\hat{\mu}^\ell\right)_{\ell \geq k+1} \in \bigcap_{n \in \mathbb{N}} (sR(n))_{(z,\theta,\mu^k)}$, then we have, by construction,

$$\left(z, \theta, , \mu^k, \left(\hat{\mu}^\ell\right)_{\ell \geq k+1}\right) \in \bigcap_{n \in \mathbb{N}} (sR(n)) = sR(\infty)$$

showing that $\left(z, \theta, , \mu^k\right) \in \mathrm{proj}_{\Omega^k} sR(\infty)$. ■

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