Abstract

We analyze strictly competitive information design games between two designers and an agent. Before the agent makes a decision, designers disclose public information about persistent state parameters during multiple stages. We consider environments with arbitrary constraints on feasible information disclosure policies. Our main results characterize equilibrium payoffs and strategies for various timings of the game: simultaneous or alternating disclosures and having a deadline or not. When policies are unconstrained, we show that there exists an equilibrium in which information is disclosed in a single stage. As an application, we study competition in a product demonstration and show that more information is revealed if there is a deadline. The format that provides the buyer with the most information is the sequential game with a deadline in which the ex-ante strongest seller is the last mover.

Keywords: Bayesian persuasion; concavification; convexification; information design; Mertens-Zamir solution; product demonstration; splitting games; statistical experiments; stochastic games.

JEL Classification: C72; D82.
1 Introduction

In many environments, economic agents spend time acquiring information before making decisions, and information often comes from multiple and interested sources. This paper analyzes strategic interactions between two information providers (called the designers) who compete for influencing the action of a decision-maker (called the agent). The designers have opposite preferences and control the access to independent pieces of information. The first key feature of our model is that designers are able to disclose information in multiple stages. Specifically, at each stage, designers disclose information publicly and choose new disclosure policies at the next stage. Hence, in each stage, a designer can react to the information disclosed by himself and by the other designer. Once designers have finished releasing information, the agent chooses an action. The payoffs of designers and of the agent depend on the realized state parameters and on the action taken. The second feature of our model is that it introduces general technological restrictions on available information disclosure policies. The model encompasses the standard information design setting in which a designer can choose any public information policy, but it also covers more realistic environments in which a designer may be constrained to choosing information policies from an exogenous subset. For example, in most real world information design problems, a designer may be constrained to choose deterministic information structures, may run experimental tests with unavoidable false-positive or false-negative results, or may rely on imperfectly reliable experts and reviewers.

As in the literature on Bayesian persuasion initiated by Kamenica and Gentzkow (2011), the term “information designer” refers to a player who is uninformed about a state parameter but is able to choose an information disclosure policy (a statistical experiment) about this parameter to modify agents’ information. Contrary to cheap-talk communication, there is no issue of credibility in information disclosure: the statistical experiment is publicly observable and verifiable. In addition, since an information designer is not privately informed about the state, the choice of the disclosure policy has no signaling effect. In practice, information designers can represent competing sellers who release information about a new product to influence buyers’ expected valuations. For example, sellers can offer free samples or trial periods to social media influencers, control the access to and content of online, press or magazine reviews, organize product testing and trade fairs, or make announcements about future product development. Alternatively, information designers could be lobbyists who control the informativeness of some studies to influence a policymaker. In these examples, each designer may want to release additional information as a function of information publicly revealed by the competing designer.

The aim of the paper is to study designers’ equilibrium information disclosure policies and payoffs in such multistage information design problems. Beyond the theoretical interest, our analysis is motivated by the question of how a decision maker should choose to acquire information from competing designers. For example, in a trial or in a committee, the judge or the regulator could decide in which order (i) evidences should be presented, (ii) experts and reviewers should be consulted and whether to set a deadline to communication. Similarly, a buyer could decide how to acquire information from competing sellers by committing to a purchasing period or by choosing in which order to visiting them. We consider various possible timings: designers may be able to disclose information simultaneously in each stage (the simultaneous-move game) or only a single designer may be able to disclose information in each
stage (the sequential-move game). In addition, there may be a deadline, i.e., an a priori bound on the
number of disclosure stages. We will refer to the case of no deadline as “long” information design even
though, under some conditions on the primitives of the game (see Proposition 2), information disclosure
ends very quickly in equilibrium, but differently than when there is a deadline.

Contributions  The main result of this paper is a characterization of equilibrium payoffs and strategies
of information design games for various timings, obtained under the following assumptions on the
environment. First, there are continuity, convexity and compactness conditions: designers’ utility func-
tions are continuous with respect to the agent’s beliefs, and the correspondences of feasible disclosure
policies are continuous (in some suitable sense) with convex and compact values. Next, we make as-
sumptions on the nature of feasible policies. We assume that noninformative policies are always feasible
and, importantly, that feasible policies are closed under iteration. That is, a distribution of beliefs that
can be obtained by a two-step combination of experiments, can also be obtained by a single feasible
experiment. Under this latter condition, which holds in standard Bayesian persuasion models, multiple
stages of information disclosure is irrelevant if there is only one designer. However, with more than one
designer, multiple stages of information disclosure become relevant because they allow each designer to
react to the information released by the competitor.

Under this set of assumptions, we show that each version of the multistage information design game
admits a Markovian equilibrium, i.e., such that designers’ strategies depend only on the current beliefs
of the agent and on calendar time. Furthermore, if there is no deadline, there exists a stationary
equilibrium; i.e., it depends only on current beliefs. In addition, if there is no constraint on the
set of available information disclosure policies, there exists an equilibrium in which information is
disclosed at the first stage only. That is, along the equilibrium path, at most one designer discloses
some information at the first stage, and information disclosure policies are uninformative thereafter. If
information disclosure policies are constrained, we provide an example in which, in every equilibrium,
the number of disclosure stages is unbounded, even under the maintained assumption that feasible
policies are closed under iteration (see Section 3.4.2).

As in the usual case of one designer of Kamenica and Gentzkow (2011), the value of the game
(the equilibrium payoff of Designer 1) is derived from the expected payoff of Designer 1, denoted by
$u(p, q)$, as a function of the beliefs $(p, q)$, where $p$ is the public belief about the information controlled
by Designer 1, and $q$ the public belief about the information controlled by Designer 2. Equilibrium
strategies can be directly backed out from the characterization of the value of the game. First, suppose
that only Designer 1 is active (e.g., because Designer 2 is constrained to be silent); then, according to
Kamenica and Gentzkow (2011), Designer 1 would “concavify” the payoff function with respect to $p$,
obtaining what we denote by $\text{cav}_p u(p, q)$. Similarly, if only Designer 2 were active, we would obtain
the “convexification” with respect to $q$, which we denote by $\text{vex}_q u(p, q)$.

Consider now the information design game in which designers make alternating moves until deadline
$N$. By backward induction, the value of the game is $\text{vex}_q \text{cav}_p u(p, q)$ if Designer 1 moves last, and
$\text{cav}_p \text{vex}_q u(p, q)$ if Designer 2 moves last. In the first case, the optimal strategy of Designer 1 is to play

\footnote{We show how this assumption can be relaxed in Section B.2 of the online Appendix.}
nonrevealingly until the last stage and to then concavify optimally for the current beliefs \((p, q)\).

In the information design game in which designers move simultaneously until deadline \(N\), in equilibrium both remain silent up to stage \(N - 1\). The value, which we call the splitting game value, is thus the same as for the one-shot simultaneous move game.

An important contribution of the paper is the equilibrium characterization of games with no deadline. Surprisingly, the values and equilibrium strategies are the same for games with simultaneous or alternating moves. We show that this value is the unique function \(v(p, q)\), which we call the Mertens-Zamir function, that satisfies the following system:\(^2\)

\[
v(p, q) = \text{cav}_p \min(u, v)(p, q) = \text{vex}_q \max(u, v)(p, q).
\]

This system is key to the study of discounted zero-sum repeated games with incomplete information on both sides (Mertens and Zamir, 1971, 1977) and of zero-sum dynamic gambling games (Laraki and Renault, 2020). It allows simple optimal strategies to be derived directly: Designer 1 plays noninformatively if \(u(p, q) \geq v(p, q)\) and Designer 1 concavifies \(\min(u, v)\) if \(u(p, q) < v(p, q)\); Designer 2 plays noninformatively if \(u(p, q) \leq v(p, q)\) and Designer 2 convexifies \(\max(u, v)\) if \(u(p, q) > v(p, q)\).

The value of every information design game is between \(\text{cav}_p \text{vex}_q u(p, q)\) and \(\text{vex}_q \text{cav}_p u(p, q)\), so if these two quantities are equal, all multistage information design games have the same value. Hence in this case, the information revealed to the agent in equilibrium is the same for any information disclosure protocol. Hence, when \(\text{cav}_p \text{vex}_q u(p, q) = \text{vex}_q \text{cav}_p u(p, q)\), it doesn’t matter how the agent acquires information from competing designers.\(^3\)

We provide examples and economic applications in which \(\text{cav}_p \text{vex}_q u\), \(\text{vex}_q \text{cav}_p u\), the splitting game value and the Mertens-Zamir function are all different. In such situations, the protocol of information disclosure determines the information revealed to the agent in equilibrium. Hence, our equilibrium characterizations allow to derive directly the agent’s preference over disclosure protocols. This comparison is especially relevant if the agent can commit to an information acquisition strategy, or if the disclosure protocol can be regulated.

**Application to a Product Demonstration by Two Sellers** We apply our results and methodology to a stylized model of competitive and public product demonstration in which two sellers disclose information about their respective products to a representative buyer. Depending on the preferences of the different parties, this model could also apply to political campaigning, legal trials, or competition between two pharmaceutical labs who perform medical tests to obtain approval to release new drugs on the market. All players are initially uncertain about the buyer’s valuations for the two products. The buyer decides to buy from the seller for whom the expected valuation conditional on the public information is highest (see Boleslavsky and Cotton, 2015 for the analysis of the one-shot game). We show that regardless of the prior expected valuations, sellers’ equilibrium strategies are always more

\(^2\)In Section B.1, we show how to approximate the Mertens-Zamir function with finite sets of admissible posterior beliefs and present an algorithm for computing it.

\(^3\)The way the agent acquires information when the two designers have access to the same information (i.e., states are perfectly correlated) and all information disclosure policies are feasible is also irrelevant because full information disclosure is always an equilibrium in this case (see Section 5.3 where we discuss the general correlated case).
informative with a deadline than without a deadline. Without a deadline, only one designer discloses information, and the expected payoff of the buyer is the same as without information disclosure. Additionally, if there is a deadline, the buyer always prefers to acquire information sequentially, and that the seller with the highest ex-ante value discloses last.

**Related Literature**  The methodology and results of Bayesian persuasion (Kamenica and Gentzkow, 2011) and information design (e.g., Bergemann and Morris, 2016a,b, Mathevet, Perego, and Taneva, 2020, and Taneva, 2019) are deeply related to repeated games with incomplete information on one side (Aumann and Maschler, 1966, 1967, Aumann, Maschler, and Stearns, 1995) and to the literature on generalized principal-agent problems, correlated and communication equilibria (Aumann, 1974, Myerson, 1982, 1986, Forges, 1986, 1993). See, e.g., the literature reviews in Kamenica (2018), Bergemann and Morris (2019) and Forges (2020). Bayesian persuasion models with constraints on information disclosure policies appear in Perez-Richet (2014), in Salamanca (2016) and Boleslavsky and Kim (2018), where the designer chooses from Bayes-plausible distributions that satisfy some incentive constraints, in Le Treust and Tomala (2019), where the designer is constrained to sending noisy messages, in Wu (2020), where the accuracy of feasible test designs is bounded due to exogenous false-negative errors, and in Montes (2019), where the receiver is able to gather additional information.

The strategic interaction between multiple information designers has been studied under the assumption of simultaneous and one-stage information disclosure by, among others, Gentzkow and Kamenica (2017), Albrecht (2017), Au and Kawai (2020, 2019), Boleslavsky and Cotton (2015, 2018), and Koessler, Laclau, and Tomala (2019). Gentzkow and Kamenica (2017) consider the case in which each designer is always able to choose an information policy that is more informative than that of the other designer. Albrecht (2017), Au and Kawai (2020, 2019), and Boleslavsky and Cotton (2015, 2018) consider the case in which designers control independent pieces of information in applied examples. Koessler, Laclau, and Tomala (2019) provide existence results and properties of equilibria in games with multiple designers and multiple agents. This latter paper assumes that designers disclose information simultaneously, followed by agents making decisions simultaneously as well.

Multistage information design with a single designer has been studied in dynamic decision problems by, among others, Doval and Ely (2016), Ely (2017), Renault, Solan, and Vieille (2017) and Makris and Renou (2018). Since we assume that the state of nature is persistent and the decision problem is static (the agent makes a decision only once), multistage information design would be irrelevant in our model if there were only one designer. The dynamics of information design is interesting in our setting precisely because there are multiple designers. Sequential information design with multiple information designers has been studied by Li and Norman (2020) (see also Wu, 2020), but there are important differences with our work. In particular, they assume that each designer discloses information only once, at a predetermined period. In addition, as in Gentzkow and Kamenica (2017), they assume that each designer is always able to choose an information policy that is more informative than the other designers. Under these assumptions, they show that the sequential game cannot generate a more

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4In our model, if designers can both reveal all the information about the payoff-relevant state (i.e., if their private states are perfectly correlated and all information disclosure policies are available), then the value of every multistage information design game coincides with the expected payoff under full information for the agent. See Section 5.3.
informative equilibrium than the simultaneous game. This property is not true in our model.\footnote{See, for example, Section 4.}

Our methodology and results are closely related to the contributions in the literature on repeated games with incomplete information on both sides, splitting games and acyclic gambling games. The Mertens-Zamir function has been introduced by Mertens and Zamir (1971) (see also Mertens and Zamir, 1977, Sorin, 2002 and Mertens, Sorin, and Zamir, 2015) for the value function \( u(p,q) \) of the one-shot incomplete information zero-sum game, which we replace by the indirect utility function of Designer 1. Mertens and Zamir (1971) have shown that the Mertens-Zamir function is the limit of the value of the infinitely repeated and discounted game as the discount factor tends to one.\footnote{It is also the limit of the value of the undiscounted \( N \)-stage repeated game as \( N \to \infty \). If \( \text{cav}_p, \text{vex}_q \) differs from \( \text{vex}_q, \text{cav}_p \), the undiscounted infinitely repeated game has no value (see Aumann et al., 1995).} It is also the value of zero-sum splitting games studied in Laraki (2001a,b) and Oliu-Barton (2017). Laraki and Renault (2020) have recently extended these results from splitting games to more general stochastic games, called acyclic gambling games. We consider the indirect utility function of designers, given the beliefs and sequentially rational actions of the agent, and obtain our results by adapting the methodology of Renault and Venel (2017) and Laraki and Renault (2020). One main difference with Laraki and Renault (2020) is that we consider terminal payoffs (when the agent makes the decision), while payoffs in splitting games and acyclic gambling games are cumulated and discounted. We also consider and compare various possible timings of the game. Finally, a technical contribution with respect to Laraki and Renault (2020) is that we introduce a continuity condition on the feasible disclosure policies correspondences which is adapted to information design and is different from the non-expansivity condition of Laraki and Renault (2020).

The timing of our multistage information design game is inspired by that of long cheap-talk games (Forges, 1990, Aumann and Hart, 2003). In one of our examples, the equilibrium martingale of posteriors does not reach its limit within a bounded number of disclosure stages, similarly to the “four frogs” example in Aumann and Hart (1986) and Forges (1984, 1990).

The one-stage version of our application to a competitive product demonstration by two sellers has been studied, among others, by Albrecht (2017), Au and Kawai (2020) and Boleslavsky and Cotton (2015, 2018). Independently to our work, Whitmeyer (2020) considers a multistage extension of this example with a finite horizon \( N \) and discounted payoffs. Differently from our setting, he assumes that there is a sequence of short lived buyers, each buyer takes a decision once at a pre-determined period. He directly solves the game by backward induction and, similarly to us, shows that less information is revealed by the designers in long information design game than in the static game. Precisely, the solution of Whitmeyer (2020) of this example, converges to the solution of our long information design game when the time horizon \( N \) tends to infinity and the discount factor tends to one.

Structure of the Paper In Section 2, we present the model. The main results are in Section 3. The application to competition in product demonstration is developed in Section 4. Several generalizations and extensions are studied in Section 5: we provide sufficient conditions under which our results apply with discontinuous indirect utility functions (Section 5.1); we also introduce stopping rules (Section 5.2), allow for correlated private states (Section 5.3), and consider the case in which the messages generated by information disclosure policies are unobservable by the designers (Section 5.4). All proofs that are
not in the text are in Appendix A. Additional results and an extended example can be found in the Online Appendix.

2 Model

2.1 Environment

There are two information designers and a single agent. There is a finite set of states \( K \times L \) that is endowed with a common prior probability distribution. At the start of the game, no player is informed about the state. Designer 1 is able to design public information about \( K \), and Designer 2 is able to design public information about \( L \). To simplify the exposition, we assume that the prior probability distribution is the product of its marginal distributions, \( p^0 \otimes q^0 \), where \( p^0 \in \Delta(K) \), \( q^0 \in \Delta(L) \), and \( \Delta(X) \) denotes the set of Borel probability measures over a compact set \( X \). The extension to correlated priors, which includes the extreme case in which designers can choose information disclosure policies on a common state space, is studied in Section 5.3.

Designers produce and disclose publicly independent pieces of information; thus, Designer 1 controls the public belief \( p \in \Delta(K) \) in a Bayes-plausible way, and similarly Designer 2 controls the public belief \( q \in \Delta(L) \).

2.2 Information Disclosure and Admissible Splittings

An information disclosure policy for Designer 1 is a public information structure for \( K \). When the public belief about \( K \) is \( p \in \Delta(K) \), the policy induces a probability distribution over posterior beliefs with expectation \( p \). Such a mean-preserving spread is called a splitting of \( p \). The set of splittings of \( p \in \Delta(K) \) for Designer 1 is denoted:

\[
S(p) = \left\{ s \in \Delta(\Delta(K)) : \int_{\tilde{p} \in \Delta(K)} \tilde{p} ds(\tilde{p}) = p \right\}.
\]

Similarly, the set of all splittings of \( q \in \Delta(L) \) for Designer 2 is:

\[
\mathcal{T}(q) = \left\{ t \in \Delta(\Delta(L)) : \int_{\tilde{q} \in \Delta(L)} \tilde{q} dt(\tilde{q}) = q \right\}.
\]

Our analysis covers situations in which designers are able to induce any splitting, i.e., any Bayes-plausible distribution of posteriors, as in the benchmark information design models. We also consider restricted sets of feasible information policies for both or just one designer: this is important for the model to represent more realistic environments. Restricting the set of feasible information policies also helps to illustrate our results with tractable examples. For instance, one extreme and simple restriction is to allow designers to choose between fully revealing and nonrevealing policies, as in the simple illustrative example of Section 3.4.1. In Section 2.3 we discuss richer environments with constrained information policies, including restrictions common in the literature.
If designers face technological constraints when choosing information disclosure policies, then the set of splittings of beliefs that designers are able to induce is constrained as well. Let \( P \subseteq \Delta(K) \) and \( Q \subseteq \Delta(L) \) be two compact sets, with \( p^0 \in P \) and \( q^0 \in Q \). We call \( P \times Q \) the set of admissible posteriors. For every \((p, q) \in P \times Q\), we let \( S(p) \subseteq \Delta(P) \cap \mathcal{S}(p) \) and \( T(q) \subseteq \Delta(Q) \cap \mathcal{T}(p) \) be the set of admissible splittings, where the correspondences \( S : P \rightrightarrows \Delta(P) \) and \( T : Q \rightrightarrows \Delta(Q) \) are continuous (i.e., upper hemi-continuous and lower hemi-continuous) and have nonempty convex and compact values. Since the sets \( S(p) \) and \( T(q) \) are convex, randomizing over splittings does not extend the set of feasible information policies.

Throughout the paper, we make two assumptions on admissible splittings. These assumptions are satisfied in the case in which the set of information disclosure policies is unconstrained. First, we assume that for every \((p, q) \in P \times Q\), \( \delta_p \in S(p) \) and \( \delta_q \in T(q) \), where \( \delta_x \) denotes the Dirac measure at \( x \), i.e., each designer is always able to choose a noninformative disclosure policy. Second, we suppose that iterating the admissible splittings does not enlarge the splitting possibilities. This is the case, e.g., if all splittings are admissible. More generally, consider \( p \in P \), \( s \in S(p) \) and let \( f : \Delta(P) \rightarrow \Delta(P) \) be a measurable selection of \( S \), \( f(p') \in S(p') \) for each \( p' \in P \). Define a splitting \( f * s \) where Designer 1 first draws a posterior \( p' \) from \( s \), and then a second posterior \( p'' \) from \( f(p') \). Specifically, \( f * s \) is the probability distribution on \( \Delta(P) \) defined by \( f * s(B) = \int f(B|p')ds(p') \) for each Borel set \( B \subseteq \Delta(P) \), where \( f(B|p') \) denotes the probability of Borel set \( B \) for distribution \( f(p') \). It easy to see that \( f * s \in \mathcal{S}(p) \) (i.e., \( \int\int p''df(p''|p')ds(p') = p \)). Let us denote by the following:

\[
S^2(p) = \{ f * s : s \in S(p), f \text{ measurable selection of } S \},
\]

the set of splittings of \( p \) obtained by iterating \( S \) twice. We then suppose that \( \forall (p, q) \in P \times Q \), \( S^2(p) = S(p) \) and \( T^2(q) = T(q) \).\(^7\)

### 2.3 Examples of feasible information policies

We describe below some natural constraints on information policies available to a designer (say, Designer 1), and show how these constraints translate into constraints on the set of admissible posteriors and admissible splittings.

When information policies are unconstrained, Designer 1 has access to any (statistical or Blackwell) experiment \( x : K \rightarrow \Delta(M) \), where \( M \) is any set of publicly observed messages. In this case, he is able induce any splitting of any \( p \in \Delta(K) \): the sets of admissible posteriors and splittings are then the entire sets: \( P = \Delta(K) \), \( S(p) = \mathcal{S}(p) \).

In most real world environments, information policies are constrained because only a limited number of statistical experiments are available. For example, a medical test usually has unavoidable false-negative or false-positive results. Bringing a witness to the court, an expert to defend a policy, asking a social media influencer to test a new product, or asking a referee to evaluate a research project or a candidate, is always subject to some errors or predictable biases in the distribution of messages that are generated. Alternatively, a designer may be constrained by providing, for each state, hard

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\(^7\)We relax this assumption in Section B.2 in the online appendix when the sets \( P \) and \( Q \) are finite.
evidence about the state from an exogenous set of evidences. In addition, the information policy might be required to be ex-post verifiable, in which case an admissible information policy should be a deterministic mapping from states to messages.

To be more explicit about such constrained information disclosure policies, consider an exogenous compact subset of experiments \( X \subseteq \{x : K \rightarrow \Delta(M)\} \), where \( M \) is a finite set of messages. The designer might be able to repeatedly run experiments in \( X \) as many times as he wants, or he might be able to run a single experiment from \( X \) only once.

For example, if only deterministic experiments are available, then \( P = \{p \in \Delta(K) : \exists E \subseteq K, \text{s.t. } p = p^0(\cdot | E)\} \) is finite, and \( S(p) = \Delta(P) \cap S(p) \). An interesting subset of deterministic experiments is obtained by following the literature on disclosure games with evidences (e.g., Milgrom, 1981) and mechanism design with evidences (e.g., Green and Laffont, 1986). Precisely, for each state \( k \), Designer 1 can only choose messages from a nonempty set \( M(k) \subseteq M \), interpreted as the set of evidences available in state \( k \). In such a setting, a natural restriction on information policies is to assume that the designer chooses any deterministic experiment \( x : K \rightarrow M \) such that \( x(k) \in M(k) \) for every \( k \in K \).\(^8\) Then, the assumption that the designer is always able to choose a noninformative disclosure policy \((\delta_p \in S(p))\) is satisfied iff there exists a message \( m_0 \in M \) such that \( m_0 \in M(k) \) for every \( k \), i.e., there exists a message \( m_0 \) that provides trivial evidence. Note that in this model, the assumption that \( S^2(p) = S(p) \) is the analogue of the standard “normality” condition in disclosure games and mechanism design with evidences, according to which the collection of sets \( \{M^{-1}(m) : m \in M\} \) is closed under intersection. This assumption is satisfied, e.g., if there is evidence for every subset \( K' \subseteq K \), or in the classical disclosure model of Dye (1985), where there is a set of states \( K_E \subseteq K \) is which there exists either full or no evidence \((M(k) = \{m_0, m_k\} \text{ for every } k \in K_E)\), and in the remaining states there is no evidence \((M(k) = \{m_0\} \text{ for every } k \in K \setminus K_E)\).

As another example, take \( K = \{k_1, k_2\} \), \( X = \{x_\varepsilon, x_0\} \), \( M = \{m_1, m_2\} \), where \( x_0 \) is a non informative experiment, \( x_\varepsilon(m_1 | k_1) = 1 - \varepsilon \), \( x_\varepsilon(m_2 | k_2) = 1 \), and \( \varepsilon \in (0, 1) \). Let \( p^0 = \frac{1}{2} \) be the prior probability of state \( k_1 \). For instance, let \( k \) be the characteristic of a product which could be bad \((k = k_1)\) or good \((k = k_2)\) and \( \varepsilon \) is the probability that the experiment \( x_\varepsilon \) produces a false-positive result. First, assume that Designer 1 can run the experiment \( x_\varepsilon \) only once. Then from Bayes’ rule, the set of admissible posterior beliefs on \( k_1 \) is simply \( P = \{\frac{1}{\varepsilon + 1}, \frac{1}{2}, 1\} \), and the sets of admissible splittings are given by

\[
S\left(\frac{1}{2}\right) = \Delta(P) \cap \mathbb{S}\left(\frac{1}{2}\right) \quad \text{and} \quad S(p) = \delta_p \text{ for every } p \neq \frac{1}{2}.
\]

Alternatively, assume that Designer 1 can repeatedly run \( x_\varepsilon \) as many times as he wants. Then, the set of admissible posteriors is \( P = \{0, \frac{n\varepsilon}{n\varepsilon + 1}, 1 : n = 0, 1, 2, \ldots\} \), and for every \( p \in P \) the set of admissible splittings is

\[
S(p) = \Delta(\{p' \in P : p' = 1 \text{ or } p' \leq p\}) \cap S(p).
\]

Our model applies more generally when the designers have access to an arbitrary compact subset \( X \) of experiments (see Proposition 4 in Section 3.3 for more details).

\(^8\)Stochastic evidence production could be added to this setting as well, by allowing stochastic experiments \( x : K \rightarrow \Delta(M) \) such that \( x(m | k) = 0 \text{ if } m \notin M(k) \).
2.4 Information Design Games

Now, we define multistage games in which designers disclose information at stages $n = 1, 2, \ldots, N$ with $N \leq \infty$, and the agent makes a decision at the end of stage $N$. We consider both simultaneous-move games in which both designers disclose information in each stage, and sequential-move games in which Designer 1 is active at odd stages and Designer 2 at even stages.

Simultaneous-move information design game. The initial public belief is $(p^0, q^0)$. At each stage $n = 1, 2, \ldots, N$ and for each current belief $(p^{n-1}, q^{n-1})$, Designer 1 chooses an admissible splitting $s^n \in S(p^{n-1})$ and Designer 2 chooses an admissible splitting $t^n \in T(q^{n-1})$. The new posteriors $p^n$ and $q^n$ are drawn according to $s^n$ and $t^n$. All players observe $s^n$, $t^n$, $p^n$ and $q^n$, and the game proceeds to stage $n + 1$. The agent chooses an action after stage $N$.

For $n \leq N$, a $n$-stage history is a sequence of splittings and posteriors as follows:

$$h^n = (p^0, q^0, s^1, t^1, \ldots, p^{n-1}, q^{n-1}, s^n, t^n, p^n, q^n).$$

We denote by $H^n$ the set of such histories. A strategy of Designer 1 is a sequence $\sigma$ of measurable functions $(\sigma^n)_{n \leq N}$, where for every $n \leq N$, $\sigma^n : H^{n-1} \rightarrow \Delta(P)$ and for every history $h^{n-1} \in H^{n-1}$, $\sigma^n(h^{n-1}) \in S(p^{n-1})$, where $p^{n-1}$ is the $(n-1)$-period posterior on $K$. Similarly, a strategy of Designer 2 is a sequence $\tau$ of measurable functions $(\tau^n)_{n \leq N}$, where for every $n \leq N$, $\tau^n : H^{n-1} \rightarrow \Delta(Q)$ and for every history $h^{n-1} \in H^{n-1}$, $\tau^n(h^{n-1}) \in T(q^{n-1})$, where $q^{n-1}$ is $(n-1)$-period posterior on $L$.

Sequential-move information design game. The description of the game is the same except that Designer 1 moves only at odd stages, and Designer 2 moves only at even stages. That is, Designer 1 is restricted to playing a strategy $\sigma$ such that for all histories and even stage $n$, $\sigma^n(p^0, q^0, s^1, t^1, \ldots, p^{n-1}, q^{n-1}) = \delta_{p_{n-1}}$, and Designer 2 is restricted to playing a strategy $\tau$ such that for all histories and odd stage $n$, $\tau^n(p^0, q^0, s^1, t^1, \ldots, p^{n-1}, q^{n-1}) = \delta_{q_{n-1}}$.

We will distinguish games with deadlines ($N < \infty$) from games without deadlines ($N = \infty$). In Section 5.2, we also consider games with random termination in which at each stage the game between the designers continues to the next stage with probability $\delta \in (0, 1)$ or terminates with probability $(1 - \delta)$.

2.5 Agent’s Decision Problem and Players’ Payoffs

The agent observes all splittings chosen by designers and all induced posteriors. After every play path $h^n$ of the designers, the agent chooses an action $z$ from a compact and convex subset $Z$ of a Euclidean space. The payoff of each player depends on the state $(k, l) \in K \times L$ and on the action $z \in Z$. The payoff of the agent is denoted by $\tilde{u}_A(z; k, l)$, and the payoff of Designer 1 is denoted by $\tilde{u}(z; k, l)$. We assume that the game is strictly competitive between the designers, so the payoff of Designer 2 is $-\tilde{u}(z; k, l)$. For every state $(k, l) \in K \times L$, utility functions $\tilde{u}(z; k, l), \tilde{u}_A(z; k, l)$ are continuous in $z$. If the agent has a finite set of actions $A$, we let $Z = \Delta(A)$ be the set of mixed actions and define payoffs linearly by taking the expectation. We extend $\tilde{u}$ and $\tilde{u}_A$ as usual as follows: for $z$ in $Z$, $p \in P$ and $q \in Q$,

$$\tilde{u}(z; p, q) = \sum_{(k, l) \in K \times L} p(k)q(l)\tilde{u}(z; k, l) \text{ and } \tilde{u}_A(z; p, q) = \sum_{(k, l) \in K \times L} p(k)q(l)\tilde{u}_A(z; k, l).$$
For every pair of admissible posteriors \( p \in P \) and \( q \in Q \), we assume that the decision problem of the agent \( \max_{z \in Z} \tilde{u}_A(z; p, q) \) has a unique solution \( z(p, q) \), which is continuous in \((p, q)\) according to the Maximum Theorem. This assumption is satisfied, e.g., if \( \tilde{u}_A(z; k, l) \) is strictly concave in \( z \). If \( \max_{z \in Z} \tilde{u}_A(z; p, q) \) has multiple solutions, our results also apply as long as there is a continuous selection. Such a selection typically does not exist if the agent has a finite set of actions, \( Z \) is the set of mixed actions, and all posteriors are admissible (i.e., \( P = \Delta(K) \) and \( Q = \Delta(L) \)). This is the case in the example presented in Section 4. Importantly, our results and methods can be extended to analyze this example (proofs have to be amended to account for the discontinuities), so the study of continuous games also gives insights for discontinuous models.

In all versions of the information design game, we assume that the agent’s decision is sequentially rational, i.e., the agent chooses \( z(p, q) \) for all possible terminal posteriors \((p, q) \in P \times Q \). Thus, for each play path \( h^N \) of the designers, there is a uniquely defined decision of the agent. Let

\[
u(p, q) = \sum_{(k, l) \in K \times L} p(k)q(l)\tilde{u}(z(p, q); k, l),\]

be Designer 1’s expected payoff induced by the optimal action of the agent when public beliefs are given by \((p, q) \in P \times Q \). Note that \( u(p, q) \) is continuous in \((p, q)\) since \( \tilde{u}(z; k, l) \) is continuous in \( z \) and \( z(p, q) \) is continuous in \((p, q)\). We extend \( u \) and denote by \( u(s, t) \) the expected value of \( u \) with respect to \( s \in \Delta(P) \) and \( t \in \Delta(Q) \),

\[
u(s, t) = \int \int u(p, q)ds(p)dt(q).
\]

We use a similar notation for the expected utility \( u_A(p, q) \) of the agent.

### 2.6 Games Between Designers, Equilibria and Values

Our main objects of interest are two-player multistage games induced by the sequentially rational decision made by the agent at the terminal stage. For \( N \leq \infty \), we denote by \( G_N(p^0, q^0) \) the game in which designers play simultaneously for \( N \) stages and by \( G_N^{seq}(p^0, q^0) \) the game in which designers play sequentially for \( N \) stages and Designer 1 moves first.

**Remark 1.** The game \( G_1^{seq}(p^0, q^0) \) is a one-designer game and corresponds to the model of Bayesian persuasion of Kamenica and Gentzkow (2011). The game \( G_1(p^0, q^0) \) is a special class of the interactive information design games studied in Koessler et al. (2019), with a single agent and two designers with opposite preferences. The timing of the simultaneous-move game without a deadline \( G_\infty(p^0, q^0) \) is similar to that of a “long cheap talk” game, and the timing of the game \( G_N^{seq}(p^0, q^0) \) is similar to that of a “long polite talk” game, as defined in Aumann and Hart (2003).

For any version of the multistage game, a pair of strategies \((\sigma, \tau)\) of designers induces a distribution \( P_{\sigma, \tau} \) over play paths, which makes the random sequence of posteriors \((p^n, q^n)_{n \leq N} \) a martingale:

\[
\forall \ n < N, \ \forall h^n = (p^0, q^0, ..., p^n, q^n), \ \mathbb{E}_{\sigma, \tau}[p^{n+1} | h^n] = p^n, \ \mathbb{E}_{\sigma, \tau}[q^{n+1} | h^n] = q^n.
\]

The martingale convergence theorem ensures that the random variable \((p^\infty, q^\infty) = (\lim_{n \to \infty} p^n, \lim_{n \to \infty} q^n)\)
exists almost surely. Therefore, the terminal beliefs \((p^N, q^N)\) are well defined for any \(N \leq \infty\).

The expected payoff of Designer 1 in each version of the information design game is given by \(U(\sigma, \tau) = E_{\sigma, \tau} u(p^N, q^N)\) and the expected payoff of Designer 2 is \(-U(\sigma, \tau)\). A pair of strategies \((\sigma^*, \tau^*)\) is an \(\varepsilon\)-equilibrium of the information design game if

\[
U(\sigma, \tau^*) - \varepsilon \leq U(\sigma^*, \tau^*) \leq U(\sigma^*, \tau) + \varepsilon,
\]

for every \(\sigma\) and \(\tau\). It is an equilibrium if it is an \(\varepsilon\)-equilibrium for \(\varepsilon = 0\). The information design game has value \(V\) if the sup inf and inf sup payoffs coincide:

\[
V = \sup_{\sigma} \inf_{\tau} U(\sigma, \tau) = \inf_{\tau} \sup_{\sigma} U(\sigma, \tau).
\]

As is well known, the zero-sum information design game has a value if and only if it has \(\varepsilon\)-equilibria for every \(\varepsilon > 0\), and \((\sigma^*, \tau^*)\) is an equilibrium if and only if \(V = \min_{\tau} U(\sigma^*, \tau) = \max_{\sigma} U(\sigma, \tau^*)\). In this case, \(\sigma^*\) and \(\tau^*\) are called optimal strategies of Designers 1 and 2, respectively.

3 Main Results

In this section, we characterize equilibrium strategies and values for the multistage information design games presented in the previous section. We also provide several examples to illustrate the described properties and the impact of the timing of the game on equilibrium strategies and payoffs.

3.1 Definitions and Preliminary Results

It is well known from the theory of repeated games (Aumann and Maschler, 1966; Aumann et al., 1995) and of Bayesian persuasion (Kamenica and Gentzkow, 2011) that the concavification (or concave closure) of a function is a key concept, as it captures the optimal splitting for a single designer. In our setting, payoffs depend on two variables \(p, q\) and are zero-sum between designers. Thus, ideally, Designer 1 would like to concavify with respect to \(p\), and Designer 2 would like to convexify with respect to \(q\). Before studying the interplay between these notions, we need to define what concave and convex mean in our setting since \(P\) and \(Q\) are not necessarily convex sets and splittings are constrained.

In the following Definitions 1 and 2, all functions defined on \(P\) (or \(Q\), or \(P \times Q\)) are assumed to be measurable and bounded, and extended to elements of \(\Delta(P)\) (or \(\Delta(Q)\), or \(\Delta(P) \times \Delta(Q)\)) by (multi)-linearity.

**Definition 1.** A function \(w : P \to \mathbb{R}\) is \(S\)-concave if for all \(p \in P\) and all \(s \in S(p)\), \(w(p) \geq w(s)\), where \(w(s) = \int w(p') ds(p')\). A function \(w : Q \to \mathbb{R}\) is \(T\)-convex if for all \(q \in Q\) and all \(t \in T(q)\),...
$w(q) \leq w(t)$. A function $w : P \times Q \to \mathbb{R}$ is $S$-concave if $w(\cdot, q)$ is $S$-concave for all $q$, and is $T$-convex if $w(p, \cdot)$ is $T$ convex for all $p$.

If $P$ is a convex set and $S(p) = \Delta(P) \cap S(p)$ for each $p$, $w$ is concave in the usual sense: $w(p)$ is greater than or equal to the expectation of $w$ under any distribution with mean $p$. Thus, the definition is the generalization to all admissible distributions with mean $p$. For functions of two variables $w(p, q)$, $S$-concave means concave with respect to variable $p$ given the admissible splittings $S$.

**Definition 2.** The concavification of $w : P \to \mathbb{R}$ is the smallest $S$-concave function that is pointwise greater than or equal to $w$. We denote it by the following:

$$\text{cav}_p w = \inf \{ g : g \geq w, \ g \text{ is } S\text{-concave} \}.$$  

The convexification of $w : Q \to \mathbb{R}$ is the largest $T$-convex function that is pointwise smaller than or equal to $w$. We denote it by the following:

$$\text{vex}_q w = \sup \{ g : g \leq w, \ g \text{ is } T\text{-convex} \}.$$  

For a function $w : P \times Q \to \mathbb{R}$, $\text{cav}_p w$ will denote the concavification with respect to $p$ of $w(\cdot, q)$ for fixed $q$, and $\text{vex}_q w$ will denote the convexification with respect to $q$ of $w(p, \cdot)$ for fixed $p$.

The next lemma shows that the concavification with respect to $p$ (the convexification with respect to $q$) of a continuous function is continuous and corresponds to the optimal choice of information policy for Designer 1 (Designer 2) when the posterior belief about the other designer is fixed.

**Lemma 1.** For a continuous function $w : P \times Q \to \mathbb{R}$, the functions $\text{cav}_p w$ and $\text{vex}_q w$ are continuous on $P \times Q$ and for every $(p, q) \in P \times Q$:

$$\text{cav}_p w(p, q) = \max \{ w(s, q) : s \in S(p) \} \text{ and } \text{vex}_q w(p, q) = \min \{ w(p, t) : t \in T(q) \}.$$  

### 3.2 Values and Equilibria with a Deadline

Consider the one-shot zero-sum game $G_1(p^0, q^0)$ where the sets of strategies are $S(p^0)$, $T(q^0)$ for Designers 1 and 2, respectively, and the payoff is $u(s, t)$. Since the strategy sets are convex and compact, and the payoff $(s, t) \mapsto u(s, t)$ is continuous and bilinear, by the minmax theorem $G_1(p^0, q^0)$ has a value, denoted by $\text{SV}(u)(p^0, q^0)$, and both designers have optimal strategies:

$$\text{SV}(u)(p^0, q^0) = \max_{s \in S(p^0)} \min_{t \in T(q^0)} u(s, t) = \min_{t \in T(q^0)} \max_{s \in S(p^0)} u(s, t).$$

We call $\text{SV}(u)$ the splitting value function of $u$.

**Lemma 2.** Functions $\text{SV}(u)$, $\text{cav}_p \text{vex}_q u$ and $\text{vex}_q \text{cav}_p u$ are continuous, $S$-concave and $T$-convex. In addition, for every $(p^0, q^0) \in P \times Q$:

$$\text{vex}_q u(p^0, q^0) \leq \text{cav}_p \text{vex}_q u(p^0, q^0) \leq \text{SV}(u)(p^0, q^0) \leq \text{vex}_q \text{cav}_p u(p^0, q^0) \leq \text{cav}_p u(p^0, q^0).$$
Note that all inequalities are equalities for Dirac measures \((p^0, q^0) = (\delta_k, \delta_l)\) since in this case \(S(p^0) = \{\delta_p\}\) and \(T(q^0) = \{\delta_p\}\).

The main result for multistage information design games with deadline is the following theorem.

**Theorem 1.** Information design games \(G_N(p^0, q^0), G^\text{seq}_N(p^0, q^0)\) have Markovian equilibria for all \(N < \infty\). The values of these games are characterized as follows:

1. The value of game \(G_N(p^0, q^0)\) is \(V_N = SV(u)(p^0, q^0)\) for every \(N < \infty\).
2. The value of game \(G^\text{seq}_1(p^0, q^0)\) is \(V^\text{seq}_1 = \text{cav}_p u(p^0, q^0)\).
3. The value of game \(G^\text{seq}_N(p^0, q^0)\) is \(V^\text{seq}_N = \text{vex}_q\text{cav}_p u(p^0, q^0)\) for every finite and odd \(N > 1\).
4. The value of game \(G^\text{seq}_N(p^0, q^0)\) is \(V^\text{seq}_N = \text{cav}_p \text{vex}_q u(p^0, q^0)\) for every finite and even \(N\).

An informal sketch of the proof is as follows. The first point states that when designers move simultaneously and there is a deadline, then in equilibrium they disclose no information until the last stage where they play the equilibrium of the one-shot game. No designer wants to deviate from the above strategy because the splitting value is \(S\)-concave and \(T\)-convex, and therefore revealing information earlier can only be detrimental.

The remaining points deal with sequential games with deadlines. First, only the last two periods matter; before that, designers simply wait. Now, if Designer 1 moves last, Designer 1 will concavify with respect to \(p\), thereby getting \(\text{cav}_p u\). By backward induction, Designer 2 convexifies this function and gets \(\text{vex}_q\text{cav}_p u\). If Designer 2 moves last, we obtain \(\text{cav}_p \text{vex}_q u\).

**Proof of Theorem 1.**

1. Define a strategy \(\sigma\) of Designer 1 as follows. At stages \(n \leq N - 1\), Designer 1 plays the nonrevealing splitting \(\delta_{p_{n-1}} \in S(p^{n-1})\) irrespective of the history. At stage \(n = N\), Designer 1 chooses a splitting \(s \in S(p^{N-1})\) that maximizes \(\min_{t \in T(q^{N-1})} u(s, t)\). For any strategy of Designer 2, we have \(p^{N-1} = p^0\), and the expected payoff is \(E\text{SV}(p^0, q^{N-1}) \geq \text{SV}(p^0, q^0)\) by \(T\)-convexity of \(\text{SV}(u)\). Hence, the payoff is at least \(\text{SV}(p^0, q^0)\) for any strategy of Designer 2. Similarly, Designer 2 can guarantee that the payoff is at most \(\text{SV}(p^0, q^0)\).

2. This follows directly from Lemma 1. If all posteriors and all splittings are allowed, this is the single-designer problem of Kamenica and Gentzkow (2011).

3. Define a strategy \(\sigma\) of Designer 1 as follows. At stages \(n = 1, \ldots, N - 2\), Designer 1 chooses the nonrevealing splitting. At stage \(n = N\), Designer 1 chooses a splitting \(s \in S(p^{N-1})\) that maximizes \(u(s, q^{N-1})\). For any strategy of Designer 2, the expected payoff is \(E\text{cav}_p u(p^0, q^{N-1}) \geq \text{vex}_q\text{cav}_p u(p^0, q^0)\).

Hence, Designer 1 has a strategy that guarantees a payoff of at least \(\text{vex}_q\text{cav}_p u(p^0, q^0)\) irrespective of the strategy of Designer 2.

Define a strategy \(\tau\) of Designer 2 as follows. At stages \(n = 2, \ldots, N - 3\), Designer 2 chooses the nonrevealing splitting. At stage \(n = N - 1\), Designer 2 chooses a splitting \(t \in S(q^{N-2})\) that minimizes \(E\text{cav}_q u(p^{N-2}, q)\). Regardless of the strategy of Designer 1, the expected payoff is at most \(E\text{vex}_q\text{cav}_p u(p^{N-2}, q^0) \leq \text{vex}_q\text{cav}_p u(p^0, q^0)\) by \(S\)-convavity of \(\text{vex}_q\text{cav}_p(p, q)\). Hence, Designer 2 has a strategy that guarantees that Designer 1’s payoff is at most \(\text{vex}_q\text{cav}_p u(p^0, q^0)\) for any strategy of
There exists a function such that $v(p, q) = \text{cav}_vu(p, q)$, and there exists $s$ such that $v(p, q) = v(s, q)$ and $v(p', q) \leq u(p', q)$, $\forall p' \in \text{supp}(s)$.

(C2) There exists $t \in T(q)$ such that $v(p, q) = v(p, t)$ and $v(p, q') \geq u(p, q')$, $\forall q' \in \text{supp}(t)$.

This result is very useful since it allows deriving intuitive strategies from the Mertens-Zamir function (see the discussion after the next theorem).

**Theorem 2.** If there exists a continuous MZ function of $u$, denoted by $MZ(u)(p^0, q^0)$, then the information design games $G_\infty(p^0, q^0)$ and $G^\text{seq}_\infty(p^0, q^0)$ have stationary equilibria and have the same value $V_\infty = V^\text{seq}_\infty = MZ(u)(p^0, q^0)$.

An informal sketch of the proof is as follows. From properties (C1) and (C2), we know that, setting $v = MZ(u)$, there exists $s \in S(p)$ such that $v(p, q) = v(s, q)$ and $v(p', q) \leq u(p', q)$ for all $p' \in \text{supp}(s)$, and there exists $t \in T(q)$ such that $v(p, q) = v(p, t)$ and $v(p, q') \geq u(p, q')$ for all $q' \in \text{supp}(t)$. These properties define naturally stationary strategies for both designers that turn out to form an equilibrium. Assume that $v(p, q)$ is the value of the continuation game starting at $(p, q)$. From the point of view of Designer 1, (C1) means that it is possible to choose a splitting that preserves the expected continuation value. Hence, suppose that $v(p, q) < u(p, q)$; then, Designer 1 can play nonrevealingly without reducing the equilibrium continuation payoff. Intuitively, if $v(p, q) < u(p, q)$, Designer 1 would be content if the
game stopped, as Designer 1 would receive more than the value. If \(v(p, q) > u(p, q)\), then Designer 1 does not want the game to stop and chooses a splitting \(s\) such that \(v(p', q) \leq u(p', q)\) with probability one to reach a point where the realized payoff \(u\) would potentially be greater than the value. The symmetry of (C1) and (C2) implies that \(v(p, q)\) can be enforced by both designers. Thus, this is the equilibrium payoff of this zero-sum game.

**Proof of Theorem 2.** Let \(v = \text{MZ}(u)\), and define a strategy \(\sigma\) of Designer 1 as follows. Given posteriors \((p, q) \in P \times Q\) at stage \(n\), Designer 1 chooses the nonrevealing splitting \(\delta_p\) if \(u(p, q) \geq v(p, q)\), and otherwise chooses a splitting \(s \in S(p)\) such that \(v(p, q) = v(s, q)\) and \(v(p', q) \leq u(p', q), \forall p' \in \text{supp}(s)\).

According to Proposition 1, this strategy is well defined. It has the property that for any strategy of Designer 2, \(u(p^{n+1}, q^n) \geq v(p^{n+1}, q^n)\) almost surely. Considering the limit gives \(u(p^\infty, q^\infty) \geq v(p^\infty, q^\infty)\) almost surely, and thus \(E(u(p^\infty, q^\infty)) \geq E(v(p^\infty, q^\infty))\). By \(T\)-convexity of \(v\), \(E[v(p^{n+1}, q^{n+1}) | h^n] \geq E[v(p^n, q^n)|h^n] = v(p^n, q^n)\) by construction of \(\sigma\). It follows that \(E(v(p^{n+1}, q^{n+1}) \geq E(v(p^n, q^n))\), and by induction \(E(v(p^\infty, q^\infty)) \geq E(v(p^n, q^n)) \geq v(p^n, q^n)\). Thus, there is a strategy of Designer 1 such that for any strategy of Designer 2, \(E(u(p^\infty, q^\infty)) \geq E(v(p^\infty, q^\infty))\). By symmetry, this is the value of the game. We have constructed a stationary equilibrium in infinite-horizon games.

From these results we deduce that if \(\text{cav}_q \text{vex}_u u(p^0, q^0) = \text{vex}_q \text{cav}_p u(p^0, q^0)\), then the values are the same for all versions of the game in which both designers are active (i.e., all but \(G^{seq}_1(p^0, q^0)\)). In particular, if \(u(p, q)\) is concave in \(p\), then \(V_{\infty} = V_1 = V_N = V_{N} = \text{vex}_u u(p^0, q^0)\) for every \(N \geq 2\). If \(u(p, q)\) is convex in \(q\), then \(V_{\infty} = V_1 = V_N = V_{N} = \text{cav}_p u(p^0, q^0)\) for every \(N \geq 1\).

In games without deadlines \(G_{\infty}(p^0, q^0), G_{\infty}(p^0, q^0)\), it follows from the above proof that under the equilibrium strategies we have \(u(p^\infty, q^\infty) = v(p^\infty, q^\infty)\) almost surely. Thus, if the martingale stops, it must be at points where \(u(p, q) = v(p, q)\). Furthermore, if designers can always reach points where \(u(p, q) = v(p, q)\), the martingale actually stops after the first period. This is the case under the conditions below.

**Proposition 2.** Assume that \(P\) and \(Q\) are convex, that all splittings are admissible, and that there exists a continuous MZ function \(v = \text{MZ}(u)\). There is an equilibrium such that:

- If \(u(p^0, q^0) < v(p^0, q^0)\), then Designer 1 plays revealing at the first stage, and both designers play non-revealing thereafter;

- If \(u(p^0, q^0) > v(p^0, q^0)\), then Designer 2 plays revealing at the first stage, and both designers play non-revealing thereafter;

- If \(u(p^0, q^0) = v(p^0, q^0)\), then both designers play non-revealing at all stages.

This is a direct consequence of the following lemma.

**Lemma 4.** Assume that \(P\) and \(Q\) are convex and that all splittings are admissible, \(S(p) = \Delta(P) \cap S(p)\), \(T(q) = \Delta(Q) \cap T(q)\). Then, for any \((p, q) \in P \times Q\),

- if \(u(p, q) \leq v(p, q)\), there exists \(s \in S(p)\) such that \(v(p', q) = u(p', q), \forall p' \in \text{supp}(s)\),

- if \(u(p, q) \geq v(p, q)\), there exists \(t \in T(q)\) such that \(v(p, q') = u(p, q'), \forall q' \in \text{supp}(t)\).

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This lemma is a direct extension of a result in Heuer (1992) and in Oliu-Barton (2017). For the sake of completeness, the proof is recalled in the Appendix. It follows that in games without a deadline, in equilibrium both designers play non-revealing and the martingale is constant if \(u(p, q) = v(p, q)\), Designer 1 splits to a point \(p'\) such that \(u(p', q) = v(p', q)\) and the martingale is constant thereafter if \(u(p, q) < v(p, q)\), and Designer 2 splits to a point \(q'\) such that \(u(p, q') = v(p, q')\) and the martingale is constant thereafter if \(u(p, q) > v(p, q)\). Thus, if the sets of admissible posteriors are convex and all splittings are admissible, information disclosure lasts one period at most. In Section 3.4.2, we provide an example with finite sets of posteriors in which the number of disclosure periods is unbounded.

**Existence.** A natural question arising from Theorem 2 is whether the \(\text{MZ}(u)(p^0, q^0)\) function exists. To answer this, we introduce the following assumption, which strengthens continuity of splitting correspondences. A related condition of “non-expansivity” appears in Renault and Venel (2017).

**Assumption 1.**

1. The splitting correspondence of Designer 1 satisfies the following. There exists \(\theta \in (0, 1]\) such that:

\[
\forall p, p' \in P, \forall s \in S(p), \forall a, b \geq 0, \exists s' \in S(p') \text{ s.t. } \forall f \in D_\theta, |af(s) - bf(s')| \leq \|ap - bp'\|_1, \tag{1}
\]

where \(D_\theta = \{f : P \to [-\theta, \theta], \forall p, p' \in P, \forall a, b \geq 0, |af(p) - bf(p')| \leq \|ap - bp'\|_1 \}\).

2. The splitting correspondence of Designer 2 satisfies the analog condition.

This assumption can be reformulated as follows.

**Lemma 5.** Assumption 1.1 is equivalent to: \(\exists \theta \in (0, 1], \forall f \in D_\theta, \text{cav}_p f \in D_\theta\).

This reformulation is easy to state in terms of concavification of functions and shows that the set of functions \(D_\theta\) is chosen to be closed under the concavification operator. Here are important cases where the assumption holds.

**Proposition 3.** If \(P = \Delta(K)\) and all splittings are allowed, or if \(P\) is finite, then Assumption 1 is satisfied.

We now consider the interesting case of splitting correspondences induced by constraints on experiments. Suppose that Designer 1 is given a finite set of messages \(M\) and a compact subset of experiments \(X \subseteq \{x : K \to \Delta(M)\}\), which contains a non-revealing experiment. Consider all the splittings that Designer 1 can generate by repeatedly choosing experiments in \(X\). For instance, if \(X\) contains only a non-revealing experiment \(x_0\) and a noisy experiment \(x_1\), the designer can run \(x_1\) as many times as he wants, depending on the outcomes of previous trials.

More generally, define an “auxiliary strategy” for Designer 1 as a sequence of measurable mappings \(\sigma = (\sigma_t)\), where for each \(t \geq 1\), \(\sigma_t : (X \times M)^{t-1} \to \Delta(X)\). For each \(x \in X\), \(\sigma_t(x|x_1, m_1, \ldots, x_{t-1}, m_{t-1})\) is the probability that the designer runs experiment \(x\) conditionally on previous experiments and message realizations \(x_1, m_1, \ldots, x_{t-1}, m_{t-1}\). A prior \(p\) and an auxiliary strategy \(\sigma\) induce a probability
distribution over sequences of experiments and messages: conditional on state \(k\), \(x_1\) is selected according to \(\sigma_1\), then \(m_1\) according to \(x_1(|k)\), then \(x_2\) according to \(\sigma_2(x_1,m_1)\), then \(m_2\) according to \(x_2(|k)\), and so on. Denote by \(\mu_T(p,\sigma)\) the induced distribution of posteriors after \(T\) trials and \(S_T(p)\) the set of all such distributions as \(\sigma\) varies. Finally let \(S(p)\) be the weak-* closure of \(\cup_{T\geq 1} S_T(p)\). It is easy to see that \(S(p)\) is a convex compact set and the correspondence \(S\) satisfies \(S^2 = S\).

**Proposition 4.** The splitting correspondence generated by a set of experiments as above satisfies Assumption 1.

We then have the following existence result.

**Theorem 3.** Under Assumption 1, there exists a continuous MZ function of \(u\), denoted by \(MZ(u)\) which is therefore the value of the information design games \(G_\infty(p^0,q^0)\) and \(G_{seq}\infty^\sigma(p^0,q^0)\).

The logic of the proof is the following. Introduce a “discounted game” where Designers 1 and 2 move simultaneously and in each period, the game ends with exogenous probability \(1 - \delta\) and the receiver takes an action, or the game continues to the next period with probability \(\delta < 1\) (see Section 5.2 for a discussion). Let \(v_\delta(p^0,q^0)\) be the value of this game. The main argument is to show that the family of functions \((v_\delta)_{\delta}\) is equicontinuous and therefore admits a limit point \(v\) as \(\delta \to 1\). This uses the fact that function \(v_\delta\) is the fixed point of an appropriate Bellmann equation. We then show that \(v\) satisfies all the conditions ensuring that \(v = MZ(u)\). The proof follows the steps of the proof of Theorem 1 in Laraki and Renault (2020), albeit using a different “continuity” condition (Assumption 1).

Laraki and Renault (2020) use a “non expansivity” assumption which guarantees also the existence of a continuous MZ function. Their assumption is satisfied if \(P = \Delta(K)\) and all splittings allowed or if \(P\) is finite. However, the splitting correspondence generated by a set of experiments need not be non-expansive. We provide a counter-example in Appendix B.4.

### 3.4 Examples

#### 3.4.1 Illustrative Example

As a simple illustration, consider binary states \(K = L = \{0, 1\}\). Identify \(p \in \Delta(K)\) with \(p(1) \in [0, 1]\), and \(q \in \Delta(L)\) with \(q(1) \in [0, 1]\), and let \(p^0 = q^0 = \frac{1}{2}\). Suppose that each designer has only two available disclosure policies: nonrevealing or fully revealing, or equivalently that each designer can only use deterministic experiments. The possible posteriors are thus \(P = Q = \{0, \frac{1}{2}, 1\}\), and all splittings are available on those sets. For each pair of feasible posteriors, the agent takes some (optimal) action that we abstract away from. We assume that the induced payoff for Designer 1 is given by the following:

\[
\begin{array}{ccc}
  u & p = 1 & 0 \\
  p = 1/2 & 1 & 0 \\
  p = 0 & 0 & 1 \\
  q = 0 & q = 1/2 & q = 1 \\
\end{array}
\]

Note that Designer 1 would like to fully reveal his own state when Designer 2 is silent at \(q = 1/2\), and Designer 2 would like to reveal his own state if Designer 1 has already revealed. Payoff \(u\) is neither
S-concave nor $T$-convex. The concavification and convexification of $u$ are given by the following:

$\text{cav}_p u = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ \\
$\text{vex}_q u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

If designers can disclose information simultaneously at a single stage, then Designer 1 can guarantee an expected payoff of $\frac{1}{2}$ by revealing the state with probability $\frac{1}{2}$ and remaining silent with probability $1/2$. Indeed, in this case the posterior belief of the agent on $K$ is $p = 0$ with probability $\frac{1}{4}$, is $p = 1$ with probability $\frac{1}{4}$ and is $p = \frac{1}{2}$ with probability $\frac{1}{2}$. Hence, the probability of obtaining the payoff of 1 is equal to $\frac{1}{2}$ for any disclosure policy of Designer 2. Similarly, Designer 2 can guarantee that Designer 1’s expected payoff is not higher than $\frac{1}{2}$ by revealing the state with probability $\frac{1}{2}$. This is the splitting value of the model, i.e., the equilibrium payoff of Designer 1 of the one-stage simultaneous move game.

Suppose now that designers play sequentially before a fixed deadline and that Designer 1 moves last before the agent makes the decision. In that case, Designer 1 can guarantee a payoff of 1 by simply waiting for the last stage and playing the opposite of what Designer 2 did (disclose if Designer 2 has not disclosed before, and do not disclose if Designer 2 has disclosed). This is the $\text{vex cav}$ value. Similarly, if Designer 2 moves last, Designer 2 can guarantee that Designer 1’s payoff is 0 by waiting for the last stage and playing the same way as Designer 1 (disclose if Designer 1 has disclosed before, and do not disclose otherwise). This is the $\text{cav vex}$ value.

What is the equilibrium if there is no deadline? The game without a deadline is not symmetric. Designer 2 can still apply the strategy above: when Designer 1 discloses, disclose right after, and do not disclose otherwise. Clearly, the resulting payoff is 0 regardless of what Designer 1 does. This is the Mertens-Zamir value of this example.

Summarizing, we have the following:

$\text{vex}_q \text{cav}_p u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ \\
$\text{cav}_p \text{vex}_q u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and

$\text{SV}(u) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$ \\
$\text{MZ}(u) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

In the online Appendix B.3, we present an extension of this example where $P = \Delta(K)$, $Q = \Delta(L)$, all splittings are admissible, and $\text{SV}(u) \neq \text{MZ}(u)$.

### 3.4.2 Unbounded Disclosure Periods

Let $|K| = |L| = 2$, $P = Q = \{0, 1/3, 2/3, 1\}$ and assume that all splittings are admissible on those sets. Consider the following utility function for Designer 1:
Consider, for example, the situation in which \((p, q) = (1, \frac{2}{3})\). If Designer 2 does not disclose any information, then the utility of Designer 1 is \(u(1, \frac{2}{3}) = 1\). The optimal information policy of Designer 2 is to induce the posteriors \(q' = 1\) and \(q'' = \frac{1}{3}\) with the same probability and we have \(\text{vex}_q u(1, \frac{2}{3}) = \frac{1}{2}u(1, \frac{1}{3}) + \frac{1}{2}u(1, 1) = \frac{1}{2}0 + \frac{1}{2}1 = \frac{1}{2}\). More generally, we have the following:

\[
\begin{array}{|c|c|c|c|c|}
\hline
p & 1 & 0 & 1 & 1 \\
\hline
p = 2/3 & 0 & 0 & 1 & 1 \\
\hline
p = 1/3 & 1 & 1 & 0 & 0 \\
\hline
p = 0 & 1 & 1 & 0 & 0 \\
\hline
q = 0 & q = 1/3 & q = 2/3 & q = 1 \\
\hline
\end{array}
\]

Using \(\text{cav}_p \text{vex}_q u \leq MZ(u) \leq \text{vex}_q \text{cav}_p u\) and the symmetries of the example, the MZ value function can be written as follows:

\[
\begin{array}{|c|c|c|}
\hline
\text{vex}_q u = & 0 & 0 \frac{1}{2} 1 \\
\hline
& 0 & 0 \frac{1}{2} 1 \\
& 1 \frac{1}{2} 0 0 \\
& 1 \frac{1}{2} 0 0 \\
\hline
\text{cav}_p \text{vex}_q u = & 0 & 0 \frac{1}{2} 1 \\
\hline
& \frac{1}{2} \frac{1}{2} \frac{1}{2} 1 \\
& 1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
& 1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
\hline
\text{vex}_q \text{cav}_p u = & 0 & 0 \frac{1}{2} 1 \\
\hline
& \frac{1}{2} \frac{1}{2} \frac{1}{2} 1 \\
& 1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
& 1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
\hline
\end{array}
\]

\[
\text{MZ}(u) =
\begin{array}{|c|c|c|}
\hline
0 & 0 \frac{1}{2} 1 \\
\hline
\frac{1}{2} x y \frac{1}{2} \\
\hline
1 & 1 y x \frac{1}{2} \\
\hline
1 \frac{1}{2} 0 0 \\
\hline
\end{array}
\]

where \(\frac{1}{4} \leq x \leq \frac{1}{2}\) and \(\frac{1}{2} \leq y \leq \frac{3}{4}\). It is easy to see that the solution of the Mertens-Zamir system \(\text{MZ}(u)(p, q) = \text{cav}_p \text{min}(u, \text{MZ}(u))(p, q) = \text{vex}_q \text{max}(u, \text{MZ}(u))(p, q)\) gives \(x = \frac{1}{3}\) and \(y = \frac{2}{3}\). From this solution, we immediately obtain the following equilibrium strategies. If \(p = \frac{2}{3}\) and \(q \in \{0, \frac{1}{3}\}\), or if \(p = \frac{1}{3}\) and \(q \in \{\frac{2}{3}, 1\}\), then Designer 1 splits the belief \(p\) to the two neighborhood posteriors with the same probability; otherwise, he discloses no information. If \(q = \frac{2}{3}\) and \(p \in \{0, \frac{1}{3}\}\), or if \(q = \frac{1}{3}\) and \(p \in \{0, \frac{1}{3}\}\), then Designer 2 splits the belief \(q\) to the two neighborhood posteriors with the same probability; otherwise, he discloses no information. The equilibrium martingale of posteriors is represented by Figure 1. Note that, as in the “four frogs” example of Forges (1984, 1990), the number of equilibrium disclosure periods in the infinite-horizon information design game is unbounded, but disclosure stops with probability one in finite time, at a profile of posterior beliefs \((p, q)\) in \(\{(0, 1), (0, \frac{2}{3}), (0, 0), (\frac{1}{3}, 0), (\frac{2}{3}, 1), (1, 1), (1, \frac{1}{3}), (1, 0)\}\), represented by a “∗” symbol in Figure 1. It is important to emphasize that an unbounded number of stages is required for this equilibrium, even though every designer is able to induce in a single stage all

\[\text{An algorithm for computing the solution more generally is provided in Section B.1.}\]
distributions of posteriors that he would be able to induce by iterating information policies in multiple stages; that is, \(S^2(p) = S(p)\) and \(T^2(q) = T(q)\) is satisfied in the example. However, Proposition 2, which guarantees that all the equilibrium information is disclosed in a single stage, does not apply because the sets of admissible posteriors \(P\) and \(Q\) are not convex.

![Figure 1: Equilibrium martingale of posteriors in the example of Section 3.4.2.](image)

It is also easy to check that the one-shot splitting value is as follows:

\[
SV(u) = \begin{pmatrix}
0 & 0 & \frac{1}{2} & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\
1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\
1 & \frac{1}{2} & 0 & 0
\end{pmatrix}
\]

Indeed, if a designer splits an interior belief to the two closest posteriors with the same probability when his payoff is different from his first best payoff then, whatever the belief induced by the other designer, the expected utility is \(\frac{1}{2}\). For example, when \((p, q) = (\frac{1}{3}, \frac{2}{3})\), if Designer 1 splits \(p = \frac{1}{3}\) to \(p' = \frac{2}{3}\) with probability \(\frac{1}{2}\) and to \(p'' = 0\) with probability \(\frac{1}{2}\), then the expected utility of Designer 1 is \(\frac{1}{2}\), and if Designer 2 plays a nonrevealing strategy, then the expected utility of Designer 1 is at most \(\frac{1}{2}\). Hence, in these situations, the value of the game is \(SV(u)(p, q) = \frac{1}{2}\). Otherwise, the value is 0 or 1. For example, when \((p, q) = (\frac{2}{3}, 1)\), Designer 2 cannot modify \(q\) and Designer 1 gets his first best, so \(SV(u)(\frac{2}{3}, 1) = 1\) and no information is disclosed. Similarly, when \((p, q) = (1, \frac{1}{3})\), Designer 1 cannot modify \(p\) and Designer 2 gets his first best, so \(SV(u)(1, \frac{1}{3}) = 0\) and no information is disclosed. In this example, for every interior posteriors \((p, q)\) we have \(\text{cav}_p \text{vex}_q u(p, q) < MZ(u)(p, q) \neq SV(u)(p, q) < \text{vex}_q \text{cav}_p u(p, q)\).

## 4 Competition in Product Demonstration

In this section, we study in detail the case of two designers acting as sellers of products of variable quality and the agent making a decision as to the seller from whom to buy the product. The agent prefers to buy from the seller with the highest expected quality. The static simultaneous-move game \(G_1(p^0, q^0)\) is studied in Boleslavsky and Cotton (2015), who characterize the equilibrium and the value.
of the game. The model is as follows.

The quality of the product of each seller can be either high \((H)\) or low \((L)\): \(K = L = \{L, H\}\). Let \(p^0 \in (0, 1)\) be the prior probability that the state is \(H\) for Designer 1 and \(q^0 \in (0, 1)\) be the prior probability that the state is \(H\) for Designer 2. Denote by \(p \in [0, 1]\) and \(q \in [0, 1]\) the corresponding posteriors.

The agent must choose either Designer 1 or Designer 2. The agent’s payoff is 1 if the chosen designer’s state is \(H\) and is 0 otherwise. Hence, Designer 1 is chosen if \(p > q\); Designer 2 is chosen if \(p < q\), and we assume that the agent randomizes uniformly if \(p = q\). Designer 1’s payoff is equal to 1 if that designer is chosen and is \(-1\) otherwise. Given the posteriors \((p, q)\), the expected payoff of Designer 1 is thus as follows:

\[
 u(p, q) = 1\{p > q\} - 1\{p < q\}.
\]

Note that \(u(p, q)\) is discontinuous at \(p = q\). The expected payoff of the agent is as follows:

\[
 u_A(p, q) = \max\{p, q\}.
\]

This example can fit the following economic scenario. Designers are two competing firms (e.g., Canon and Nikon) that are about to release a new product or system (e.g., a new camera technology together with plans for lens development and compatibility). The agent is a representative consumer (e.g., a professional photographer) who plans to switch in the near future to one of these two new products. Ex-ante, the agent does not know own valuation for the products (it can be high \((H)\) or low \((L)\) for each product). Suppose that the agent already owns a similar product from an older product generation and is ready to spend some time acquiring information before making a decision. Firms can release public information to consumers about their respective products through product demonstration (e.g., press events, reviews, trade fairs, product testing or announcements of future compatible lenses). When choosing how to disclose information, the firm does not know the consumer’s valuation for the product. Additionally, it does not know in advance the public information feedback (e.g., the ratings of reviews) its product will receive. Finally, firms cannot significantly adjust theirs prices but are able to adjust their information policies to signals generated by their competitors.\(^{10}\) It follows that they have strictly opposite preferences, and both want to attract the consumer.

Below, we characterize equilibrium strategies and values for various timings of the information design games. We use those results to analyze informativeness of firm’s strategies and the resulting welfare of the consumer. If we consider finite sets of admissible posteriors, the example fits all our assumptions. For instance, this represents situations in which the consumer reads reviews that apply some rating system (e.g., from one to five stars, or scores corresponding to relevant features of the product). To compare our results with the equilibria of static game \(G_1(p^0, q^0)\) studied in Boleslavsky and Cotton (2015), we perform the equilibrium analysis assuming that all splittings of all possible posteriors are admissible. Note that if all splittings are admissible, designers’ utility functions are discontinuous on the diagonal \(p = q\).\(^{11}\) To deal with this discontinuity, we extend some of the proofs of the previous

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\(^{10}\)See, e.g., Boleslavsky, Cotton, and Gurnani (2016) for more details on product demonstration and on price flexibility in a similar scenario with a single information designer.

\(^{11}\)The designers’ utility functions are discontinuous at \(p = q\) for any tie-breaking rule adopted by the agent.
section to this particular example. In simultaneous-move games (with and without a deadline), we show that equilibria exist. Sequential games with a deadline admit ε-equilibria for all ε > 0 but not exact equilibria.

Before proceeding to equilibrium analysis, let us first consider the fully revealing (FR) and nonrevealing (NR) benchmarks. If information is fully revealed to the agent, then the payoff of the latter is 1 if the state is $H$ for Designer 1 or 2, and 0 if the state is $L$ for both designers. The ex-ante expected payoff of the agent is therefore equal to the following:

$$U_{A}^{FR} = 1 - (1 - p^{0})(1 - q^{0}) = p^{0} + q^{0}(1 - p^{0}).$$

This is an upper bound on the ex-ante expected payoff of the agent. A lower bound is obtained with no revelation of information. When no information is revealed, the agent’s ex-ante expected payoff is as follows:

$$U_{A}^{NR} = u_{A}(p^{0}, q^{0}) = \max\{p^{0}, q^{0}\} < U_{A}^{FR}.$$

**Simultaneous Product Demonstration with a Deadline ($G_{N}(p^{0}, q^{0})$)**

The equilibrium strategies and payoffs for the one-period simultaneous-move game have been characterized by Boleslavsky and Cotton (2015). Assume without loss of generality that $p^{0} \geq q^{0}$. The equilibrium strategies $s \in S(p^{0})$ and $t \in T(q^{0})$ are unique and given as follows:

If $p^{0} \leq \frac{1}{2}$,

$$s = U[0, 2p^{0}] \quad \text{and} \quad t = \begin{cases} U[0, 2p^{0}] \text{ with prob. } \frac{p^{0}}{p^{0} - q^{0}} , \\
0 \text{ with prob. } 1 - \frac{p^{0}}{p^{0} - q^{0}} .
\end{cases}$$

If $p^{0} > \frac{1}{2}$,

$$s = \begin{cases} U[0, 2(1 - p^{0})] \text{ with prob. } \frac{1}{p^{0} - q^{0}} - 1 , \\
1 \text{ with prob. } 2 - \frac{1}{p^{0} - q^{0}} ,
\end{cases} \quad \text{and} \quad t = \begin{cases} 0 \text{ with prob. } 1 - \frac{q^{0}}{p^{0} - q^{0}} , \\
U[0, 2(1 - p^{0})] \text{ with prob. } \frac{q^{0}}{p^{0} - q^{0}}(\frac{1}{p^{0} - q^{0}} - 1) , \\
1 \text{ with prob. } \frac{q^{0}}{p^{0} - q^{0}}(2 - \frac{1}{p^{0} - q^{0}}) .
\end{cases}$$

Hence, the one-shot splitting value exists and is given by the following:

$$SV(u)(p^{0}, q^{0}) = 1 - \frac{q^{0}}{p^{0}}.$$

The proof of Theorem 1 extends directly to this game: for $N < \infty$, the value of game $G_{N}(p^{0}, q^{0})$ is the splitting value $SV(u)(p^{0}, q^{0})$: the equilibrium strategies of designers are nonrevealing at the first $N - 1$ stages, and they use the splittings above at the last stage.

---

12 An alternative approach would be to assume a continuous utility by letting the agent tremble and choose Designer 1 with probability that is a continuous function of $p - q$, increasing from 0 to 1 (e.g., following a logit rule).

13 The solution is similar to those of lotto and electoral competition games (see, e.g., Bell and Cover, 1980, Sahuguet and Persico, 2006 and Hart, 2008).
If $p^0 \leq \frac{1}{2}$, the ex-ante expected utility of the agent is as follows:

$$U^{SV}_A = \left(1 - \frac{q^0}{p^0}\right)p^0 + \frac{q^0}{p^0}\left(\frac{4}{3}p^0\right) = p^0 + \frac{1}{3}q^0.$$  

If $p^0 > \frac{1}{2}$, the ex-ante expected utility of the agent is as follows:

$$U^{SV}_A = \left(\frac{1}{p^0} - 1\right)\left[\left(1 - \frac{q^0}{p^0}\right)(1 - p^0) + \frac{q^0}{p^0}\left(\frac{1}{p^0} - 1\right)\frac{4}{3}(1 - p^0) + \frac{q^0}{p^0}\left(2 - \frac{1}{p^0}\right)\right] + \left(2 - \frac{1}{p^0}\right).$$

**Product Demonstration with no Deadline ($G_\infty(p^0, q^0)$ and $G^{seq}_\infty(p^0, q^0)$)**

For every $p$ and $q$, let

$$v(p, q) = \frac{p - q}{\max(p, q)}.$$

It is easy to verify that for every $p$ and $q$ we have $v(p, q) = \text{cav}_p \min(u, v)(p, q) = \text{vex}_q \max(u, v)(p, q)$ (see Figure 2). Hence, despite the discontinuity of $u$, there exists a solution to the Mertens-Zamir system. We have the following:

**Proposition 5.** The value of the product demonstration games $G_\infty(p^0, q^0)$ and $G^{seq}_\infty(p^0, q^0)$ is $v(p^0, q^0) = \frac{p^0 - q^0}{\max(p^0, q^0)}$.

**Proof.** See Section 5.1, where we derive a more general result in Proposition 6, dealing with discontinuous utilities.

The equilibrium strategies are quite intuitive: if the product of a seller is perceived as better than the product of his competitor, for example because his product demonstrations received good reviews and feedbacks, then the latter is forced to react by revealing additional information. On the contrary, if the product of a seller is perceived as worse than the product of his competitor, then the latter is better off not revealing additional information. Such strategies naturally arise in other strategic information disclosure environments such as political campaigning or information disclosure in committees. Precisely, the (stationary) equilibrium strategy $\sigma$ of Designer 1 is such that, given posteriors $(p, q)$ at stage $n$, he plays the nonrevealing splitting $\delta_q$ if $p \geq q$, and the splitting $s = \frac{p}{q}\delta_q + (1 - \frac{p}{q})\delta_0$ if $p < q$. The equilibrium strategy $\tau$ of Designer 2 is such that, given posteriors $(p, q)$ at stage $n$, he plays the nonrevealing splitting $\delta_q$ if $p \leq q$, and the splitting $s = \frac{q}{p}\delta_p + (1 - \frac{q}{p})\delta_0$ if $p > q$. Note that for a fixed $q$, we have $v(p, q) = \frac{p - q}{\max(p, q)} > u(p, q)$ if $p < q$ and $v(\cdot, q)$ is linear on $[0, q)$. If $p \geq q$, $v(p, q) \leq u(p, q)$. Therefore, $\sigma$ is as in the proof of Theorem 2 and satisfies $v(s, q) = v(p, q) \leq u(s, q)$.

Note that with an initial prior $p^0 \geq q^0$, Designer 1 uses a nonrevealing splitting, and Designer 2 splits the prior $q^0$ to $q = p^0$ with probability $\frac{q^0}{p^0}$ and to $q = 0$ with probability $1 - \frac{q^0}{p^0}$. Even though the values of games $G_1(p^0, q^0)$ and $G_\infty(p^0, q^0)$ are the same for the designers, the induced equilibrium outcomes are very different. In particular, the designers’ strategies are always more informative in the simultaneous-move game with a deadline than in games without a deadline. The equilibrium payoffs of the agent in $G_\infty(p^0, q^0)$ and $G^{seq}_\infty(p^0, q^0)$ are actually equal to $p^0$, as in the nonrevealing case:

For $p^0 \geq q^0$, $U^{MZ}_A = U^{NR}_A = p^0$.  

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Sequential Product Demonstration with a Deadline \((G^\text{seq}_N(p^0, q^0), N < \infty)\)

Now, we consider the sequential-move games with deadlines and calculate \(\text{cav}_p \text{vex}_q u\) and \(\text{vex}_q \text{cav}_p u\). The concavification of \(u(p, q)\) with respect to \(p\) and its convexification with respect to \(q\) are given by the following:

\[
\text{cav}_p u(p, q) = \begin{cases} 
1 & \text{if } p > q \\
-1 + \frac{2p}{q} & \text{if } p \leq q \text{ and } 0 < q < 1 \\
0 & \text{if } p = q = 0 \\
-1 + p & \text{if } q = 1
\end{cases}
\]

\[
\text{vex}_q u(p, q) = \begin{cases} 
-1 & \text{if } q > p \\
1 - \frac{2q}{p} & \text{if } q \leq p \text{ and } 0 < p < 1 \\
0 & \text{if } p = q = 0 \\
1 - q & \text{if } p = 1.
\end{cases}
\]

Note that \(\text{cav}_p u\) is discontinuous at \(p = q = 0\) and \(q = 1\), and \(\text{vex}_q u\) is discontinuous at \(p = q = 0\) and \(p = 1\). If only Designer 1 can disclose information at \((p^0, q^0)\), the equilibrium is nonrevealing. If only Designer 2 can disclose information, that designer splits \(q^0\) to \(p^0\) (more precisely, to \(p^0 + \varepsilon\), with \(\varepsilon \to 0\)) with probability \(\frac{q^0}{p^0}\) and to 0 with probability \(1 - \frac{q^0}{p^0}\). In both cases, the agent’s expected payoff is the nonrevealing payoff:

\[
U^\text{cav}_A = U^\text{vex}_A = U^NR_A = p^0.
\]

To compute \(\text{cav}_p \text{vex}_q u\), fix \(q\) and consider the optimal splitting of \(p\) for Designer 1 when the utility of the latter is given by \(\text{vex}_q u(p, q)\). There are three cases (see Figure 3).

(i) If \(q \geq \frac{2 - \sqrt{2}}{2}\), then the optimal splitting of Designer 1 is full revelation: the designer splits \(p\) to
the posterior 1 with probability $p$ and to 0 with probability $1 - p$.

(ii) If $0 < q < \frac{2 - \sqrt{2}}{2}$, then the optimal splitting of Designer 1 depends on $p$. If $p < 2q$, the designer splits $p$ to the posterior $2q$ with probability $\frac{p}{2q}$ and to 0 with probability $1 - \frac{1}{2q}$. If $2q \leq p \leq 2 - \sqrt{2}$, the designer plays a nonrevealing splitting. If $p > 2 - \sqrt{2}$, the designer splits $p$ to the posterior $2 - \sqrt{2}$ with probability $\frac{1 - p}{\sqrt{2} - 1}$ and to 1 with the complementary probability.

(iii) If $q = 0$, then the optimal splitting of Designer 1 is nonrevealing.

Hence,

$$cav_p \vex_q u(p, q) = \begin{cases} 0 & \text{if } p = q = 0 \\ -1 + p(2 - q) & \text{if } 2 - \sqrt{2} \leq 2q \\ 1 - \frac{2q}{p} & \text{if } 2q \leq p \leq 2 - \sqrt{2} \text{ and } p > 0 \\ -1 + \frac{2q}{2} & \text{if } p \leq 2q < 2 - \sqrt{2} \\ 1 - \frac{q(4 - p - 2\sqrt{2})}{3 - 2\sqrt{2}} & \text{if } 2q \leq 2 - \sqrt{2} \leq p. \end{cases}$$

Note that $cav_p \vex_q u$ is discontinuous at $p = q = 0$. Additionally, $cav_p \vex_q u(p, q) < SV(u)(p, q)$.
for every $p, q \in (0, 1)$. Similarly, we obtain the following:

$$
vex_q \text{cav}_p u(p, q) = \begin{cases} 
0 & \text{if } p = q = 0 \\
1 - q(2 - p) & \text{if } 2 - \sqrt{2} \leq 2p \\
-1 + \frac{2q}{q} & \text{if } 2p \leq q \leq 2 - \sqrt{2} \text{ and } q > 0 \\
1 - \frac{q}{2} & \text{if } q \leq 2p < 2 - \sqrt{2} \\
-1 + \frac{p^4 - q - 2\sqrt{2}}{3 - 2\sqrt{2}} & \text{if } 2p \leq 2 - \sqrt{2} \leq q.
\end{cases}
$$

These computations allow a direct description of $\varepsilon$-equilibrium strategies of $G_N^{\text{seq}}(p^0, q^0)$ as $\varepsilon \to 0$. Below, we describe the designers’ behavior along the equilibrium path. In all cases, designers play nonrevealingly up to stage $N - 2$.

Assume that $N \geq 2$ is even (i.e., Designer 2 plays last) and $(p^0, q^0) \in (0, 1)^2$.

(i) $q^0 > \frac{2 - \sqrt{2}}{2}$. At stage $N - 1$, Designer 1 fully discloses information. Afterwards, at stage $N$ Designer 2 fully discloses information if $p^{N-1} = 1$, and plays nonrevealingly if $p^{N-1} = 0$. The agent always chooses the best designer, so the agent’s expected payoff is $U_A^{vex_q \text{cav}_p} = U_A^{FR}$.

(ii) $q^0 < \frac{2 - \sqrt{2}}{2}$.

(a) $p^0 < 2q^0$. At stage $N - 1$, Designer 1 splits $p^0$ to $2q^0$ and 0. Afterwards, at stage $N$ Designer 2 splits $q^0$ to $2q^0$ ($+\varepsilon$) and 0 if $p^{N-1} = 2q^0$, and plays nonrevealingly ($q^N = q^0$) if $p^{N-1} = 0$. In the former case, the expected payoff of the agent is $2q^0$, and in the latter case it is $q^0$. Hence, the expected utility of the agent is $U_A^{\text{cav}_p, \text{vex}_q} = \frac{p^0}{2} + q^0 \in (U_A^{NR}, U_A^{FR})$. Note that if $p^0 < \frac{1}{2}$, we have $U_A^{\text{cav}_p, \text{vex}_q} > U_A^{SV}$ if $p^0 < \frac{4q^0}{3}$. If $p^0 > \frac{1}{2}$ we have $U_A^{\text{cav}_p, \text{vex}_q} > U_A^{SV}$.

(b) $p^0 \in (2q^0, 2 - \sqrt{2})$. At stage $N - 1$, Designer 1 does not disclose information (hence, $p^{N-1} = p^0$).

Afterwards, at stage $N$ Designer 2 splits $q^0$ to $p^0$ ($+\varepsilon$) and 0. The agent’s expected payoff is then equal to $U_A^{\text{cav}_p, \text{vex}_q} = p^0 = U_A^{NR}$.

(c) $p^0 > 2 - \sqrt{2}$. At stage $N - 1$, Designer 1 splits $p^0$ to 1 with probability $\frac{p^0 - (2 - \sqrt{2})}{\sqrt{2} - 1}$ and to $2 - \sqrt{2}$ with probability $\frac{1 - p^0}{\sqrt{2} - 1}$. Afterwards, at stage $N$ Designer 2 reveals fully ($q^N = 0$ or $q^N = 1$) if $p^{N-1} = 1$, and splits $q^0$ to $2 - \sqrt{2}$ ($+\varepsilon$) and 0 if $p^{N-1} = 2 - \sqrt{2}$. In the former case, the expected payoff of the agent is 1, and in the latter case it is $2 - \sqrt{2}$. The agent’s expected payoff is then equal to $U_A^{\text{cav}_p, \text{vex}_q} = p^0 = U_A^{NR}$.

By symmetry, the equilibrium strategies are similar if $N \geq 3$ is odd (i.e., if Designer 1 plays last). The expected equilibrium payoff of the agent in the $N$-stage sequential game as a function of the priors is summarized below for even $N$ ($U_A^{\text{cav}_p, \text{vex}_q}$) and odd $N$ ($U_A^{\text{vex}_q \text{cav}_p}$):

$$
U_A^{\text{cav}_p, \text{vex}_q} = \begin{cases} 
U_A^{FR} & \text{if } q^0 > 2 - \sqrt{2} \\
U_A^{NR} & \text{if } q^0 < 2 - \sqrt{2} \text{ and } p^0 > 2q^0 \\
\frac{p^0}{2} + q^0 \in (U_A^{NR}, U_A^{SV}) & \text{if } q^0 < 2 - \sqrt{2} \text{ and } p^0 < 2q^0 \\
\frac{p^0}{2} + q^0 \in (U_A^{SV}, U_A^{FR}) & \text{if } q^0 < 2 - \sqrt{2} \text{ and } p^0 < \frac{4q^0}{3}.
\end{cases}
$$

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If \( p^0 \) and \( q^0 \) are greater than \( \frac{2 - \sqrt{2}}{2} \), the equilibrium expected payoff of the agent in \( G_{N}^{\text{seq}}(p^0, q^0) \) is the highest for all \( N \geq 2 \). The agent receives the fully revealing payoff \( U_{A}^{FR} \) that is higher than the agent’s payoff \( U_{A}^{SV} \) in simultaneous-move games with a deadline, which itself is higher than the agent’s expected payoff in information design games with no deadline. Every sequential game also gives a higher expected payoff to the agent than does the simultaneous game if \( p^0 \) and \( q^0 \) are less than \( \frac{2 - \sqrt{2}}{2} \) but close to each other (if \( \frac{3}{4}q^0 < p^0 < \frac{4}{3}q^0 \)). Otherwise, if \( p^0 \) and \( q^0 \) are less than \( \frac{2 - \sqrt{2}}{2} \), the agent prefers the sequential game in which Designer 1 plays last to the simultaneous game if \( p^0 > \frac{2}{3}q^0 \), and prefers the sequential game in which Designer 2 plays last to the simultaneous game if \( p^0 > \frac{4}{3}q^0 \).

We summarize the above comparisons as follows:

1. The buyer prefers that sellers have a deadline. Indeed, designers’ equilibrium strategies are more informative with a deadline than without a deadline. With no deadline, regardless of whether information disclosure is simultaneous or sequential, only one designer discloses information, and the expected payoff of the agent is the same as that without information disclosure.

2. The buyer prefers that the seller with the highest ex-ante expected valuation disclose last. Indeed, if there is a deadline, the agent is better off in the sequential game than in the simultaneous game whenever the designer with the highest ex-ante value has the last-mover advantage.

5 Extensions and Further Results

5.1 Discontinuous Utilities

In the example studied in Section 4, the utility function of designers is a discontinuous function of posteriors. Nevertheless, we are able to characterize the value of the game and determine equilibrium strategies using the intuition from Theorems 1 and 2. The next result gives sufficient conditions under which the existence of an MZ function implies that it is the value of the game.

**Proposition 6.** Assume that there exists a function \( v : P \times Q \rightarrow \mathbb{R} \) that is \( S \)-concave, \( T \)-convex and satisfies (C1) and (C2) for all \((p, q) \in P \times Q\). Suppose further that \( v \) is u.s.c. with respect to \( q \) and l.s.c. with respect to \( p \), that sets

\[
\{(p, q) \in P \times Q : v(p, q) \leq u(p, q)\} \quad \text{and} \quad \{(p, q) \in P \times Q : v(p, q) \geq u(p, q)\},
\]

are closed and that \( v \) is l.s.c. on the closure of \( \{(p, q) \in P \times Q : v(p, q) < u(p, q)\} \) and u.s.c. on the closure of \( \{(p, q) \in P \times Q : v(p, q) > u(p, q)\} \). Then, \( v(p^0, q^0) \) is the value of games \( G_{\infty}(p^0, q^0) \) and \( G_{\infty}^{\text{seq}}(p^0, q^0) \), and the equilibrium is as constructed in the proof of Theorem 2.
There, we show that the stationary strategies given by (C1) and (C2) form an equilibrium of the game under the above weakening of continuity. These conditions are met in the example of Section 4.

5.2 Stopping Rules

In games $G_\infty(p^0, q^0)$ and $G_{\text{seq}}^\infty(p^0, q^0)$, there is no deadline, so the number of times information is disclosed could potentially be unbounded. Next, we study two new variations of the game in which the game stops either exogenously or endogenously.

In the first variation of the game, we assume that the game stops whenever both designers choose a nonrevealing splitting at some stage. For instance, in a debate or a committee discussion, participants are asked if they have additional evidence at each period, and if they do not, the discussion stops. We have the following proposition.

**Proposition 7.** A no-deadline simultaneous information design game that stops whenever both designers choose a nonrevealing splitting at some stage has a stationary equilibrium. The value of this game is $\text{MZ}(u)(p^0, q^0)$.

The same holds for a no-deadline sequential game that stops whenever both designers play nonrevealingly consecutively. The proof is identical to that of Theorem 2.

In the second variation of the game denoted by $G_\delta(p^0, q^0)$, the game terminates after each stage with exogenous probability $1 - \delta \in (0, 1)$, and the agent takes an action. With probability $\delta$, the game continues to the next stage. It is easy to see that this game is equivalent to a discounted game in which a short-lived agent makes a decision at each period and designers maximize/minimize the discounted average payoff. This game is a discounted stochastic game called a splitting game in Laraki (2001a) and a gambling game in Laraki and Renault (2020).

**Proposition 8.** Game $G_\delta(p^0, q^0)$ has value $V_\delta$, and $\lim_{\delta \to 1} V_\delta = \text{MZ}(u)(p^0, q^0)$.

**Proof.** See Lemma 9 and Proposition 11 in Appendix A.

Thus, our games $G_\infty(p^0, q^0)$ and $G_{\text{seq}}^\infty(p^0, q^0)$ can be approximated by discounted games with a high discount factor.

5.3 Correlated Information

Our results also extend to correlated priors. The first approach consists of analyzing a modified game with independent priors. Specifically, given any prior $\mu^0 \in \Delta(K \times L)$ and utility functions $\tilde{u}_i(z; k, l)$, where $i$ denotes Designer 1, Designer 2 or the agent, we define a modified game with stochastically independent states by letting $\hat{\mu}(k, l) = (\sum_{i \in L} \mu^0(k, l)) \times (\sum_{k \in K} \mu^0(k, l))$ and for every $i$,

$$\hat{\tilde{u}}_i(z; k, l) = \frac{\mu^0(k, l)}{\hat{\mu}(k, l)} \tilde{u}_i(z; k, l),$$

for every $(k, l)$ in the support of $\mu^0$.

Such transformations can be found in Aumann et al. (1995, Chap. 2, section 4.2, pages 100–104) and in Myerson (1985).
modified multistage information design game (without correlation) coincide with the equilibria of the original multistage information design game (with correlation). The second approach consists of defining concavification and convexification for correlated distributions and extending the notions of \( \text{cav}_p \), \( \text{vex}_q \) and \( \text{MZ} \). The reader is referred to Ponssard and Sorin (1980), Sorin (1984), Mertens et al. (2015) and Oliu-Barton (2017).

Consider now the particular case in which designers disclose information about a common payoff-relevant state; i.e., their private states are perfectly correlated, as in Gentzkow and Kamenica, 2017. Denote by \( \Omega = K = L \) the common set of states and assume that all information disclosure policies are available. In all versions of the information design game, full revelation is an equilibrium. Indeed, when one designer fully discloses the state, the other is indifferent and may fully disclose as well. Thus, the value of each version of the game is that obtained by full revelation. It is easy to verify that this value coincides with the \( \text{cav}_p \), \( \text{vex}_q \) and \( \text{cav}_p \), \( \text{vex}_q \) values, and thus also with the \( \text{SV} \) and \( \text{MZ} \) values. Indeed, all of those functions have to be both concave and convex with respect to the prior. Therefore, they must be linear and equal to \( \sum_\omega \mu(\omega)u(z(\omega), \omega) \), where \( z(\omega) \) is the optimal action of the agent when the agent knows that the state is \( \omega \).

5.4 Unobserved Messages

Let us now consider a variation of our model where posteriors are not observed at each period. More precisely, designers only observe splittings chosen at each stage, until the game between the designers stops. Then, the sequence of splittings chosen is drawn and posteriors are finally privately observed before the agent chooses an action. Denote by \( G_{\text{unobs}}^N \), \( G_{\text{unobs},\text{seq}}^N \), \( G_{\text{unobs}}^\infty \) and \( G_{\text{unobs},\text{seq}}^\infty \) the different games of this version when designers play simultaneously or sequentially, with or without a deadline.

Contrary to the case where posteriors are publicly observed at each period by the designers, all these games (except \( G_{\text{unobs},\text{seq}}^1 \), whose value is \( \text{cav}_p u(p^0, q^0) \)) have the same value and optimal strategies, which are the one-shot splitting value and the one-shot optimal strategies. In particular, when posteriors are privately observed by the agent, it does not matter for him how he would acquire information from competing designers.

**Proposition 9.** The value (unique equilibrium utility of Designer 1) of all information design games \( G_{\text{unobs}}^N \), \( G_{\text{unobs},\text{seq}}^N \) with \( N \geq 2 \), \( G_{\text{unobs}}^\infty \) and \( G_{\text{unobs},\text{seq}}^\infty \) is \( \text{SV}(u)(p^0, q^0) \).

**Proof.** Directly from the fact that \( \text{SV}(u)(p, q) = \max_{s \in S(p)} \min_{t \in T(q)} u(s, t) = \min_{t \in T(q)} \max_{s \in S(p)} u(s, t) \): in every version of the game, Designer 1 can guarantee that his expected utility is at least \( \text{SV}(u)(p, q) \) by playing his one-shot equilibrium strategy in any single period of the game chosen arbitrarily, and Designer 2 can guarantee that the expected utility of Designer 1 is at most \( \text{SV}(u)(p, q) \) by playing his one-shot equilibrium strategy in any single period of the game chosen arbitrarily. Hence, \( \text{SV}(u)(p, q) \) is the value.

It is worth noting that the previous proposition does not apply if one considers more general utility functions. To see this, consider the following non-zero sum example, where the set of admissible
Since this holds for any \( s \) we obtain cav pointwise inequality implies that for any \( s \in S(p) \). Hence, take \( f \rightarrow P \) since \( w \), we have cav pointwise inequality.

### A Appendix: Proofs

#### Proof of Lemma 1

We show that \( \text{cav}_p w(p, q) = \max\{w(s, q) : s \in S(p)\} \). Fix \( q \) and let \( h(p) := \max\{w(s, q) : s \in S(p)\} \). Consider a function \( g : P \rightarrow \mathbb{R} \), \( S \)-concave such that \( g(\cdot) \geq w(\cdot, q) \). This pointwise inequality implies that for any \( s \in S(p) \), \( g(s) \geq w(s, q) \) and since \( g \) is \( S \)-concave,

\[
g(p) \geq g(s) \geq w(s, q).
\]

Since this holds for any \( s \in S(p) \), this implies \( g(p) \geq h(p) \), and since this holds for any \( g(\cdot) \geq w(\cdot, q) \), we obtain \( \text{cav}_p w(p, q) \geq h(p) \).

To obtain the converse inequality, since \( h(p) \geq w(p, q) \), it is enough to prove that \( h \) is \( S \)-concave. Hence, take \( s \in S(p) \) and consider \( h(s) = \int h(p') ds(p') \). Define a measurable selection of \( S \), \( f : P \rightarrow \Delta(P) \) such that \( f(p') \in \arg \max\{u(s', p') : s' \in S(p')\} \). We have for each \( p' \in P \) that \( h(p') = w(f(p'), q) = \int w(p''', q) df(p''|p') \). We then obtain the following:

\[
h(s) = \int w(p'', q) df(p''|p') ds(p') = \int w(p, q) df(f \ast s)(\bar{p}) = w(f \ast s, q) \leq h(p)
\]

since \( f \ast s \in S(p) \) by assumption. We thus have \( \text{cav}_p w(p, q) = \max\{w(s, q) : s \in S(p)\} \) for all \( p \) and \( q \).

#### Proof of Lemma 2

1. **Continuity.** From Lemma 1, \( \text{vex}_u u, \text{cav}_p u, \text{cav}_u \text{vex}_q u, \text{vex}_q \text{cav}_p u \) are continuous. The correspondence \( (p, q) \mapsto S(p) \times T(q) \) is both upper and lower hemicontinuous; hence, by the Maximum Theorem \( \text{SV}(u) \) is continuous.
2. Function $\text{cav}_p \text{vex}_q u$ is $T$-convex. Denote $F(p, q) = \text{vex}_q u(p, q)$ and take $s \in S(p)$ such that

$$\text{cav}_p \text{vex}_q u(p, q) = F(s, q) \leq \int F(p', q) ds(p').$$

Since $F$ is $T$-convex, $F(p', q) \leq F(p', t)$ for each $t \in T(q)$. Thus,

$$\text{cav}_p \text{vex}_q u(p, q) = \int F(p', q) ds(p') \leq \int F(p', t) ds(p') \leq \int \text{cav}_p \text{vex}_q u(p', t) ds(p') \leq \text{cav}_p \text{vex}_q u(p, t),$$

where the last inequality holds due to $\text{cav}_p \text{vex}_q u$ being $S$-concave. By symmetry, $\text{vex}_q \text{cav}_p u$ is $S$-concave.

3. Function $SV(u)$ is $S$-concave. Fix $(p, q) \in P \times Q$ and $s \in S(p)$; we have to show that $SV(u)(p, q) \geq SV(u)(s, q)$. Let $f : P \rightarrow \Delta(P)$ be a measurable selection of $S$ such that for each $p' \in P$, $f(p') \in S(p')$ is an optimal strategy of Designer 1 in game $G_1(p', q)$. We have for each $p' \in P$ that $\forall t \in T(q)$, $u(f(p'), t) \geq SV(u)(p', q)$. Taking expectation with respect to $s$ implies the following:

$$\forall t \in T(q), \int u(f(p'), t) ds(p') \geq \int SV(u)(p', q) ds(p') = SV(u)(s, q).$$

We get $\forall t \in T(q), u(f * s, t) \geq SV(u)(s, q)$. Since $f * s \in S(p)$, we obtain that the value $SV(u)(p, q)$ of $G_1(p, q)$ is at least $SV(u)(s, q)$. Hence, $SV(u)$ is $S$-concave.

4. For each $(p, q) \in P \times Q$, $\text{cav}_p \text{vex}_q u(p, q) \leq SV(u)(p, q)$. It suffices to show that there exists $s \in S(p)$ such that for all $t \in T(q)$, $u(s, t) \geq \text{cav}_p \text{vex}_q u(p, q)$. Choose $s$ such that $\text{cav}_p \text{vex}_q u(p, q) = \text{vex}_q u(s, q)$. Then,

$$\forall t \in T(q), u(s, t) \geq \text{vex}_q u(s, q) = \text{cav}_p \text{vex}_q u(p, q).$$

The other inequalities are either trivial or deduced by symmetry.

**Proof of Lemma 3** If $v = \text{cav}_p \text{min}(u, v)$, then it is $S$-concave, and if $v = \text{vex}_q \text{max}(u, v)$, then it is $T$-convex. To prove the inequality $\text{cav}_p \text{vex}_q u(p, q) \leq \text{MZ}(u)(p, q)$, observe that $\text{max}(u, \text{MZ}(u)) \geq u$; thus,

$$\text{MZ}(u) = \text{vex}_q \text{max}(u, \text{MZ}(u)) \geq \text{vex}_q u.$$

Since $\text{MZ}(u)$ is $S$-concave, this implies $\text{MZ}(u) \geq \text{cav}_p \text{vex}_q u$. The other inequality is obtained by symmetry.

**Proof of Proposition 1** We first prove the following lemma that states a useful property of concavification and optimal splittings: at an optimal splitting $s$ such that $\text{cav}_p w(p) = w(s)$, it must be that $\text{cav}_p w(p') = w(p')$ on the support of $s$. This implies that

$$\text{cav}_p w(s) = \int \text{cav}_p w(p') ds(p') = \int w(p') ds(p') = \text{cav}_p w(p);$$

therefore, $\text{cav}_p w$ is “linear” on the support of $s$. 

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Lemma 6. Let \( w : P \to \mathbb{R} \) be a continuous function. For each \( p \in P \) and each \( s \in S(p) \) such that \( \text{cav}_p w(p) = w(s) = \max \{ w(s') : s' \in S(p) \} \), we have the following:

\[
s(\{ p' \in P : w(p') = \text{cav}_p w(p') \}) = 1.
\]

**Proof.** Assume the contrary that \( \text{cav}_p w(p) = w(s) \) and \( s(\{ p' \in P : \text{cav}_p w(p') > w(p') \}) > 0 \). Then, there exist \( \varepsilon, \alpha > 0 \) such that \( s(\{ p' \in P : \text{cav}_p w(p') \geq w(p') + \varepsilon \}) = \alpha > 0 \). Let us denote \( B = \{ p' \in P : \text{cav}_p w(p') \geq w(p') + \varepsilon \} \); define a measurable selection \( f \) of \( S \) such that \( \text{cav}_p w(p') = w(f(p')) \) for all \( p' \in B \) and \( f(p') = \delta_{p'} \) for all \( p' \notin B \), and consider the splitting \( f \ast s \). We have the following:

\[
w(f \ast s) = \int_B \text{cav}_p w(p')ds(p') + \int_{P \setminus B} w(p')ds(p') \geq \varepsilon \alpha + \int w(p')ds(p').
\]

Since \( f \ast s \in S(p) \), this contradicts \( w(s) = \max \{ w(s') : s' \in S(p) \} \).

To complete the proof of the proposition, we first show that if \( v = \text{MZ}(u) \), then \( v \) is \( S \)-concave and \( T \)-convex, and (C1) and (C2) hold. \( \text{MZ}(u) \) is \( S \)-concave and \( T \)-convex; by symmetry, it is enough to prove (C1). Suppose that \( v = \text{cav}_p \min(u, v) \) and consider some \( (p, q) \in P \times Q \). According to Lemma 6, there exists \( s \in S(p) \) such that \( v(p, q) = \min(u, v)(s, q) \) and \( v(p', q) = \min(u, v)(p', q) \), \( \forall p' \in \text{supp}(s) \). It follows that \( v(p', q) \leq u(p', q) \), \( \forall p' \in \text{supp}(s) \) and \( v(p, q) = v(s, q) \leq \text{cav}_p w(p, q) \). Thus, \( v = \text{cav}_p \min(u, v) \) and by symmetry, \( v = \text{vex}_q \max(u, v) \).

**Proof of Lemma 4** Consider a continuous function \( w : P \to \mathbb{R} \). For each \( p \) such that \( w(p) < \text{cav}_p w(p) \), there exists a splitting with finite support \( s = \sum \lambda_m \delta_{p_m} \) such that \( \text{cav}_p w(p) = \sum \lambda_m w(p_m) \) and \( w(p') < \text{cav}_p w(p') \) for each \( p' \) in the relative interior of the convex hull of \( \{ p_m : m \} \). The reason is that \( (p, \text{cav}_p w(p)) \) lies on a face of the convex hull hypograph of function \( w \) if \( w(p) < \text{cav}_p w(p) \). One just has to obtain this point as a “minimal” convex combination of extreme points of the hypograph (minimal in the sense that the convex hull of its extreme points is minimal for inclusion) among the splittings \( s \) such that \( \text{cav}_p w(p) = w(s) \). Fix \( q \) and apply this logic to \( v(p, q) = \text{cav}_p \min(u, v)(p, q) \). We obtain \( v(p, q) = \sum \lambda_m \min(u, v)(p_m, q) \), with \( v(p_m, q) \leq u(p_m, q) \) for all \( m \) and \( \min(u, v)(p', q) < v(p', q) \) for all \( p' \) in the relative interior of the convex hull of \( \{ p_m : m \} \). By continuity, this implies that \( v(p_m, q) = u(p_m, q) \) for all \( m \).

**Proof of Lemma 5** \( \implies \) Take \( f \) in \( D_\theta, p, p' \) in \( P, a, b \geq 0 \). Also, let \( s \in S(p) \) be such that \( f(s) = \text{cav}_p f(p) \). By Assumption 1, there exists \( s' \in S(p') \) such that \( af(s) - bf(s') \leq \| ap - bp' \|_1 \). Thus, \( a \text{cav}_p f(p) - b \text{cav}_p f(p') \leq af(s) - bf(s') \leq \| ap - bp' \|_1 \). Applying this logic to \( -f \) yields the result.

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Fix $p, p'$ in $P$, $s \in S(p)$, $a, b \geq 0$. Define $\gamma : S(p') \times D_\theta \rightarrow \mathbb{R}$ by

$$\gamma(s', f) = af(s) - bf(s') - \|ap - bp'\|_1.$$ 

For each $f$, there exists $s' \in S(p')$ such that $f(s') = \text{cav}_p f(p')$. Then

$$\gamma(s', f) \leq a \text{cav}_p f(p) - b \text{cav}_p f(p') - \|ap - bp'\|_1 \leq 0.$$ 

From Sion’s minmax theorem, there exists $s' \in S(p'), \forall f \in D_\theta, \gamma(s', f) \leq 0$. Considering $f$ and $-f$ gives the result. \qed

**Proof of Proposition 3**

(1) $P = \Delta(K)$, all splittings admissible. Fix $p, p'$ in $\Delta(K)$, $f \in D_\theta$, $a, b \geq 0$. Consider $s = \sum_m \lambda_m \delta_{p_m}$ a splitting of $p$ with finite support such $\text{cav}_p f(p) = f(s)$. This splitting is induced by the experiment which sends message $m$ with probability $\lambda_m = \frac{p_m(k)}{p(k)}$ in state $k$. Consider now the same experiment used at prior $p'$, and denote by $s' = \sum_m \lambda'_m \delta_{p'_m}$ the splitting induced by that experiment at $p'$. We have $p' = \sum_m \lambda'_m \delta_{p'_m}$, with $\lambda'_m = \sum_k \lambda_m p(k) \frac{p(k)}{p(k)}$ and $p'_m(k) = \frac{p(k)}{\lambda_m}$. Then,

$$a \text{cav}_p f(p) - b \text{cav}_p f(p') \leq af(s) - bf(s') = \sum_m a\lambda_m f(p_m) - b\lambda'_m f(p'_m)$$

$$\leq \sum_m \|a\lambda_m p_m - b\lambda'_m p'_m\|_1 = \sum_m \sum_k |a\lambda_m p^k_m - b\lambda'_m p^k'_m| = \sum_m \sum_k |ap^k - b\lambda'_m - b\lambda^k_m| = \|ap - bp'\|_1.$$ 

(2) $P$ is finite. Assume that $P$ contains at least two points. Define $\theta \in (0, 1]$ by

$$\theta = \min\{\|ap - (1 - \alpha)p'\|_1, \alpha \in [0, 1], p, p' \in P, p \neq p'\}.$$ 

Fix $p, p'$ in $P$, $f \in D_\theta$, $a, b \geq 0$. If $p \neq p'$, we have for any $s \in S(p)$, $s' \in S(p')$,

$$af(s) - bf(s') \leq (a + b)\theta$$ 

and \(\|ap - bp'\|_1 \geq (a + b)\theta\).

If $p = p'$, given $s \in S(p)$, we have $af(s) - bf(s) \leq |a - b|\theta \leq |a - b||p||_1$, since $\theta \leq 1$. \qed

**Proof of Proposition 4**

**Lemma 7.** Let $M$ be a non empty compact subset of an Euclidean space (set of messages), and let $x : K \rightarrow \Delta(M)$ be fixed. For $p$ in $\Delta(K)$, let $\varphi(p)$ in $\Delta(\Delta(K))$ be the splitting of $p$ induced by $x$, i.e.,

$$\varphi(p) = \int_{m \in M} \delta_{\nu(m)}d\zeta(m),$$

where $\zeta$ is the distribution of messages induced by $p$ and $\nu(m)$ is a conditional probability on $K$ given $m$. We have for any $\theta \in (0, 1]$:

$$\forall p, p' \in \Delta(K), \forall f \in D_\theta, \forall a, b \geq 0, |af(\varphi(p)) - bf(\varphi(p'))| \leq \|ap - bp'\|_1.$$ 

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Proof. Define the probability \( \lambda \) on \( M \) by \( \lambda(B) = \frac{1}{|K|} \sum_{k \in K} x(B|k) \) for all Borel subsets \( B \) of \( M \). By the Radon-Nikodym theorem, for each \( k \in K \), there exists a measurable \( g^k : M \to \mathbb{R}_+ \) such that \( dx(m|k) = g^k(m)d\lambda(m) \), i.e. for each \( B \), \( x(B|k) = \int_{m \in B} g^k(m)d\lambda(m) \).

Fix now \( p \) and \( p' \). Let \( \zeta \) be the distribution of \( m \) induced by \( p \): \( d\zeta(m) = g(m)d\lambda(m) \), with \( g(m) = \sum_{k \in K} p(k)g^k(m) \). A conditional distribution of \( m \) given \( m \) is given by the measurable map \( \nu : M \to \Delta(K) \) such that for all \( k \) in \( K \) and Borel \( B \subseteq M \):

\[
p(k)x(B|k) = \int_{m \in B} \nu(k|m)d\zeta(m).
\]

So for all \((k, m)\), \( p(k)g^k(m) = \int_{m \in B} \nu(k|m)d\zeta(m) \). Similarly, let \( \zeta' \) be the distribution of \( m \) induced by \( p' \): \( d\zeta'(m) = g'(m)d\lambda(m) \), with \( g'(m) = \sum_{k \in K} p'(k)g^k(m) \). Consider \( \nu' : M \to \Delta(K) \) such that for all \( k \) in \( K \) and Borel \( B \subseteq M \):

\[
p'(k)x(B|k) = \int_{m \in B} \nu'(k|m)d\zeta'(m).
\]

So for all \((k, m)\), \( p'(k)g^k(m) = \int_{m \in B} \nu'(k|m)d\zeta'(m) \).

Fix \( f \) in \( D_\theta, a, b \geq 0. \) Since \( \varphi(p) = \int_{m \in M} \delta_{\nu(m)}d\zeta(m) \),

\[
f(\varphi(p)) = \int_{m \in M} f(\nu(m))d\zeta(m) = \int_{m \in M} f(\nu(m))g(m)d\lambda(m).
\]

Therefore,

\[
a f(\varphi(p)) - b f(\varphi(p')) = \int_{m \in M} (a f(\nu(m))g(m) - b f(\nu'(m))g'(m))d\lambda(m),
\]

\[
\leq \int_{m \in M} \|ag(m)\nu(m) - b\nu'(m)g'(m)\|_1d\lambda(m),
\]

\[
= \int_{m \in M} \sum_{k \in K} |ap(k)g^k(m) - bp'(k)g^k(m)|d\lambda(m),
\]

\[
= \sum_{k \in K} \int_{m \in M} g^k(m)|ap(k) - bp'(k)|d\lambda(m),
\]

\[
= \|ap - bp'\|_1.
\]

We now show that Assumption 1 is satisfied for \( S \). Fix \( p \) and \( p' \) in \( P, s \in S(p) \) and a sequence \( T_n \) and \( \sigma_n \) such that \( (\mu_{T_n}(p, \sigma_n))_n \) converges to \( s \). We have \( \mu_{T_n}(p', \sigma_n) \in S(p') \) for each \( n \). Let \( s' \) be a limit point of the sequence \( (\mu_{T_n}(p', \sigma_n))_n \). Take \( f \in D_\theta, a, b \geq 0 \). The auxiliary strategy \( \sigma_n \) defines an experiment \( z : K \to \Delta((X \times M)^{T_n}) \) with compact set of messages. From Lemma 7:

\[
\forall n, a f(\mu_{T_n}(p, \sigma_n)) - b f(\mu_{T_n}(p', \sigma_n)) \leq \|ap - bp'\|_1.
\]

So

\[
a f(s) - b f(s') \leq a f(s) - a f(\mu_{T_n}(p, \sigma_n)) + (a f(\mu_{T_n}(p, \sigma_n)) - b f(\mu_{T_n}(p', \sigma_n))) + b f(\mu_{T_n}(p', \sigma_n)) - b f(s')
\]

\[
= a f(s) - a f(\mu_{T_n}(p, \sigma_n)) + b f(\mu_{T_n}(p', \sigma_n)) - b f(s').
\]
Lemma 8. Proposition 4, which implies:

Passing to the limit as \( n \to \infty \), we get \( af(s) - bf(s') \leq \|ap - bp'\|_1 \), concluding the proof.

Proof of Theorem 3 First note that uniqueness has been proved by Laraki and Renault (2020) in Proposition 4, which implies:

**Lemma 8.** Let \( u : P \times Q \to \mathbb{R} \) be a continuous function. There exists at most one continuous function \( v : P \times Q \to \mathbb{R} \) which is \( S \)-concave \( T \)-convex and satisfies (C1) and (C2) of Proposition 1 for \( u \).

The reader is referred to Laraki and Renault (2020) for the proof. We focus then on existence. From Assumption 1, there exists \( \theta \in (0, 1] \) such that condition (1) is satisfied for Designer 1 and the analog condition for Designer 2. Define \( D_\theta' \) as the set of functions from \( P \times Q \) to \([-\theta, \theta]\) such that \( \forall p, p' \in P, \forall q, q' \in Q, \forall a, b \geq 0 \),

\[
|af(p, q) - bf(p', q)| \leq \|ap - bp'\|_1 \quad \text{and} \quad |af(p, q) - bf(p, q')| \leq \|aq - bq'\|_1.
\]

Notice that any \( f \) in \( D_\theta' \) is 1-Lipschitz in each variable, therefore \( D_\theta' \) is a set of equicontinuous functions from \( P \times Q \) to \([-\theta, \theta]\). The following proposition shows existence of \( \text{MZ}(u) \) for \( u \in D_\theta' \).

**Proposition 10.** Assume \( u \in D_\theta' \). There exists a function \( v = \text{MZ}(u) \in D_\theta' \) which is \( S \)-convex, \( T \)-concave and satisfies (C1) and (C2) of Proposition 1. Moreover, for \( u, u' \in D_\theta' \), \( \|\text{MZ}(u) - \text{MZ}(u')\|_\infty \leq \|u - u'\|_\infty \).

**Proof.** Proposition 10 follows from a series of several lemmas and propositions, where \( u \in D_\theta' \) is assumed. For \( \delta \in [0, 1] \) and \( f \in D_\theta' \), define \( \Psi(f) : P \times Q \to \mathbb{R} \) by

\[
\Psi(f)(p, q) = (1 - \delta)u(p, q) + \delta \text{Val}_{S(p) \times T(q)} f(s, t),
\]

where \( \text{Val}_{S(p) \times T(q)} f(s, t) = \max_{s \in S(p)} \min_{t \in T(q)} f(s, t) = \min_{t \in T(q)} \max_{s \in S(p)} f(s, t) \).

**Lemma 9.** For any \( f \) in \( D_\theta' \), \( \Psi(f) \) is well defined and belongs to \( D_\theta' \). Also, \( \Psi \) has a unique fixed point in \( D_\theta' \) which we denote by \( v_\delta \).

**Proof.** Fix \( \delta \) and \( f \) in \( D_\theta' \). For every \( p, q \), \( \text{Val}_{S(p) \times T(q)} f(s, t) \) is well defined by Sion’s minmax Theorem. So the function \( \Psi(f) : P \times Q \to \mathbb{R} \) is well defined. We now show that \( \Psi(f) \in D_\theta' \) for \( f \in D_\theta' \). Fix \( p, p', q, a \geq 0, b \geq 0 \), we want to prove that \( a\Psi(f)(p, q) - b\Psi(f)(p', q) \leq \|ap - bp'\|_1 \). Since \( u \in D_\theta' \), we have to prove that \( A \leq \|ap - bp'\|_1 \), where

\[
A = \max_{s \in S(p)} \min_{t \in T(q)} af(s, t) - \max_{s' \in S(p')} \min_{t \in T(q)} bf(s', t).
\]

Let \( s \in S(p) \) be optimal in \( \max_{s \in S(p)} \min_{t \in T(q)} f(s, t) \). By Assumption 1, there exists \( s' \in S(p') \) s.t. for all \( t \in \Delta(Q) \), \( bf(s', t) \geq af(s, t) - \|ap - bp'\|_1 \). So \( b\text{Val}_{S(p') \times T(q)} f \geq a\min_{t \in T(q)} f(s, t) - \|ap - bp'\|_1 \). Hence \( A = \min_{t \in T(q)} af(s, t) - b\text{Val}_{S(p') \times T(q)} f \leq \|ap - bp'\|_1 \) and \( \Psi(f) \in D_\theta' \).

It is easy to see that the operator \( \Psi \) is a \( \delta \)-contraction, so by the contracting fixed point Theorem, \( \Psi \) has a unique fixed point \( v_\delta \).
The family of functions \((v_\delta)_\delta\) is equicontinuous since all belong to \(D'_g\). By Ascoli’s Theorem, it admits a limit point \(v \in D'_g\) for uniform convergence. Taking limit in the fixed point equation as \(\delta \to 1\) implies that for all \((p, q) \in P \times Q\), \(v(p, q) = \text{Val}_{S(p) \times T(q)} v(s, t)\). We now have the following proposition.

**Proposition 11.** Any such limit point \(v\) is \(S\)-concave, \(T\)-convex and satisfies \((C1)\) and \((C2)\) of Proposition 1 for \(u\). Thus, an MZ function \(MZ(u)\) exists.

**Proof.** The proofs of Propositions 2 and 3 in Laraki and Renault (2020) show that any limit point \(v\) of \((v_\delta)_\delta\) is \(S\)-concave and \(T\)-convex and has the following property: for all \((p, q) \in P \times Q\), there exists \(s \in S(p)\) such that \(v(p, q) = v(s, q) \leq u(s, q)\) and there exists \(t \in T(q)\) such that \(v(p, q) = v(p, t) \geq u(p, t)\). We show now that \(v\) satisfies \((C1), (C2)\) being obtained by a symmetric argument.

Fix \((p_0, q_0)\) in \(P \times Q\). Consider the correspondence \(S'\) from \(P\) to \(\Delta(P)\) defined for \(p\) in \(P\) by,

\[
S'(p) = \{ s \in S(p) : v(p, q_0) = v(s, q_0) \leq u(s, q_0) \}.
\]

This correspondence admits measurable selections and any such selection \(f\) defines a strategy for Designer 1: at each \(p\), play \(f(p)\). Starting from \(p_0\), the strategy \(f\) induces a martingale \((p_n)_n\) which converges a.s. to some random variable \(p_\infty\) with \(E(p_\infty) = p_0\) and \(E(v(p_\infty, q_0)) = v(p_0, q_0)\). Thus, the distribution of \(p_\infty\) denoted \(s_\infty\) belongs to \(S(p_0)\) and satisfies \(v(s_\infty, q_0) = v(p_0, q_0)\).

The proof of Proposition 11 is concluded by the following lemma.

**Lemma 10.** \(v(p_\infty, q_0) \leq u(p_\infty, q_0)\) almost surely.

**Proof.** Let \(g : P \to \mathbb{R}\) be the continuous function given by \(g(p) = v(p, q_0) - u(p, q_0)\). We want to show that \(g(p_\infty) \leq 0\) a.s. From the choice of \(f\), for each \(p\) in \(P\), \(g(f(p)) = E_f(g) \leq 0\). Since \(g\) is continuous, \(g(p_{n+1}) - g(p_n) \to_{n \to \infty} 0\) a.s., so by the dominated convergence theorem \(E((g(p_{n+1}) - g(p_n))^2) \to_{n \to \infty} 0\). Define the random variable:

\[
Y_n = E[(g(p_{n+1}) - g(p_n))^2 | p_n].
\]

We have then \(Y_n \geq 0\), \(Y_n\) is bounded and \(E(Y_n) \to_{n \to \infty} 0\), which imply that there exists a subsequence \((Y_{\varphi(n)})_n\) which converges to 0 almost surely. Moreover, since \(E[g(p_{n+1}) - g(f(p_n))|p_n] = 0\), we have,

\[
Y_n = E[(g(p_{n+1}) - g(f(p_n)))^2 | p_n] + E[(g(f(p_n)) - g(p_n))^2 | p_n]
\]

\[
= E[(g(p_{n+1}) - g(f(p_n)))^2 | p_n] + (g(f(p_n)) - g(p_n))^2.
\]

Thus \(Y_{\varphi(n)}\) tends to 0 and is the sum of two positive terms, so each tend to 0. Thus

\[
(g(f(p_{\varphi(n)})) - g(p_{\varphi(n)}))^2 \to_{n \to \infty} 0\ a.s.
\]

Since \(g(f(p_{\varphi(n)})) \leq 0\) for all \(n\) and \(g(p_{\varphi(n)}) \to_{n \to \infty} g(p_\infty)\), we get \(g(p_\infty) \leq 0\) a.s.

This proves Proposition 11.

We know then that any limit point of \(v_\delta\) is the unique MZ function (Lemma 8). This implies that \(v_\delta\) converges to \(v = MZ(u)\) as \(\delta \to 1\). Since \(\|v_\delta(u) - v_\delta(u')\|_\infty \leq \|u - u'\|_\infty\) for all \(\delta\), we get

\[
\|MZ(u) - MZ(u')\|_\infty \leq \|u - u'\|_\infty.
\]

This ends the proof of Proposition 10.
To complete the proof of Theorem 3, we must extend the existence result to all continuous functions. The argument is that functions in $D_\theta'$ (or their multiples) are dense within continuous functions.

**Lemma 11.**

1. The set $\mathcal{L} = \{lf, l \geq 0, f \in D_\theta\}$ is dense (for the uniform norm) in the set of continuous functions from $P$ to $\mathbb{R}$.

2. The set $\mathcal{L}' = \{lf, l \geq 0, f \in D_\theta'\}$ is dense (for the uniform norm) in the set of continuous functions from $P \times Q$ to $\mathbb{R}$.

**Proof.**

1. First observe that $\mathcal{L}$ is a lattice. Take $f, g$ in $\mathcal{L}$ and $l, l'$ such that $\frac{f}{l}$ and $\frac{g}{l'}$ belong to $D_\theta$ and let $l'' = \max\{l, l'\}$. Then $\frac{f}{l'}, \frac{g}{l'}$ belong to $D_\theta$ and $\frac{\max(f, g)}{l''}$ and $\frac{\min(f, g)}{l''}$ as well. We get that $\max(f, g)$ and $\min(f, g)$ are in $\mathcal{L}$.

   Second, for every $p_0 \neq p_1$ in $P$ and real numbers $\alpha, \beta$, there exists $f \in \mathcal{L}$ such that $f(p_0) = \alpha$ and $f(p_1) = \beta$. We know that there exists a linear mapping $\varphi$ on $\mathbb{R}^K$ such that $\varphi(p_0) = \alpha$ and $\varphi(p_1) = \beta$. Because $\varphi$ is linear, there exists $l \geq 0$ such that for every $p, p'$ and $a, b \geq 0$,

$$|a\varphi(p) - b\varphi(p')| = |\varphi(ap - bp')| \leq l||ap - bp'||_1$$

Thus $\varphi$ restricted to $P$ belongs to $\mathcal{L}$. By the (lattice) Stone-Weierstrass Theorem, $\mathcal{L}$ is dense in continuous functions.

2. As in the previous point, it is easy to see that $\mathcal{L}'$ is also a lattice. It is thus enough to prove that for $(p_0, q_0)$ and $(p_1, q_1)$, with $(p_0, q_0) \neq (p_1, q_1)$, in $P \times Q$, $\alpha, \beta \in \mathbb{R}$, there exists $f$ in $\mathcal{L}'$ such that $f(p_0, q_0) = \alpha$ and $f(p_1, q_1) = \beta$. Assume $p_0 \neq p_1$ From the previous point, we know that there exists $g$ in $\mathcal{L}$ such that $g(p_0) = \alpha$ and $g(p_1) = \beta$. Define $f(p, q) = g(p)$ for all $p, q$. To see that $f \in \mathcal{L}'$, consider $p, p', q, q'$ and $\alpha, 0 \geq b \geq 0$. We have

$$|af(p, q) - bf(p', q)| = |ag(p) - bg(p')| \leq l||ap - bp'||_1$$

since $g \in \mathcal{L}$. Then,

$$|af(p, q) - bf(p, q')| = |g(p)||a - b| = |g(p)| \cdot ||aq||_1 - ||bq'||_1 \leq l'||aq - bq'||_1$$

with $l' = \max_p |g(p)|$.

We may now conclude the proof of Theorem 3. Consider a continuous payoff function $u : P \times Q \to \mathbb{R}$. For each $n \geq 1$, there exists $u_n$ in $\mathcal{L}'$ such that $||u - u_n||_1 \leq \frac{1}{n}$ and $l_n > 0$ such that $u_n/l_n \in D_\theta'$. By Proposition 11, $\text{MZ}(u_n/l_n)$ exists and we let $v_n = l_n\text{MZ}(u_n/l_n)$. This function is $S$-concave, $T$-convex and satisfies (C1) and (C2) of Proposition 1 for $u_n$. So $v_n = \text{MZ}(u_n)$. Fix now $n, m$ and set $l := \max\{l_n, l_m\}$. We have $u_n/l$ and $u_m/l$ are in $D'_\theta$, $v_n = l\text{MZ}(u_n/l)$ and $v_m = l\text{MZ}(u_m/l)$. Also we know that $||v_n - v_m||_1 \leq l||u_n/l - u_m/l||_1 = ||u_n - u_m||_1$. Since $(u_n)_n$ is a Cauchy sequence, so is $(v_n)_n$. Hence $(v_n)_n$ converges to a continuous $v$ which is $S$-concave and $T$-convex. It is also easy to see by
taking limits that $v$ satisfies (C1) and (C2) for $u$. Therefore $v = \text{MZ}(u)$ exists. This ends the proof of Theorem 3. □

**Proof of Proposition 6**  Consider the stationary strategy $\sigma$ of Designer 1 who plays nonrevelingly if $v(p, q) \leq u(p, q)$, and $s$ is given by (C1) otherwise. Consider any strategy $\tau$ of Designer 2. The definition of $\sigma$ implies that for each $n$,

$$
E[v(p^{n+1}, q^n)|p^n, q^n] = v(p^n, q^n).
$$

Since $v$ is $T$-convex,

$$
E[v(p^{n+1}, q^{n+1})|p^{n+1}, q^n] \geq v(p^{n+1}, q^n).
$$

Taking expectation, for each $n$, $E[v(p^{n+1}, q^{n+1})] \geq E[v(p^{n+1}, q^n)] \geq \limsup E[v(p^n, q^n)] \geq v(p_0, q_0)$ by induction. Denote $X = \{(p, q) \in P \times Q : v(p, q) \leq u(p, q)\}$; by construction, $(p^{n+1}, q^n) \in X$ almost surely for each $n$. Since by assumption $X$ is closed, $(p^\infty, q^\infty) \in X$ a.s., i.e., $u(p^\infty, q^\infty) \geq v(p^\infty, q^\infty)$ a.s.

**Claim 1.** $\limsup_n v(p^n, q^n) \leq v(p^\infty, q^\infty)$ a.s.

**Proof.** Fix a realized play path and consider a converging subsequence of $(v(p^n, q^n))_n$ denoted by $(v(p^n, q^n))_n$. We show that $\lim_n v(p^n, q^n) \leq v(p^\infty, q^\infty)$. There are 2 cases.

1) Suppose that there exists $n_0$ such that for all $n \geq n_0$, $(p^n, q^n) \in X$. Then, for each $n \geq n_0$, $p^{n+1} = p^n = p^\infty$ and $\lim_n v(p^n, q^n) = \lim_n v(p^\infty, q^n)$. Since $v$ is u.s.c. in $q$, $\lim_n v(p^n, q^n) \leq v(p^\infty, q^\infty)$.

2) Otherwise, there exists a subsequence $(p^n, q^n)$ of $(p^n, q^n)$ with values in $\{(p, q) : v(p, q) > u(p, q)\}$. Since $v$ is u.s.c. on the closure of this set, we obtain $v(p^\infty, q^\infty) \geq \lim_n v(p^n, q^n)$.□

It follows that

$$
v(p_0, q_0) \leq \limsup_n E(v(p_n, q_n)) \leq E(\limsup_n v(p_n, q_n)) \leq \lim_n E(v(p_n, q_n)) \leq E(u(p^\infty, q^\infty)),
$$

which concludes the proof. □

**References**

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B.1 Computing the Mertens-Zamir Value

The Mertens-Zamir function is defined as a fixed point $v = \text{cav}_p \min(u, v) = \text{vex}_q \max(u, v)$ in a set of functions, and its computation is difficult in general. We present several algorithmic results for computing it. The first result shows an iterative procedure converging to the MZ value.

**Proposition 12.** Let $u : P \times Q \to \mathbb{R}$ be a continuous function. Define inductively two sequences of functions $\{\mu_k\}_{k=1}^{\infty}, \{\nu_k\}_{k=1}^{\infty}$ as follows:

$$
\mu_0 = \nu_0 = u, \quad \mu_{k+1} = \text{cav}_p \text{vex}_q \max(u, \mu_k), \quad \nu_{k+1} = \text{vex}_q \text{cav}_p \min(u, \nu_k).
$$

Then, $\{\mu_k\}_{k=1}^{\infty}$ is monotonically increasing, $\{\nu_k\}_{k=1}^{\infty}$ is monotonically decreasing, and both sequences converge uniformly to $\text{MZ}(u)$.

**Proof of Proposition 12.** These properties are proven by Mertens and Zamir (1977) if $P$ and $Q$ are convex and compact and all splittings are admissible, $S(p) = P \cap S(p)$, $T(q) = Q \cap T(q)$. A minor adaptation of the proof in Mertens and Zamir (1977) shows that these properties also hold in our more general context. 

In particular, Proposition 12 gives an algorithm for calculating the MZ value if $P$ and $Q$ are finite and all splittings are admissible. Each step of the algorithm is a series of simple cav$_p$ and vex$_q$ operations. The algorithm can be made more explicit for binary states $K = L = \{0, 1\}$. We identify $\Delta(K)$ and $\Delta(L)$ with the interval $[0, 1]$, and identify $p \in \Delta(K)$ with $p(1) \in [0, 1]$ and $q \in \Delta(L)$ with $q(1) \in [0, 1]$. Let $P = \{p_0, p_1, \ldots, p_{M-1}, p_M\} \subset [0, 1]$ with $p_i < p_{i+1}$ for $i = 0, \ldots, M - 1$. Similarly, $Q = \{q_0, q_1, \ldots, q_{N-1}, q_N\} \subset [0, 1]$ with $q_j < q_{j+1}$ for $j = 0, \ldots, N - 1$. An algorithm for computing cav$_p u$ is as follows. Start from function $u$. For each $j = 0, \ldots, N$,

- for each $i = 1, \ldots, M - 1$, if

$$
    u(p_i, q_j) < \frac{p_{i+1} - p_i}{p_{i+1} - p_i - 1} u(p_{i-1}, q_j) + \frac{p_i - p_{i-1}}{p_{i+1} - p_i - 1} u(p_{i+1}, q_j),
$$

    replace $u(p_i, q_j)$ by

$$
    \frac{p_{i+1} - p_i}{p_{i+1} - p_i - 1} u(p_{i-1}, q_j) + \frac{p_i - p_{i-1}}{p_{i+1} - p_i - 1} u(p_{i+1}, q_j);
$$

- repeat until $u(p_i, q_j) \geq \frac{p_{i+1} - p_i}{p_{i+1} - p_i - 1} u(p_{i-1}, q_j) + \frac{p_i - p_{i-1}}{p_{i+1} - p_i - 1} u(p_{i+1}, q_j)$ for all $i = 1, \ldots, M - 1$.

We now present another algorithm for computing the MZ value if states are binary ($K = L = \{0, 1\}$), $P$ and $Q$ are finite, and all splittings are admissible. To simplify the exposition, we assume that $P$ and $Q$ are uniform grids, i.e., there exist positive integers $M$ and $N$ such that $P = \{\frac{i}{M}, i \in I\}$ and $Q = \{\frac{j}{N}, j \in J\}$, with $I = \{0, 1, \ldots, M\}$ and $J = \{0, 1, \ldots, N\}$ (the approach can be easily generalized to irregular grids). The payoff function $u$ can here be simply represented by a matrix in $\mathbb{R}^{I \times J}$ that we denote with some abuse of notation by $u = (u_{i,j})$, where $u_{i,j} = u(p_i, p_j)$ for every $(i, j) \in I \times J$. Given
a matrix $w$ in $\mathbb{R}^{I \times J}$, we say that

$$w \text{ is column-concave if } \forall j \in J, \forall i \in \{1, \ldots, M - 1\}, \ w_{i,j} \geq \frac{1}{2}(w_{i-1,j} + w_{i+1,j}),$$

$$w \text{ is row-convex if } \forall i \in I, \forall j \in \{1, \ldots, N - 1\}, w_{i,j} \leq \frac{1}{2}(w_{i,j-1} + w_{i,j+1}).$$

We have the following characterization of the MZ value.

**Lemma 12.** Given a matrix $u$ in $\mathbb{R}^{I \times J}$, there exists a unique matrix $v$ in $\mathbb{R}^{I \times J}$ that is column-concave and row-convex, and such that for all $(i, j) \in I \times J$,

$$\text{if } v_{i,j} > u_{i,j}, \text{ then } 0 < i < M \text{ and } v_{i,j} = \frac{1}{2}(v_{i-1,j} + v_{i+1,j}), \text{ and}$$

$$\text{if } v_{i,j} < u_{i,j}, \text{ then } 0 < j < N \text{ and } v_{i,j} = \frac{1}{2}(v_{i,j-1} + v_{i,j+1}).$$

Moreover, $v_{i,j} = \text{MZ}(u)(i/M, j/N)$ for all $i$ and $j$, where $\text{MZ}(u)$ is the value of information design games with payoff $u : P \times Q \to \mathbb{R}$ and no deadline. Abusing notation, we write $v = \text{MZ}(u)$.

**Proof.** We have to show that the matrix $v$ corresponding to $\text{MZ}(u)$ is the unique matrix satisfying the conditions of the lemma. We first show that $v$ satisfies these conditions. By Proposition 1, $v$ is column-concave and row-convex, and satisfies (C1) and (C2) for all $(p, q) \in P \times Q$. Consider $(i, j)$ in $I \times J$ such that $v_{i,j} > u_{i,j}$. The splitting $s$ obtained from (C1) at $(p, q) = (i/M, j/N)$ must be informative, and we can find $i'$ and $i''$ in $I$ such that $i' \leq i - 1 < i + 1 \leq i''$, and $i'/M$ and $i''/M$ belong to the support of $s$. Since $v(p, q) = v(s, q)$ and $v$ is column-concave, $v(\cdot, w)$ has to be affine on $[i'/M, i''/M]$ and in particular on $[(i - 1)/M, (i + 1)/M]$. Hence, $v_{i,j} = \frac{1}{2}(v_{i-1,j} + v_{i+1,j})$. The case of $v_{i,j} < u_{i,j}$ is treated symmetrically.

Now, we prove uniqueness. Assume that two matrices $v$ and $w$ satisfy the conditions of Lemma 12. Define $\alpha = \max_{(i,j)\in I \times J} \{v_{i,j} - w_{i,j}\}$ and $Z = \arg \max_{(i,j)\in I \times J} \{v_{i,j} - w_{i,j}\}$ and let $(i_0, j_0)$ be an element of $Z$ minimizing the sum of coordinates $i + j$.

Suppose that $v_{i_0,j_0} > w_{i_0,j_0}$. Then, $v_{i_0,j_0} = \frac{1}{2}(v_{i_0-1,j_0} + v_{i_0+1,j_0})$, and since $w$ is column-concave, $w_{i_0,j_0} \geq \frac{1}{2}(w_{i_0-1,j_0} + w_{i_0+1,j_0})$. We obtain $\alpha = v_{i_0,j_0} - w_{i_0,j_0} \leq \frac{1}{2}((v_{i_0-1,j_0} - w_{i_0-1,j_0}) + (v_{i_0+1,j_0} - w_{i_0+1,j_0})) \leq \frac{1}{2}((v_{i_0-1,j_0} - w_{i_0-1,j_0}) + \alpha)$; hence, $(i_0 - 1, j_0) \in Z$, which contradicts the minimality of $(i_0, j_0)$. Thus, $v_{i_0,j_0} \leq w_{i_0,j_0}$.

Suppose now that $w_{i_0,j_0} < u_{i_0,j_0}$. Then, $w_{i_0,j_0} = \frac{1}{2}(w_{i_0,j_0-1} + w_{i_0,j_0+1})$ and since $v$ is row-convex, $v_{i_0,j_0} \leq \frac{1}{2}(v_{i_0,j_0-1} + v_{i_0,j_0+1})$. We obtain $\alpha = v_{i_0,j_0} - w_{i_0,j_0} \leq \frac{1}{2}((v_{i_0,j_0-1} - w_{i_0,j_0-1}) + (v_{i_0,j_0-1} - w_{i_0,j_0+1}))$, so $(i_0, j_0 - 1) \in Z$ which contradicts the minimality of $(i_0, j_0)$. Therefore, $w_{i_0,j_0} \geq u_{i_0,j_0}$. We deduce that $\alpha = v_{i_0,j_0} - w_{i_0,j_0} \leq u_{i_0,j_0} - u_{i_0,j_0} = 0$; hence, $v \leq w$. By symmetry, $v = w$. \[\Box\]

Given Lemma 12, the algorithm for computing $\text{MZ}(u)$ is as follows: First, concavify $u_0$ and $u_{i,N}$ w.r.t. $i$, and convexify $u_0j$ and $u_{i,j}$ w.r.t. $j$. Take two subsets $A_+, A_-$ of $I \times J$ and postulate that for all $i, j$, $v_{ij} > u_{ij}$ if and only if $(i, j) \in A_+$, and $v_{ij} < u_{ij}$ if and only if $(i, j) \in A_-$. Solve the following linear system:

$$v_{ij} = \frac{1}{2}v_{i-1,j} + \frac{1}{2}v_{i+1,j} \text{ for } (i, j) \in A_+,$$

$$v_{ij} = \frac{1}{2}v_{i,j-1} + \frac{1}{2}v_{i,j+1} \text{ for } (i, j) \in A_-,$$

$$v_{ij} = u_{ij} \text{ for } (i, j) \notin A_- \cup A_+.$$
If the solution of the system does not satisfy \( v_{ij} > u_{ij} \) for all \((i, j) \in A_+ \) and \( v_{ij} < u_{ij} \) for all \((i, j) \in A_- \), or does not satisfy \( S \)-concavity and \( T \)-convexity, try another pair of subsets. Otherwise, \( v = MZ(u) \).

The linear system above corresponds to a diagonally dominant matrix, and hence has a unique solution. Since there is a unique \( v = MZ(u) \), there is a unique pair \( A_+, A_- \) on which the algorithm will eventually stop.

Let us illustrate this algorithm with the following matrix:

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
4 & 2 & 3 & 3 \\
1 & 4 & 2 & 5 \\
0 & 3 & 4 & 7 \\
\end{array}
\]

The first step is the concavification of the first and fourth columns, and the convexification of the first and fourth rows. We obtain the following:

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
4 & 2 & 3 & 3 \\
2 & 4 & 2 & 5 \\
0 & 2 & 4 & 7 \\
\end{array}
\]

It is easy to note that \( v = MZ(u) = MZ(u') \) and that \( v \) coincides with \( u' \) on the first and fourth columns and on the first and fourth rows. Hence, \( v \) can be written as follows:

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
2 & c & d & 5 \\
0 & 2 & 4 & 7 \\
\end{array}
\]

We have to compare \( a, b, c \) and \( d \) to the corresponding entries in \( u \). Let us compute lower and upper bounds for \( v \):

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
4 & \frac{10}{3} & 3 & 3 \\
2 & \frac{5}{3} & 3 & 5 \\
0 & 2 & 4 & 7 \\
\end{array} \quad \text{cav}_p, \text{vex}_q \begin{array}{cccc}
6 & 4 & 2 & 1 \\
4 & \frac{5}{3} & \frac{8}{3} & 3 \\
2 & \frac{5}{3} & \frac{10}{3} & 5 \\
0 & 2 & 4 & 7 \\
\end{array}
\]

We have \( \text{cav}_p, \text{vex}_q u' \leq v \leq \text{vex}_q, \text{cav}_p u' \), so \( a > 2 \), \( d > 2 \), \( b \leq 3 \) and \( c < 4 \). The entries corresponding to \( a \) and \( d \) belong to the region where \((v > u)\), and the entry corresponding to \( c \) belongs to the region \((v < u)\). From Lemma 12, we obtain \( a = (4 + c)/2, c = (2 + d)/2, d = (b + 4)/2 \). There are now 2 possible cases: \( b < 3 \) and \( b = 3 \). If \( b < 3 \), we have \( b = (a + 3)/2 \), and the system has a unique solution given by \( a = 51/15, b = 48/15, c = 42/15 \) and \( d = 54/15 \). We observe a contradiction since \( 48/15 > 3 \). Hence, \( b = 3 \) must hold. The system has a unique solution given by \( a = 27/8, b = 3, c = 11/4 \) and
\[ d = 7/2. \] Finally,

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
4 & 22/3 & 24/3 & 3 \\
2 & 22/3 & 28/3 & 5 \\
0 & 2 & 4 & 7
\end{array}
\]

\[ v = \]

In the equilibrium in this example, information disclosure stops after the maximum of 3 stages.

The next result shows that computing the MZ function for finite sets allows approximating that function for models with binary states \( K = L = \{0, 1\} \) and no restrictions on admissible posteriors and admissible splittings: \( P = Q \cong [0, 1] \), \( S(p) = S(p) \), \( T(q) = T(q) \).

**Proposition 13.** Let \( u : [0, 1] \times [0, 1] \to \mathbb{R} \) be a continuous function. For each \( k \geq 1 \), denote \( P_k = Q_k = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\} \), and let \( u_k : P_k \times Q_k \to \mathbb{R} \) be the restriction of \( u \) on \( P_k \times Q_k \) and \( v_k : P_k \times Q_k \to \mathbb{R} \) be the MZ function of \( u_k \). Additionally, let \( w_k : [0, 1] \times [0, 1] \to \mathbb{R} \) be the piecewise bilinear extension of \( v_k \). In other words, \( w_k(p, q) \) is bilinear on each square \( [i/k, (i+1)/k] \times [j/k, (j+1)/k] \) and coincides with \( v_k \) in the corners of the square.

Then, \( (w_k)_k \) uniformly converges to \( v = \text{MZ}(u) \). Moreover, if \( u \) is \( C \)-Lipschitz for \( d((p, q), (p', q')) = |p - p'| + |q - q'| \), then for each \( k \geq 1 \),

\[
\|v - w_k\|_{\infty} \leq \frac{4C}{k}.
\]

As a consequence, an approximation of \( \text{MZ}(u) \) is obtained by considering a fine discretization of \([0, 1]\) and computing the MZ value of the information design game in which designers are constrained by the grid.

**Proof.** We use the distance \( d((p, q), (p', q')) = |p - p'| + |q - q'| \) on the square \([0, 1]^2\). Since the function \( u : P \times Q \to \mathbb{R} \) is continuous on compact sets, it is also uniformly continuous, and therefore there exists a function \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \lim_{x \to 0} \omega(x) = 0 \) called a modulus of continuity such that

\[
\forall p, p' \in P, q, q' \in Q, \ |u(p, q) - u(p', q')| \leq \omega(d((p, q), (p', q'))).
\]

Moreover, \( \omega \) can be chosen to be nondecreasing and concave. Laraki and Renault (2020) in Lemma 9.1 prove that if \( v = \text{MZ}(u) \), then \( v \) also has \( \omega \) as a modulus of continuity, i.e.,

\[
\forall p, p' \in P, q, q' \in Q, \ |v(p, q) - v(p', q')| \leq \omega(d((p, q), (p', q'))).
\]

In particular, if \( u \) is \( C \)-Lipschitz, \( \omega(x) = Cx \) and \( v \) is also \( C \)-Lipschitz. With the notation of Proposition 13, for each \( k \geq 1 \), \( u_k \) also has \( \omega \) as a modulus of continuity; this is also true for \( v_k \) from Laraki and Renault (2020), and it is easy to see that this holds for \( w_k \) as well.

Fix \( k \geq 1 \) and consider the information design game \( G_{\infty}(p^0, q^0) \) in which the payoff function is \( u \), \( P = P_k \), \( Q = \Delta(L) \), and there are no further restrictions on admissible splittings. Denote \( A_k = B_k = \{0, 1, \ldots, k\} \) and define a strategy \( \sigma \) of Designer 1 as follows. For the posteriors of stage \( n \), \( (p^n, q^n) \in P_k \times \Delta(L) \), let \( i^n \) in \( A_k \) and \( j^n \) in \( B_k \) be such that \( p^n = \frac{i^n}{k} \) and \( q^n \in \left[\frac{j^n}{k}, \frac{j^n+1}{k}\right) \).
a) If \( w_k(\frac{i^n}{k}, \frac{j^n}{k}) > u(\frac{i^n}{k}, \frac{j^n}{k}) \) and \( w_k(\frac{i^n+1}{k}, \frac{j^n}{k}) > u(\frac{i^n+1}{k}, \frac{j^n}{k}) \), then \( \sigma(p^n, q^n) \) splits at stage \( n \) uniformly between \( p^n - \frac{1}{k} \) and \( p^n + \frac{1}{k} \).

b) Otherwise, \( \sigma(p^n, q^n) \) is nonrevealing.

Fix any strategy \( \tau \) of Designer 2 and consider the induced random sequence of posteriors \( (p^n, q^n)_n \). Suppose that case a) holds at stage \( n \). Since \( w_k(\frac{i^n}{k}, \frac{j^n}{k}) > u(\frac{i^n}{k}, \frac{j^n}{k}) \) and \( w_k = MZ(u_k) \) on the grid, we have the following:

\[
w_k\left(\frac{i^n}{k}, \frac{j^n}{k}\right) = \frac{1}{2}w_k\left(\frac{i^n - 1}{k}, \frac{j^n}{k}\right) + \frac{1}{2}w_k\left(\frac{i^n + 1}{k}, \frac{j^n}{k}\right).
\]

Similarly, \( w_k(\frac{i^n}{k}, \frac{j^n+1}{k}) = \frac{1}{2}w_k(\frac{i^n-1}{k}, \frac{j^n+1}{k}) + \frac{1}{2}w_k(\frac{i^n+1}{k}, \frac{j^n+1}{k}) \); therefore, from bilinearity,

\[
w_k\left(\frac{i^n}{k}, q^n\right) = \frac{1}{2}w_k\left(\frac{i^n - 1}{k}, q^n\right) + \frac{1}{2}w_k\left(\frac{i^n + 1}{k}, q^n\right),
\]

and we obtain the following:

\[
\mathbb{E}[w_k(p^{n+1}, q^n)|p^n, q^n] = w_k(p^n, q^n).
\]

This equality obviously holds in case b), so it holds almost surely for every \( n \).

Next, for any given \( p \) in \( P_k \), the mapping \( q \mapsto w_k(p, q) \) is convex. To see this, note that the \( T \)-convexity of \( w_k \) implies that for \( 0 < j < k \),

\[
w_k\left(p, \frac{j+1}{k}\right) \leq \frac{1}{2}\left(w_k(p, \frac{j-1}{k}) + w_k(p, \frac{j+1}{k})\right).
\]

Thus, \( q \mapsto w_k(p, q) \) is continuous and piecewise linear with a nondecreasing slope, and hence is convex. As a consequence,

\[
\mathbb{E}(w_k(p^{n+1}, q^{n+1})) \geq \mathbb{E}(w_k(p^{n+1}, q^n)) = \mathbb{E}(w_k(p^n, q^n)) \geq w_k(p^0, q^0),
\]

and considering the limit gives the following:

\[
\mathbb{E}(w_k(p^\infty, q^\infty)) \geq w_k(p^0, q^0).
\]

Now, since \( P_k \) is finite, for almost all realized sequences \( (p^n, q^n)_n \) there exists \( n_0 \) such that \( p^n = p^\infty \) for all \( n \geq n_0 \). Hence, for \( n \geq n_0 \) we have \( w_k(p^\infty, \frac{i^n}{k}) \leq u(p^\infty, \frac{i^n}{k}) \) or \( w_k(p^\infty, \frac{i^n+1}{k}) \leq u(p^\infty, \frac{i^n+1}{k}) \). Denoting by \( \omega \) the modulus of continuity of \( u \), we have \( u(p^\infty, q^n) \geq u(p^\infty, q^\infty) \geq u(p^\infty, q^n) - \omega(\frac{1}{k}) \). Additionally, \( w_k(p^\infty, q^n) \leq w_k(p^\infty, \frac{i^n}{k}) + \omega(\frac{1}{k}) \) and \( w_k(p^\infty, q^n) \leq w_k(p^\infty, \frac{i^n+1}{k}) + \omega(\frac{1}{k}) \). We obtain that for all \( n \geq n_0 \), \( w_k(p^\infty, q^n) \leq u(p^\infty, q^n) + 2\omega(\frac{1}{k}) \). Considering the limit, we obtain that, almost surely,

\[
w_k(p^\infty, q^\infty) \leq u(p^\infty, q^\infty) + 2\omega(\frac{1}{k}).
\]

It follows that

\[
\mathbb{E}(u(p^\infty, q^\infty)) \geq w_k(p^0, q^0) - 2\omega(\frac{1}{k}).
\]

Therefore, by playing \( \sigma \), Designer 1 guarantees the payoff \( w_k(p^0, q^0) - 2\omega(\frac{1}{k}) \) in \( G_\infty(p^0, q^0) \). Hence, we
have the following:

$$\forall (p^0, q^0) \in P_k \times \Delta(L), \ v(p^0, q^0) \geq w_k(p^0, q^0) - 2\omega\left(\frac{1}{k}\right).$$

Now, consider \((p^0, q^0)\) in \(\Delta(K) \times \Delta(L)\). For \(p^0\) in \(P_k\) such that \(d(p^0, p^0') \leq \frac{1}{k}\), we have the following:

$$v(p^0, q^0) \geq v(p^0, q^0) - \omega\left(\frac{1}{k}\right) \geq w_k(p^0, q^0) - 3\omega\left(\frac{1}{k}\right) \geq w_k(p^0, q^0) - 4\omega\left(\frac{1}{k}\right).$$

Exchanging the roles of designers 1 and 2, we obtain \(v(p^0, q^0) \leq w_k(p^0, q^0) + 4\omega\left(\frac{1}{k}\right)\), and finally

$$\|v - w_k\|_{\infty} \leq 4\omega\left(\frac{1}{k}\right).$$

Minor adaptations of this proof yield the following result for irregular grids.

**Proposition 14.** Let \(u : [0, 1] \times [0, 1] \rightarrow \mathbb{R}\) be a continuous function with modulus of continuity \(\omega\) for distance \(d((p, q), (p', q')) = |p-p'| + |q-q'|\). Let \(P = \{p_0, p_1, \ldots, p_{M-1}, p_M\}\) and \(Q = \{q_0, q_1, \ldots, q_{N-1}, q_N\}\) and denote the following:

$$m(P, Q) = \max \{ \max\{|p_{i+1} - p_i| : i = 0, \ldots, M - 1\}; \max\{|q_{j+1} - q_j| : j = 0, \ldots, N - 1\} \}. $$

Let \(u_{PQ}\) be the restriction of \(u\) on \(P \times Q\), \(v_{PQ}\) be the MZ function of \(u_{PQ}\), and \(w_{PQ}\) be the piecewise bilinear extension of \(v_{PQ}\). We have the following:

$$\|MZ(u) - w_{PQ}\|_{\infty} \leq 4\omega\left(m(P, Q)\right).$$

### B.2 General Splittings on Finite Sets of Admissible Posteriors

In this section, we show that if \(P\) and \(Q\) are finite, very few assumptions are needed to prove that games \(G_{\infty}(p^0, q^0)\) and \(G_{\infty}^{eq}(p^0, q^0)\) have values. In other words, we consider finite \(P\) and \(Q\), suppose that for all \((p, q), \delta_p \in S(p)\) and \(\delta_q \in T(q)\), but make no further assumptions on correspondences \(S, T\).

For all \((p, q)\) in \(P \times Q\), define

$$v_-(p, q) = \sup \{ v(s, q) : \exists \alpha \in [0, 1], \exists s \in \Delta(P\{p\}) \text{ s.t. } \alpha\delta_p + (1-\alpha)s \in S(p) \},$$

$$v_+(p, q) = \inf \{ v(p, t) : \exists \alpha \in [0, 1], \exists t \in \Delta(Q\{q\}) \text{ s.t. } \alpha\delta_q + (1-\alpha)t \in T(q) \}.$$ 

For instance, one may think of the example where \(P = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}\), and Designer 1 at posterior \(p \in \{\frac{1}{k}, \ldots, \frac{k-1}{k}\}\) can either stay at \(p\) or split between the two closest neighbors \(p - \frac{1}{k}\) and \(p + \frac{1}{k}\). Then,

$$v_-(p, q) = \frac{1}{2}(v(p - \frac{1}{k}, q) + v(p + \frac{1}{k}, q)).$$

**Proposition 15.** Games \(G_{\infty}(p^0, q^0)\) and \(G_{\infty}^{eq}(p^0, q^0)\) admit the same value, which is the unique function \(v : P \times Q \rightarrow \mathbb{R}\) that is \(S\)-concave, \(T\)-convex, and such that for all \((p, q)\) in \(P \times Q\),

\((C1')\) if \(v(p, q) > u(p, q)\), then \(v(p, q) = v_-(p, q)\), and
\((C2')\) if \(v(p, q) < u(p, q)\), then \(v(p, q) = v_{+}(p, q)\).

Moreover, the value is obtained in pure strategies: designers have pure \(\varepsilon\)-optimal strategies for all \(\varepsilon > 0\).

This result relaxes the assumptions of compactness and convexity of correspondences \(S, T\). Importantly, it relaxes the assumption that admissible splittings are closed under iteration. In the proof, we consider correspondences \(S^{\infty}\) and \(T^{\infty}\) that are the “closures” of \(S, T\) defined by taking convex hull, topological closure and closure under repetition. The first part of the proof (Lemma 13) shows that the MZ function associated with \(S^{\infty}\) and \(T^{\infty}\) satisfies \((C1')\) and \((C2')\). The second part (Lemma 14) shows that each Designer can guarantee the value up to any \(\varepsilon > 0\). Since admissible splittings are not closed under iteration, achieving a desired splitting may require several stages. As a result, \(\varepsilon\)-optimal strategies are not Markovian.

**Lemma 13.** There exists \(v : P \times Q \to \mathbb{R}\) that is \(S\)-concave, \(T\)-convex and such that \((C1')\) and \((C2')\) are satisfied for all \((p, q)\) in \(P \times Q\).

**Proof.** Define the correspondence \(S : \Delta(P) \rightrightarrows \Delta(P)\) as follows:

\[
\forall s \in \Delta(P), \quad S(s) = \left\{ \sum_{p \in P} s(p) l(p) : \forall p \in P, \ l(p) \in \overline{co} S(p) \right\},
\]

where \(s(p) \in [0, 1]\) denotes the probability of \(p\) under splitting \(s\), and \(\overline{co} S(p)\) is the closure of the convex hull of \(S(p)\). Define next the iterates of \(S\) as follows: \(S^{0}(s) = \{s\}\), and for all \(n \geq 1\), \(S^{n+1}(s) = \{s' \in S^{n}(s') : s' \in S(s)\}\). Finally, let \(S^{\infty}(s)\) be the closure of \(\bigcup_{n=0}^{\infty} S^{n}(s)\). Since \(P\) and \(Q\) are finite, \(S\) and \(T\) are nonexpansive. According to Theorem 3.1 of Laraki and Renault (2020), there exists a unique function \(v : P \times Q \to \mathbb{R}\) that is \(S\)-concave, \(T\)-convex and such that for every \((p, q)\) in \(P \times Q\),

\((C1')\) If \(v(p, q) > u(p, q)\), there exists \(s\) in \(S^{\infty}(\delta_{p})\) such that \(v(p, q) = v(s, q) \leq u(s, q)\), and

\((C2')\) If \(v(p, q) < u(p, q)\), there exists \(t\) in \(T^{\infty}(\delta_{q})\) such that \(v(p, q) = v(p, t) \geq u(p, t)\).

We now prove that this function \(v\) satisfies \((C1')\). Consider \((p, q)\) such that \(v_{-}(p, q) < v(p, q)\) and choose \(\varepsilon > 0\) such that \(v_{-}(p, q) \leq v(p, q) - \varepsilon\). From the definition of \(v_{-}\), for any \(s\) in \(S(p)\) there exists \(s' \in \Delta(P)\) such that \(s = s(p) \delta_{p} + (1 - s(p)) s'\), \(p \notin \text{supp}(s')\) and \(v(s', q) \leq v(p, q) - \varepsilon\). It follows that

\[
v(s, q) = s(p) v(p, q) + (1 - s(p)) v(s', q) \leq v(p, q) - \varepsilon(1 - s(p)).
\]

The inequality \(v(s, q) \leq v(p, q) - \varepsilon(1 - s(p))\) then directly extends to any \(s\) in \(\overline{co} S(p) = S(\delta_{p})\). Additionally, for each \(s\) in \(S(\delta_{p})\) there exists \(s' \in \Delta(P)\) such that \(s = s(p) \delta_{p} + (1 - s(p)) s'\), \(p \notin \text{supp}(s')\) and \(v(s', q) \leq v(p, q) - \varepsilon\).

Assume now by induction that for some \(n \geq 2\), \(v(s, q) \leq v(p, q) - \varepsilon(1 - s(p))\) holds for each \(s \in S^{n-1}(\delta_{p})\) and consider a given \(s_{n}\) in \(S^{n}(\delta_{p})\). There exists \(s_{1}\) in \(S(\delta_{p})\) such that \(s_{n} \in S^{n-1}(s_{1})\), and \(s_{1}\) can be written \(s_{1} = s_{1}(p) \delta_{p} + (1 - s_{1}(p)) s'_{1}\), with \(p \notin \text{supp}(s'_{1})\) and \(v(s'_{1}, q) \leq v(p, q) - \varepsilon\). We can write \(s_{n} = s_{1}(p) s_{n-1} + (1 - s_{1}(p)) s'_{n}\), with \(s_{n-1} \in S^{n-1}(\delta_{p})\) and \(s'_{n} \in S^{n-1}(s'_{1})\). From the induction
hypothesis, \(v(s_{n-1}, q) \leq v(p, q) - \varepsilon + \varepsilon s_{n-1}(p)\). \(S\)-concavity implies \(v(\zeta', q) \leq v(\zeta, q)\) for any \(\zeta, \zeta'\) with \(\zeta' \in S(\zeta)\), so \(v(s'_n, q) \leq v(s'_{n}, q) \leq v(p, q) - \varepsilon\). This gives the following:

\[
v(s_n, q) \leq v(p, q) - \varepsilon + \varepsilon s_1(p)s_{n-1}(p) \leq v(p, q) - \varepsilon + \varepsilon s_n(p).
\]

By induction, we obtain the following:

\[
\forall s \in S^\infty(\delta_p), \ v(s, q) \leq v(p, q) - \varepsilon(1 - s(p)).
\]

Finally, assume that \(v(p, q) > u(p, q)\). By (C1”), there exists \(s \in S^\infty(\delta_p)\) such that \(v(p, q) = v(s, q) \leq u(s, q)\). We deduce that \(s \neq \delta_p\) and \(v(p, q) = v(s, q) \leq v(p, q) - \varepsilon(1 - s(p))\), a contradiction. Hence, \(v(p, q) \leq u(p, q)\), and we have proven that (C1’) holds. By symmetry, (C2’) also holds.

**Lemma 14.** \(G_\infty(p^0, q^0)\) has a value, and it is given by \(v(p^0, q^0)\).

**Proof.** We fix \(\varepsilon > 0\) and define a strategy \(\sigma = (\sigma^n)_{n \geq 1}\) for Designer 1 by a main phase and transitions phases. At each stage \(n\), the strategy depends on the history of beliefs \((p^0, q^0, \ldots, p^{n-1}, q^{n-1})\) and on the current phase. The play is initially in the main phase.

- At any stage \(n \geq 1\) in the main phase,
  - if \(u(p^{n-1}, q^{n-1}) \geq v(p^{n-1}, q^{n-1})\), play \(\sigma_n = \delta_{p^{n-1}}\) and remain in the main phase at stage \(n + 1\), and
  - if \(u(p^{n-1}, q^{n-1}) < v(p^{n-1}, q^{n-1})\), according to (C1’) there exist \(s_n \in S(p^{n-1})\), \(s'_n \in \Delta(P\setminus\{p^{n-1}\})\) and \(\alpha_n \in [0, 1]\) such that \(s_n = \alpha_n\delta_{p^{n-1}} + (1 - \alpha_n)s'_n\) and \(v(s'_n, q^{n-1}) \geq v(p^{n-1}, q^{n-1}) - \frac{\varepsilon}{2}\).

The strategy enters a transition phase where \(s_n\) is played at each stage \(m \geq n\) as long as \(p^{n-1} - p^n = \cdots = p^{n-1}\). At the first stage \(m > n\) where \(p^{m-1} \neq p^{n-1}\), \(\sigma\) returns to the main phase at stage \(m\).

Consider any strategy \(\tau\) of Designer 2, and let \((p^n, q^n)_{n \geq 0}\) be the induced martingale and \((p^\infty, q^\infty)\) be its almost sure limit. Since \(P\) and \(Q\) are finite, there almost surely exists a stage \(n_0\) such that for each \(n \geq n_0\), \((p^n, q^n) = (p^\infty, q^\infty)\). Additionally, since a transition phase starting at \((p^n, q^n)\) ends up almost surely at some posterior \(p' \neq p_n\), it must be the case that

\[
u(p^\infty, q^\infty) \geq v(p^\infty, q^\infty)\text{ almost surely.}
\]

Next, we define sequences of stopping times \((l_i)_{i \geq 1}, (m_i)_{i \geq 1}\) with values in \(\{1, 2, \ldots, \infty\}\) as follows:

- Let \(l_1\) be the first stage \(n \geq 1\) where Designer 1 is in the main phase and enters the first transition phase, and let \(m_1 \geq l_1\) be the last stage of the first transition phase. For \(i \geq 2\), let \(l_i > m_{i-1}\) be the first stage of the \(i\)-th transition phase and \(m_i \geq l_i\) be the last stage of the \(i\)-th transition phase.

Fix \(i \geq 1\). We have \(u(p^{l_i-1}, q^{l_i-1}) < v(p^{l_i-1}, q^{l_i-1})\), and at all stages \(n = l_i, \ldots, m_i\) Designer 1 plays \(s_{l_i} = \alpha_l \delta_{p^{l_i-1}} + (1 - \alpha_l)s'_{l_i}\), with \(s'_{l_i} \in \Delta(P\setminus\{p^{l_i-1}\})\), \(\alpha_l \in [0, 1]\) and \(v(s'_{l_i}, q^{l_i-1}) \geq v(p^{l_i-1}, q^{l_i-1}) - \frac{\varepsilon}{2l_i}\).

At stages \(n = m_i + 1, \ldots, l_{i+1}\), the play is in the main phase and \(p^{l_i+1-1} = p^{m_i}\). We have the following:

\[
\mathbb{E}[v(p^{l_{i+1-1}}, q^{l_{i+1-1}}) | h^{l_i}] = \mathbb{E}[v(p^{m_i}, q^{l_{i+1-1}}) | h^{l_i}] \geq \mathbb{E}[v(p^{m_i}, q^{l_i-1}) | h^{l_i}] \geq v(p^{l_i-1}, q^{l_i-1}) - \frac{\varepsilon}{2l_i},
\]
where the first inequality uses the $T$-convexity of $v$. As a consequence,

$$
\mathbb{E}(v(p^{l+1}, q^{l+1})) \geq v(p^0, q^0) - \varepsilon \sum_{j=1}^i \frac{1}{2^j},
$$

and thus $\mathbb{E}(v(p^\infty, q^\infty)) \geq v(p^0, q^0) - \varepsilon$. We obtain $\mathbb{E}(u(p^\infty, q^\infty)) \geq v(p_1, q_1) - \varepsilon$; thus, Designer 1 guarantees the payoff $v(p_0, q_0)$ in $G_\infty(p^0, q^0)$ up to any $\varepsilon > 0$. By symmetry, this is also true for Designer 2, and $v(p^0, q^0)$ is the value of the game. The proof is identical for $G_\infty^{seq}(p^0, q^0)$.

As an illustration, consider the following example with no 0-optimal strategy. Let $K = \{0, 1\}$, $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, $S(p) = \{\delta_p\}$ for $p \in \{0, \frac{1}{4}, \frac{3}{4}, 1\}$, and

$$
S\left(\frac{1}{2}\right) = \{(1 - 2\varepsilon - 2\varepsilon^2)\delta_1 + \varepsilon\delta_{\frac{1}{4}} + \varepsilon^2\delta_0 + \varepsilon^2\delta_1 : \varepsilon \in \left[0, \frac{1}{4}\right]\}. 
$$

In other words, at all posteriors but $\frac{1}{2}$, Designer 1 cannot split the belief. At $\frac{1}{2}$, Designer 1 can split it, but the posterior is likely to remain at $\frac{1}{2}$ and is much more likely to be $\frac{1}{4}$ or $\frac{3}{4}$ than 0 or 1. Suppose that for each $q$ in $Q$, $u(0, q) = u(1/2, q) = u(1, q) = 0$ and $u(1/4, q) = u(3/4, q) = 1$. Then, value $v$ satisfies $v(0, q) = v(1, q) = 0$ and $v(\frac{1}{4}, q) = v(\frac{3}{4}, q) = v(\frac{1}{2}, q) = v(\frac{1}{2}, q) = 1$. It is $S$-concave, and we have $v(\frac{1}{2}, q) = v_+(\frac{1}{2}, q)$. The point is that the splitting $\frac{1}{2}\delta_{\frac{1}{2}} + \frac{1}{4}\delta_{\frac{3}{4}}$ is infeasible at $\frac{1}{2}$ but can be approximately achieved with many stages by choosing a very small $\varepsilon$.

### B.3 Extension of the Illustrative Example of Section 3.4.1

We extend the example of Section 3.4.1 to the case of all splittings being admissible for the designers. To this end, we define designers’ payoffs for all possible posteriors rather than only for posteriors in $\{0, \frac{1}{4}, 1\}$. Consider the following payoff function, where $p' = 1 - p$ and $q' = 1 - q$:

$$
\begin{align*}
u(p, q) = & \begin{cases} 
2q - 2p' & \text{if } p > \frac{1}{2} \\
2q' - 2p & \text{if } p < \frac{1}{2} \\
2p - 2q & \text{if } p = \frac{1}{2} \\
2p' - 2q & \text{if } q = \frac{1}{2} \\
2q - 2p & \text{if } q = \frac{1}{2} \\
2p - 2q & \text{if } p = \frac{1}{2} \\
2q' - 2p & \text{if } q > \frac{1}{2} \\
2p' - 2q' & \text{if } q < \frac{1}{2} 
\end{cases} 
\end{align*}
$$

This function is the same as in example VI.7.4 in Mertens et al. (2015, page 375). Mertens et al. (2015) have shown that the MZ value function is given by Figure 4. The blue lines represent the set of values $(p, q)$ for which $MZ(u)(p, q) = u(p, q)$ (the equations of these curves in different areas are given in blue); in the red rectangle, we have $MZ(u)(p, q) = \frac{1}{4}$; black arrows represent increasing
linear functions (corresponding to cases in which optimal strategies are such that only Designer 1 is revealing information), and black lines represent constant values (corresponding to cases in which optimal strategies are such that only Designer 2 is revealing information). One can easily verify that this function satisfies conditions (C1) and (C2) of Proposition 1. Value functions of sequential games with a deadline are shown in Figures 5 and 6.

\[
MZ(u)(p, q) = \frac{1}{2}
\]

Figure 4: MZ value function in the extended example of Section 3.4.1.
\( \text{vex}_q \text{ cav}_p u(p, q) = \)

Figure 5: Value function of the sequential game with a deadline for \( N = 3, 5, 7, \ldots \)

\( \text{cav}_p \text{ vex}_q u(p, q) = \)

Figure 6: Value function of the sequential game with a deadline for \( N = 2, 4, 6, \ldots \)
The MZ value of this continuous game at \(\left(\frac{1}{2}, \frac{1}{2}\right)\) is \(MZ(u)(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}\), while it is equal to 0 if the set of admissible posteriors is \(\{0, \frac{1}{2}, 1\}\). The optimal strategy of the 1-period simultaneous-move game with admissible posteriors \(\{0, \frac{1}{2}, 1\}\) remains optimal in the extended game when the prior is \(\left(\frac{1}{2}, \frac{1}{2}\right)\).

Indeed, an optimal strategy of Designer 1 (and of Designer 2) is to play nonrevealingly with probability \(\frac{1}{2}\) and to fully reveal information with probability \(\frac{1}{2}\). Hence, the one-shot splitting value of \(u\) at \(\left(\frac{1}{2}, \frac{1}{2}\right)\) is \(SV(u)(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}\). We can also verify that \(\text{cav}_p \text{vex}_q u(\frac{1}{2}, \frac{1}{2}) = 0\) and \(\text{vex}_q \text{cav}_p u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}\). Summarizing, in this example we have the following:

\[
\text{cav}_p \text{vex}_q u\left(\frac{1}{2}, \frac{1}{2}\right) = 0 < MZ(u)\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} < \text{vex}_q \text{cav}_p u\left(\frac{1}{2}, \frac{1}{2}\right) = SV(u)\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}.
\]

This example shows that even in the standard case of \(P = \Delta(K)\), \(Q = \Delta(L)\) and all splittings being admissible, one may have \(SV(u) \neq MZ(u)\). To our knowledge, no such example can be found in the literature on splitting games.

### B.4 Example of a non-expansive splitting correspondence generated by a set of experiments

This example is inspired by Example 3.13 in Renault and Venel (2017). Let \(K = \{a, b, c, d\}\) and consider a single deterministic experiment \(x : K \rightarrow \{m_0, m_1\}\) defined by:

\[x(a) = x(c) = m_0, x(b) = x(d) = m_1\]

That is, \(x\) reveals whether the state is in \(\{a, c\}\) or \(\{b, d\}\). Denote by \(\varphi(p, x) \in \Delta(\Delta(K))\) the splitting of prior \(p\) induced by \(x\).

- For \(p' = (1/4, 1/2, 1/4, 0)\), \(\varphi(p, x) \in \Delta(\Delta(K))\) puts probability \(\frac{1}{2}\) on the posterior \((1/2, 0, 1/2, 0)\) and probability \(\frac{1}{2}\) on the posterior \((0, 1, 0, 0)\).

- For \(p'' = (1/4, 1/2, 0, 1/4)\), \(\varphi(p', x) \in \Delta(\Delta(K))\) puts probability \(\frac{1}{4}\) on the posterior \((1, 0, 0, 0)\) and probability \(\frac{3}{4}\) on the posterior \((0, 2/3, 0, 1/3)\).

Notice that \(\|p' - p''\|_1 = 1/2\). Consider now \(p_1 = (1, 0, 0, 0)\), \(p_2 = (0, 2/3, 0, 1/3)\), \(p_3 = (0, 1, 0, 0)\), \(p_4 = (1/2, 0, 1/2, 0)\) and denote \(S = \{p_1, p_2, p_3, p_4\}\). Define a 1-Lipschitz function \(f : S \rightarrow \mathbb{R}\) by such that \(f(p_1) = 2/3\), \(f(p_2) = 1\), \(f(p_3) = 1/3\) and \(f(p_4) = -1/3\). The function \(f\) can be extended to a 1-Lipschitz function from \(\Delta(K)\) to \(\mathbb{R}\) by the formula:

\[
\forall q \in \Delta(K), f(q) = \sup_{\hat{p} \in S} (f(\hat{p}) - \|q - \hat{p}\|_1).
\]

We have \(f(\varphi(p')) = \frac{1}{2} 1/3 + \frac{1}{2}(-1/3) = 0\) and \(f(\varphi(p'')) = \frac{1}{4} 2/3 + \frac{3}{4}1 = 11/12\), so

\[
|f(\varphi(p')) - f(\varphi(p''))| = 11/12 > 1/2 = \|p' - p''\|_1.
\]
The non expansivity condition of Laraki and Renault (2020) requires that

\[ |f(\varphi(p')) - f(\varphi(p''))| \leq \|p' - p''\|_1, \forall f \text{ 1-Lipschitz} \]

and is therefore not satisfied here.