Income Anonymity*

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Abstract

This paper characterizes a normative criterion for ranking income distributions based on two axioms: Pareto and Income Anonymity. Pareto requires that, if everyone supports a simultaneous change in the distribution of income and in prices, then that change is socially desirable. Income Anonymity requires that, whenever everyone faces the same prices, social welfare can be evaluated based on the anonymized distribution of income. When individuals have heterogeneous preferences, there exists at most one social preference relation that satisfies both axioms. Given current expenditure patterns in the United States, this social welfare function ranks income distributions approximately according to the sum of log incomes.

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1 Introduction

Economic analysis often aims to identify the optimal policy in various contexts, such as determining the best trade agreements or designing a fair taxation system. The success of this analysis depends on a clear understanding of the social objective. The overarching goal of welfare economics in general, and this paper in particular, is to narrow down the set of plausible social welfare criteria based on ethical principles.

One such principle is that everyone should be treated equally. In practice, applied welfare analysis usually reflects this principle by assigning equal value to the income of each individual. We study this approach through a new axiom, Income Anonymity, which states that each person's income matters equally for social welfare. Although we are the first to formally study this axiom (to the best of our knowledge), it is implicit in any welfare criterion that is based on the statistical properties of the income distribution, such as the Gini index, GDP per-capita or the Atkinson Index (see, for example, Dollar et al. [2015] or Kraay et al. [2023]).

Our main result is that, for each heterogeneous preference profile, there exists at most one social preference relation that satisfies both Pareto and Income Anonymity. This means that when we agree on the applicability of these principles, then we must also agree on everything else, from optimal trade agreements to optimal redistributive policy.

The level of inequality aversion implied by this welfare criterion depends on the profile of individual preferences. For example, when preferences are homothetic, then the social preference ranks income distributions according to the sum of log incomes. When individual preferences are quasilinear, then the social ranking exhibits no inequality aversion, and ranks income distributions according to per-capita income.

For certain individual preference profiles, the two axioms are inconsistent. Consistency requires that individual preferences satisfy a certain separability condition. Loosely speaking, the condition requires that we can *imagine* that any two individuals who face the same budget have the same marginal utility of income. That is, there is nothing about the profile of ordinal preferences that can reject this hypothesis.

We calibrate our social welfare criterion based on the cross sectional distribution of consumption expenditures, using data from the United States Consumer Expenditure survey. We find that our social welfare criterion ranks income distributions approximately according to the sum of log incomes. This means that, if we accept Pareto and Income Anonymity, then, at current prices, policymakers should strive to maximize the geometric mean of the income distribution.

This paper contributes to a rich literature on the axiomatic characterization of social

preferences. The approach here is closely related to the money-metric utility approach (Deaton and Muellbauer [1980], Fleurbaey and Maniquet [2011], Fleurbaey and Maniquet [2018]). The money-metric utility approach proposes a social preference relation that is an aggregation of individuals' equivalent incomes at common reference prices ("money-metric utilities"). There are many social welfare functions that satisfy anonymity with respect to the distribution of money-metric utilities. The anonymity condition that we consider is more restrictive, as it requires anonymity with respect to the distribution of income at any prices – not just at the reference price. The social preferences must therefore be symmetric in money-metric utilities for any reference price.

This paper is also related to the literature on price-independent welfare prescriptions (Roberts [1980], Slesnick [1991], Blackorby et al. [1993] and Fleurbaey and Blanchet [2013]). Roberts [1980] studies the conditions under which income distributions can be ranked irrespective of prices, and finds that they are "highly restrictive".[1] This sparked a debate about whether it is appropriate to require welfare prescriptions to be independent of prevailing prices (Fleurbaey and Blanchet [2013]). The social preference relation that we derive generates price-independent welfare prescriptions only in special cases.

In addition, this paper is related to the literature on the inconsistency of Pareto and other normative principles. Sen [1970a] and Kaplow and Shavell [2001] show that the Pareto principle leaves limited room for expressing a concern for non-welfarist normative principles such as liberalism, justice or procedural fairness (see also Sher [2021]). In the context of resource allocation problems, Sen [1970b], Suzumura [1981a], Suzumura [1981b], Suzumura [1983], Tadenuma [2002] and Fleurbaey and Trannoy [2003] uncover tensions between Pareto and various egalitarian principles pertaining to the fair allocation of resources. Fleurbaey and Trannoy [2003] show that, whenever preferences are heterogeneous, Pareto is inconsistent with a social preference for redistributing resources from rich to poor (the Pigou-Dalton principle).[2] We add to these impossibility results by showing that, for some preference profiles, Pareto is also inconsistent with Income Anonymity. However, in light of the negative results in this literature, our positive results are perhaps more surprising: we show that, in some special cases, there is no conflict between Income Anonymity and Pareto, even when preferences are heterogeneous.

Finally, this paper is related to the literature on the axiomatic foundations of additively-separable utility functions (Gorman [1968], Wakker [1989], Blackorby et al. [1998] and Qin

[1] He finds that, unless the welfare criterion is dictatorial, both individual and social preferences must be homothetic.

[2] More precisely, the Pigou-Dalton principle states that if one person has more of every good than another person, then any transfer of goods that maintains this ordering but reduces inequality is a socially-desirable transfer.

and Rommeswinkel [2022]). Here, we establish that if the social preference relation satisfies Income Anonymity and Pareto (and individual preferences are heterogeneous), then it must be additively separable in individual budgets. Previous results in this literature obtain additively-separable representations by assuming that it is possible to rank subgroups of variables independently from one another. In our analysis, this is embedded in the Pareto condition, which requires that each individual's consumption bundles can be ranked independently of other individuals' consumption bundles.

2 Preliminaries

There are $2 \leq I < \infty$ individuals indexed $i = 1, ..., I$, and $2 \leq J < \infty$ goods indexed $j = 1, ..., J$. Throughout, we use subscripts for indicating individuals and superscripts for indicating goods; for example, c_i^j is individual i's consumption of good j.

Individuals' preferences over goods are strictly quasi-concave and represented by utility functions, $\{u_i\}_{i=1}^I$.[3] Define the indirect utility function, v_i, as

$$v_i(m, p) = \max_{c^1,...,c^J} u(c^1, ..., c^J) \text{ s.t. } \sum_{j=1}^{J} p^j c^j \leq m \tag{1}$$

Here, $m \in \mathbb{R}_{++}$ is income and $p = (p^1, ..., p^J) \in \mathbb{R}_{++}^J$ is a vector of prices. We assume that the indirect utility functions, $\{v_i\}$, are continuously differentiable and strictly increasing in m.[4] The ranking $\preceq_i$ on $\mathbb{R}_{++} \times \mathbb{R}_{++}^J$ denotes the individual's indirect ranking of combinations of income and prices (with the corresponding relations $\succeq_i$, $\sim_i$, $\prec_i$, $\succ_i$). Note that each $(m, p) \in \mathbb{R}_{++} \times \mathbb{R}_{++}^J$ represents a budget constraint; thus, $\preceq_i$ represents the individual's preferences over budget sets.

Throughout, we use bold letters to denote arrays of length I (the number of individuals). The social preference ranking, $\preceq$ (with no subscript), is defined over elements of the form $(\mathbf{m}, \mathbf{p})$, where $\mathbf{m} = (m_1, ..., m_I) \in \mathbb{R}_{++}^I$ is the distribution of income and $\mathbf{p} = (p_1, ..., p_I) \in \left(\mathbb{R}_{++}^J\right)^I$ are individual price vectors (individual i faces the prices $p_i = (p_i^1, ..., p_i^J)$). The symbols $\succeq$, $\sim$, $\succ$ and $\prec$ (with no subscripts) are used to describe the corresponding social preference relations. For a price vector p, the notation $(\mathbf{m}, p)$ is used as a shorthand for the allocation $(\mathbf{m}, (p, ..., p))$, in which everyone faces the same prices.

[3]The assumption of strict quasiconcavity can be relaxed for the purpose of obtaining our uniqueness result. It is used only for deriving the formula for the social welfare function, which uses the assumption that each individual's consumption bundle is the unique solution to his constrained optimization problem.

[4]Note that, as indirect preferences are always homogeneous of degree 0 and strictly increasing in m, it follows that they must be locally strictly decreasing in at least one price.

2.1 Axioms

We consider two axioms on the social preference relation. The first is the standard Pareto condition, which is stated as follows.

Axiom (Pareto)**.** *For each* $\mathbf{m}, \mathbf{m}', \mathbf{p}, \mathbf{p}'$*,*

1. *If* $(m_i, p_i) \preceq_i (m_i', p_i')$ *for all* i*, then* $(\mathbf{m}, \mathbf{p}) \preceq (\mathbf{m}', \mathbf{p}')$*; and*
2. *If, in addition,* $(m_i, p_i) \prec_i (m_i', p_i')$ *for some* i*, then* $(\mathbf{m}, \mathbf{p}) \prec (\mathbf{m}', \mathbf{p}')$*.*

This axiom is sometimes referred to as "unanimity". It states that, if all individuals support a certain change in prices and incomes, then it should be considered socially desirable.

The second axiom is Income Anonymity:

Axiom (Income Anonymity)**.** *For every common price vector,* $p \in \mathbb{R}^J_{++}$*, and income distribution,* $\mathbf{m} \in \mathbb{R}^I_{++}$*, we have*

$$(\mathbf{m}, p) \sim ((m_{\sigma(1)}, ..., m_{\sigma(I)}), p)$$

for any permutation $\sigma : \{1, ..., I\} \mapsto \{1, ..., I\}$*.*

Income Anonymity states that the normative ranking of income distributions should not depend on which person receives which income. Instead, income distributions can be ranked anonymously. This anonymity condition is ubiquitous in applied welfare analysis. Below, we discuss two alternative justifications for it.

Informational constraints. According to some prominent welfare criteria, what matters is the distribution of utilities, not incomes. The problem is that we currently lack the tools to ascertain the individual mappings between consumption and cardinal utilities. Therefore, for practical reasons, we assume that any two people who face the same budget constraint are equally "productive" in converting consumption into utility; or, at least, that there is no good reason to think that one person is better at producing utils than any other. This type of assumption is captured by Income Anonymity.

The normative argument. Alternatively, this anonymity condition reflects the normative argument in Dworkin [1981].[5] Dworkin [1981] argues that fairness should be judged based on the distribution of income, regardless of individuals' preferences, and regardless of

[5] Varian [1976] also considers the question of how to define a fair allocation when people have heterogeneous preferences. He discusses the sense in which an equal distribution of income results in a better market allocation than an unequal distribution of income.

prevailing prices.[6] According to Dworkin, what matters is the value of the resources devoted to each person's life – and not how that individual chooses to use those resources, or the utility that he derives from their use.

Dworkin writes, "the true measure of the social resources devoted to the life of one person is fixed by asking how important, in fact, that resource is for others." This implies a measure of value that is based on common equilibrium prices. To reflect this, Income Anonymity requires anonymity with respect to income only when all people face the same price vectors. In this case, the value of each person's consumption bundle is given by that person's income.

Like all axioms, the normative appeal of our axioms is not universal. It is possible to come up with examples in which each of our axioms contradicts basic moral intuitions. The Pareto condition is unappealing in circumstances involving addiction, as people's choices go against their own best interests. Similarly, Income Anonymity is problematic in situations in which people have disabilities or very different needs (see Sen [1980]). Nonetheless, there are many situations in which these two moral principles seem like reasonable starting points.

3 Uniqueness

On their own, both Pareto and Income Anonymity are consistent with various degrees of inequality aversion. For example, the Pareto condition is consistent with any social welfare function of the form

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi_i(v_i(m_i, p_i))$$

where ϕ_i is strictly increasing. In this class of social welfare functions, the concavities of the functions $\{\phi_i\}_{i=1}^{I}$ determine aversion to inequalities in utilities, and hence, indirectly, aversion to income inequality.

Similarly, Income Anonymity is consistent with any social welfare function of the form

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} f(m_i, p_i)$$

In this class of social welfare functions, Income Anonymity is satisfied because, when everyone faces the same prices, then the social ranking is symmetric with respect to all incomes.

[6] See Keller [2002] for a discussion. A similar notion of equality is also reflected in the Laisser-Faire axiom in Fleurbaey and Maniquet [2006], which postulates that there is no scope for redistribution between two people who face the same earning opportunities, even when they choose to work different amounts.

In addition, if all individuals have the same preferences, then there are many social preference relations that are consistent with both axioms. For example, when $v_i = v$ for all i, then both axioms are satisfied by any social preference relation that is represented by

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi(v(m_i, p_i))$$

where ϕ is any strictly increasing function. Here, the concavity of ϕ determines the amount of inequality aversion.

The following theorem establishes that, when preferences are heterogeneous, the combination of Pareto and Income Anonymity uniquely characterizes the entire social preference relation. In particular, at most one level of inequality aversion is consistent with both axioms.

To state the theorem, it is necessary to introduce the following notation. Let $c_i^j(m, p)$ denote individual i's consumption of good j, given the budget (m, p):[7]

$$c_i(m, p) = (c_i^1(m, p), ..., c_i^J(m, p)) = \arg\max_{c^1,...,c^J} u_i(c^1, ..., c^J) \text{ s.t. } \sum_{j=1}^{J} p^j c^j \leq m$$

and let $e_i(m, p, p')$ denote the solution to the indifference condition,

$$(m, p) \sim_i (e_i(m, p, p'), p')$$

The quantity $e_i(m, p, p')$ is individual i's equivalent income at prices p', given a budget (m, p).[8]

We are now ready to state our theorem.

Theorem 1. *Assume that $c_1^j(m_0, p_0) > c_2^j(m_0, p_0)$. If there exists a social preference relation that satisfies Pareto and Income Anonymity, then it is unique and represented by*

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' \tag{2}$$

This theorem establishes that, when preferences are heterogeneous, then there is at most

[7] Note that, given the assumption that u_i is strictly concave, $c_i(m, p)$ is uniquely defined.

[8] To see that $e_i(m, p, p')$ is well-defined, note that, for $m' > 0$ sufficiently large, the consumption bundle $c_i(m, p)$ is affordable given the budget (m', p'); it follows that, in this case, $(m, p) \preceq_i (m', p')$. In addition, for $m' > 0$ sufficiently small, it holds that $(m/J)/p^j > m'/p'^j$: that is, with the budget $(m/J, p)$, the individual can buy the entire amount of each good that is affordable by the budget (m', p'). Because $c_i^j(m', p') \leq m'/p'^j$ for every j, it follows that $u_i(c_i(m', p')) \leq u_i(m'/p'^1, ..., m'/p'^J) \leq u_i((m/J)/p^1, ..., (m/J)/p^J) \leq u_i(c_i(m, p))$. Hence, for this m', it holds that $(m', p') \preceq_i (m, p)$. By the continuity of individual preferences, it follows that there exists some $m' = e_i(m, p, p')$ such that $(m', p') \sim (m, p)$. Because indirect preferences are strictly increasing income, m' is unique and hence $e_i(m, p, p')$ is uniquely defined.

one social preference relation that satisfies both of our axioms. When it exists, it is represented by (2); it particular, it must be (a) continuous and (b) additively separable in the individual budgets, $(m_1, p_1), ..., (m_I, p_I)$. These are desirable properties which we did not assume, but rather obtained as an implication of combining Pareto and Income Anonymity.

In addition, the theorem provides a formula for computing the social preference relation based on the distribution of ordinal preferences. As we illustrate in section 3.2, this formula can be used to obtain analytical characterizations of social welfare functions in some important special cases. In section 5, we show how this formula can be used for estimating a social welfare function based on consumer expenditure data.

The complete proof of the theorem is in the appendix, together with other omitted proofs. Below we sketch its key steps, under the simplifying assumption that individuals 1 and 2 are the only two individuals.[9]

Observe that, given the Pareto condition, it must always hold that

$$((m_1, m_2), (p_1, p_2)) \sim ((e_1(m_1, p_1, p_0), e_2(m_2, p_2, p_0)), p_0)$$

That is, any arbitrary allocation of incomes and prices is always Pareto-equivalent to an income distribution in which everyone faces the prices p_0. It is therefore sufficient to obtain a unique characterization of the partial social ranking over elements of the form $(\mathbf{m}, p_0)$. Note that, as $e_i(m_i, p_0, p_0) = m_i$, the social welfare function in (2) implies the restricted ranking

$$W(\mathbf{m}, p_0) = \sum_{i=1}^{I} \int_1^{m_i} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' \tag{3}$$

It is thus sufficient to show that, given our axioms, this social welfare function must represent the partial social ranking of elements of the form $(\mathbf{m}, p_0)$.

The proof proceeds using the construction in Figure 1. Figure 1 presents the indifference curves of individuals 1 and 2 over combinations of income and the price of good j, holding the prices of all other goods fixed at $(p_0^1, ...p_0^{j-1}, p_0^{j+1}, ..., p_0^J)$. In this space, the individuals' indifference curves are upward sloping, because a higher price can be compensated with a higher income.

Consider an initial allocation in which person 1 faces the budget (m_1, p_0), and person 2 faces the budget (m_2, p_0). Construct an alternative price vector, p, for which $p^j = p_0^j + 1$ and $p^k = p_0^k$ for all $k \neq j$. It will be useful to think of "1" as a very small unit, so that indifference curves are approximately linear between p_0 and p, and parallel in small neighborhoods around

[9]This construction easily extends to an environment with multiple individuals, but notation becomes more cumbersome.

m_1 and m_2. Define m_1' and m_2' based on the indifference conditions,

$$(m_1, p_0) \sim_1 (m_1', p) \text{ and } (m_2, p_0) \sim_2 (m_2', p)$$

Pareto requires that, when all individuals are indifferent, then the social preference relation is indifferent as well. Consequently, Pareto implies

$$((m_1, m_2), p_0) \sim ((m_1', m_2'), p) \tag{4}$$

Income Anonymity requires that, at any given prices, income distributions can be ranked anonymously. In particular, social preferences must be indifferent with respect to switching the incomes of the two individuals at the prices p:

$$((m_1', m_2'), p) \sim ((m_2', m_1'), p) \tag{5}$$

Next, define m_1'' and m_2'' based on the indifference conditions

$$(m_2', p) \sim_1 (m_2'', p_0) \text{ and } (m_1', p) \sim_2 (m_1'', p_0)$$

Again, using the Pareto indifference condition, it follows that

$$((m_2', m_1'), p) \sim ((m_2'', m_1''), p_0) \tag{6}$$

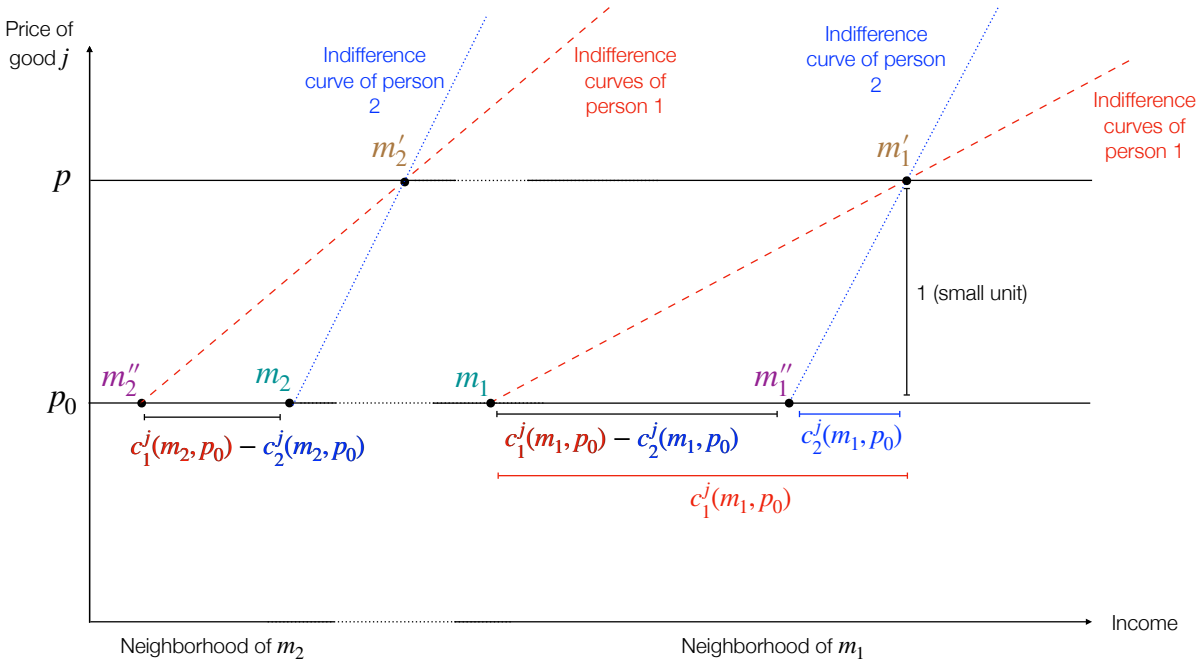
Finally, by the transitivity of the indifference relation, (4), (5) and (6) imply

$$((m_1, m_2), p_0) \sim ((m_1', m_2'), p) \sim ((m_2', m_1'), p) \sim ((m_2'', m_1''), p_0) \tag{7}$$

Consequently, given a price level of p_0, any social preferences that satisfy Pareto and Income Anonymity must be indifferent between an income distribution of (m_1, m_2) and an income distribution of (m_2'', m_1''). Note that the income distribution (m_2'', m_1'') is more dispersed than the income distribution (m_1, m_2). A sufficiently inequality-averse social preference relation would strictly prefer the less-dispersed allocation, (m_1, m_2), while a sufficiently inequality-tolerant relation would prefer the more dispersed allocation, (m_2'', m_1''). Neither of these preferences would be consistent with the combination of Pareto and Income Anonymity.

Intuitively, the uniqueness of the social preference relation follows from the uniqueness of the level of inequality aversion. Any social preference relation that satisfies our axiom must be indifferent with respect to simultaneously increasing the top income by $(m_1'' - m_1)$ and decreasing the bottom income by $(m_2 - m_2'')$. This implies that the marginal welfare gain from increasing the top income by $(m_1'' - m_1)$ must be exactly offset by the marginal welfare loss from decreasing the bottom income by $(m_2 - m_2'')$. This pins down the social marginal

Figure 1: Sketch of the proof of Theorem 1



rate of substitution between the top income and the bottom income, $SMRS(m_1, m_2)$:

$$SMRS(m_1, m_2) = \frac{m_1'' - m_1}{m_2 - m_2''}$$

Since (m_1, m_2) was chosen arbitrarily, a similar construction pins down all of the social ranking's marginal rates of substitution. Since a preference ranking is fully characterized by its marginal rates of substitution, this construction pins down a unique social ranking.

Next, we explain briefly why this social ranking must be represented by (2). In Figure 1, it is apparent that individual 1 cares more about the price of good j, because he demands a higher monetary compensation for the increase in its price. Intuitively, the extent to which an individual cares about a good's price depends on the quantity of the good that he consumes. It turns out (using Roy's identity) that the amount of compensation required is approximately equal to the individual's consumption of good j. Using this approximation, we obtain

$$m_1'' - m_1 = c_1^j(m_1, p_0) - c_2^j(m_1, p_0) \text{ and } m_2 - m_2'' = c_1^j(m_2, p_0) - c_2^j(m_2, p_0)$$

Hence, the social marginal rates of substitution are given by

$$SMRS(m_1, m_2) = \frac{m_1'' - m_1}{m_2 - m_2''} = \frac{c_1^j(m_1, p_0) - c_2^j(m_1, p_0)}{c_1^j(m_2, p_0) - c_2^j(m_2, p_0)}$$

Using expression (3), it is possible to show that these are the marginal rates of substitution implied by the social ranking represented by W:

$$\frac{\partial W((m_1, m_2), p_0)}{\partial m_1} = \frac{1}{c_1^j(m_1, p_0) - c_2^j(m_1, p_0)}$$

$$\frac{\partial W((m_1, m_2), p_0)}{\partial m_2} = \frac{1}{c_1^j(m_2, p_0) - c_2^j(m_2, p_0)}$$

$$\Rightarrow \frac{\frac{\partial W((m_1,m_2),p_0)}{\partial m_2}}{\frac{\partial W((m_1,m_2),p_0)}{\partial m_1}} = \frac{c_1^j(m_1, p_0) - c_2^j(m_1, p_0)}{c_1^j(m_2, p_0) - c_2^j(m_2, p_0)}$$

It thus follows that W represents the unique social preference ranking that satisfies our axioms.

3.1 Restrictions on the social preference relation

Generally, the functional form of the social welfare function depends on the profile of individual preferences. However, Theorem 1 is sufficient for ruling out certain social welfare functions regardless of the preference profile. For example, because any social preferences represented by (2) are continuous, we can rule out the Rawlsian social preference that cares lexicographically about the income of the worst-off individual. Similarly, as (2) is additively separable in individual budgets, we can rule out social rankings that are not separable in individual incomes, such as the Gini Index.

In addition, it turns out that our axioms place some restrictions on the possible levels of inequality aversion in the Atkinson index. The Atkinson index (Atkinson [1970]) is a ranking of income distributions (conditional on prices) that is represented by

$$W^{Atkinson}(\mathbf{m}) = \frac{1}{1-\eta} \sum_{i=1}^{I} m_i^{1-\eta}$$

where η is a parameter that governs the degree of inequality aversion. A higher η corresponds to a social ranking that is more inequality averse.

The following corollary establishes that, when the social preference relation satisfies our axioms, then the ranking of income distributions (conditional on prices) cannot be represented by an Atkinson index with an inequality aversion parameter that is greater than 1.

Corollary 1. *Assume that (a) for some price vector, p, it holds that $c_1^j(1, p) > c_2^j(1, p)$; (b) the social preference relation satisfies Pareto and Income Anonymity; and (c) the social rank-*

ing of income distributions conditional on everyone facing prices p ($W(\cdot, p)$) is represented by an Atkinson index with parameter η. Then, $\eta \leq 1$.

The restriction on the degree of inequality aversion arises because, in the Atkinson Index, the degree of inequality aversion is assumed to be constant. However, our axioms do not place any general restrictions on the local degree of inequality aversion. Our axioms cannot rule out that, over the empirically-relevant range of incomes, the social welfare function is well-approximated by an Atkinson index with $\eta > 1$.

3.2 Examples

Theorem 1 provides a formula for computing a social welfare function that represents the unique social preference relation that satisfies our axioms. In this section, we use this formula to characterize the social ranking of income distributions for three classes of individual preferences: homothetic preferences, Stone-Geary preferences, and quasilinear preferences. Here, we focus on illustrating how to compute the social welfare function in (2), and studying its implications. In Appendix D we establish that, for these preference profiles, the two axioms are, in fact, consistent, and satisfied by the social preference relation (2).

In the following examples, we assume $I \geq 2$ individuals, and that, for some price vector $p = p_0$ and good j, it holds that $c_1^j(1, p_0) > c_2^j(1, p_0)$. Given this specification, the social preferences over income distributions conditional on prices p are represented by (3).

Consider the following special cases.

Homothetic preferences. When preferences are homothetic, each individuals' consumption bundle changes proportionately with his income: that is, for each individual, i, and good, j,

$$c_i^j(m, p) = m c_i^j(1, p)$$

Substituting into (3), we have that

$$\begin{aligned} W(\mathbf{m}, p) =& \sum_{i=1}^{I} \int_1^{m_i} \frac{1}{c_1^j(1,p)m' - c_2^j(1,p)m'} dm' \\ =& \frac{1}{c_1^j(1,p) - c_2^j(1,p)} \sum_{i=1}^{I} \int_1^{m_i} \frac{1}{m'} dm' \\ =& \frac{1}{c_1^j(1,p) - c_2^j(1,p)} \sum_{i=1}^{I} \ln(m_i) \end{aligned}$$

As $c_1^j(1,p) > c_2^j(1,p)$, this social preference relation ranks income distributions according to the sum of log incomes. Note that we did not assume anywhere that individual preferences are "log" - individuals may have arbitrary constant-elasticity-of-substitution preferences, or other homothetic preferences. Regardless, the social ranking will rank income distributions according to their geometric means.

Stone-Geary preferences. Consider a modification of the homothetic case, in which preferences are non-homothetic due to the presence of a common "subsistence bundle". To survive, each individual must consume at least $\underline{c}^j \geq 0$ goods of type j. However, after consuming the subsistence bundle, individuals' preferences over their remaining consumption are homothetic; formally, there exist utility functions, $\{H_i\}$, which are homogeneous of degree 1, such that individual preferences are represented by

$$u_i(c^1, ..., c^J) = H_i(c^1 - \underline{c}^1, ..., c^J - \underline{c}^J)$$

Following similar steps as in the homothetic case (replacing m with $m - \sum_{k=1}^J p^k \underline{c}^k$), we obtain

$$W(\mathbf{m}, p) = a \sum_{i=1}^I \ln \left(m_i - \sum_{k=1}^J p^k \underline{c}^k \right) + b$$

for some $a > 0$ and constant b. Note that, in this case, the social ranking of income distributions depends on the cost of the subsistence bundle. The social preference relation is more averse to income inequality when the subsistence bundle is more expensive.

Quasilinear preferences. Assume that utility functions are of the form

$$u_i(c^1, ..., c^J) = c^1 + g_i(c^2, ..., c^J)$$

where g_i is strictly concave. Note that the functions $\{g_i\}_{i=1}^I$ may be heterogeneous and take arbitrary functional forms.

It is straightforward to verify that, in this case, consumption patterns are as follows:

$$c_i^j(m,p) = c_i^j(1,p) \ \forall j > 1$$

$$c_i^1(m,p) = \frac{m - \sum_{j=2}^J c_i^j(1,p)}{p^1}$$

In particular, for each $j > 1$, it holds that, for every m',

$$c_1^j(m',p) - c_2^j(m',p) = c_1^j(1,p) - c_2^j(1,p)$$

Substituting into (3),

$$\begin{aligned} W(\mathbf{m}, p) =& \sum_{i=1}^{I} \int_{1}^{m_i} \frac{1}{c_1^j(1,p) - c_2^j(1,p)} dm' \\ =& \frac{1}{c_1^j(1,p) - c_2^j(1,p)} \sum_{i=1}^{I} \int_{1}^{m_i} dm' \\ =& \frac{1}{c_1^j(1,p) - c_2^j(1,p)} \sum_{i=1}^{I} (m_i - 1) \end{aligned}$$

This social welfare function ranks income distributions according to the sum of individual incomes – that, is, the simple measure of per-capita income. Unlike the previous examples, this preference ranking exhibits no aversion to income inequality.

4 Existence

Theorem 1 establishes that *if* there exists a social preference relation that is consistent with our axioms, then there is only one, and it is represented by (2). However, there is also a possibility that there are no social preference relations that satisfy both of our axioms.

The following theorem establishes two equivalent conditions on the profile of individual preferences which are necessary and sufficient for the consistency of our axioms.

Theorem 2. *The following conditions are equivalent.*

(a) There exists a social preference relation that satisfies Pareto and Income Anonymity.

(b) The following two conditions hold:

(i) For every i, i', j, p_0 and m_0,

$$c_i^j(m_0, p_0) > c_{i'}^j(m_0, p_0) \Rightarrow c_i^j(m, p_0) > c_{i'}^j(m, p_0) \ \forall m$$

(ii) If $c_1^j(m_0, p_0) > c_{i'}^j(m_0, p_0)$ for some m_0, p_0, j and i', then, for every m, i and p,

$$\int_{e_1(m,p,p_0)}^{e_i(m,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_{i'}^j(m', p_0)} dm' = \int_{e_1(1,p,p_0)}^{e_i(1,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_{i'}^j(m', p_0)} dm'$$

(c) There exists a function $\mu : \mathbb{R}_{++} \times \mathbb{R}_{++}^J \mapsto \mathbb{R}$ and functions $\{\gamma_i : \mathbb{R}_{++}^J \mapsto \mathbb{R}\}_{i=1}^I$ such that, for every i, the function $(m, p) \mapsto \mu(m, p) + \gamma_i(p)$ is a representation of the indirect preferences $\preceq_i$.

The equivalence between (a) and (b) is useful for verifying whether our two axioms are consistent given a certain profile of individual preferences. Note that both conditions (b.i) and (b.ii) do not depend on a particular representation of individual preferences. Thus, they can be easily verified based on the ordinal preference relations. In section 4.1, we provide graphical illustrations of what it means to violate conditions (b.i) and (b.ii), and explain why this implies the inconsistency of our axioms.

The equivalence between (a) and (c) is more useful for developing an economic intuition for the conditions that a preference profile must satisfy in order to avoid a conflict between our axioms. Condition (c) requires that the profile of individual preferences is consistent with a model in which individuals' objective is to maximize utilities of the form $\{\mu+\gamma_i\}_{i=1}^{I}$. Note that it does not require that this representation actually captures any meaningful notion of cardinal utility; rather, it states that the profile of ordinal preferences could have been generated by a model in which cardinal utilities take this form.

In this model, if two people face the same budget, (m, p), then they also have the same marginal utility of income, $\frac{\partial \mu(m,p)}{\partial m}$. When this is the case, the social preference ranking that we characterize based on Pareto and Income Anonymity is also the standard utilitarian criterion.

Corollary 2. *When there exists a social preference relation that satisfies Pareto and Income Anonymity, then it is represented by*

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \left(\mu(m_i, p_i) + \gamma_i(p_i)\right) \tag{8}$$

where, for each i, $\mu + \gamma_i$ is a representation of individual i's preferences.

This corollary is established as a step in the proof of Theorem 1.

4.1 What can go wrong

Clause (b) of Theorem 2 establishes two necessary conditions for the consistency of our axioms. In this section, we illustrate why our axioms are in conflict when each one of them is violated.

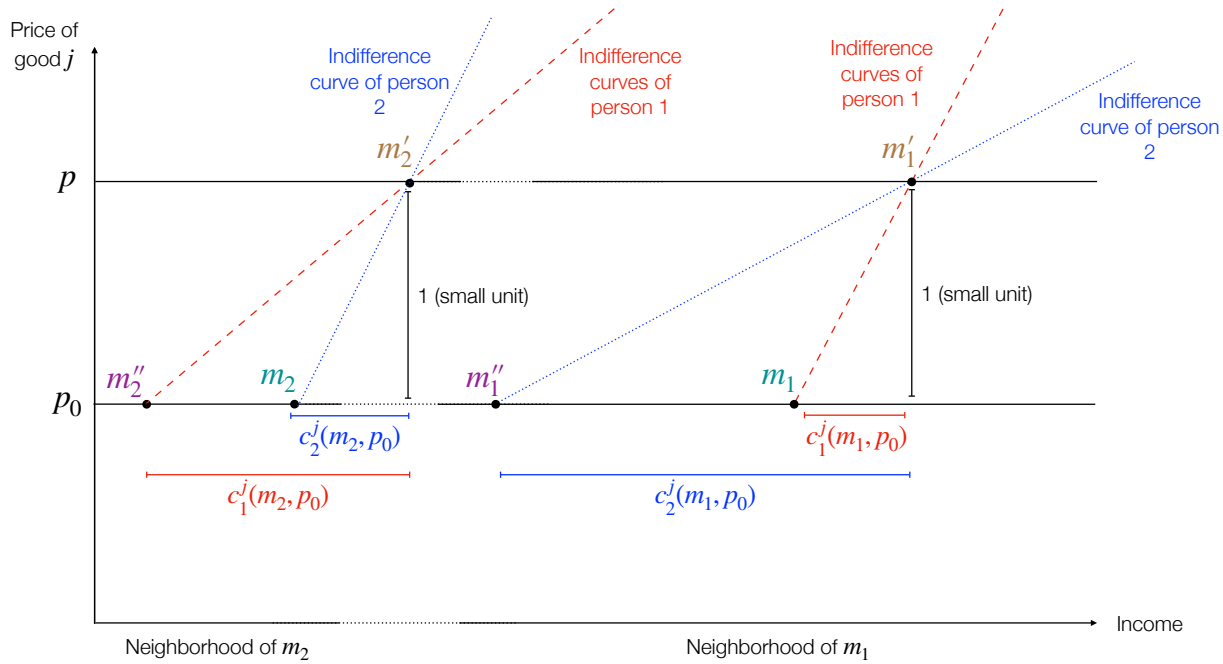
A violation of condition (b.i) is illustrated in Figure 2. In this case, we have a "preference flip": at low incomes, person 1's consumption of good j is higher than person 2's. At higher incomes, the situation is reversed, and person 2 consumes more than person 1. This is a violation of condition (b.i).

To see why this scenario implies a conflict between our axioms, observe that, following the steps laid out in (7) and applying Income Anonymity once again, we have that

$$((m_1, m_2), p_0) \sim ((m_2'', m_1''), p_0) \sim ((m_1'', m_2''), p_0) \tag{9}$$

which is a contradiction to the Pareto condition, as $m_1'' < m_1$ and $m_2'' < m_2$.

Figure 2: A violation of condition (b.i)



A violation of condition (b.ii) is illustrated in Figure 3. In this figure, we set $i' = 2$, so that individuals 1 and 2 consume different quantities of good j, as in Figure 1. The points m_1, m_2, m_1'' and m_2'' correspond to the points indicated in Figure 1, so that

$$m_2 - m_2'' = c_1^j(m_2, p_0) - c_2^j(m_2, p_0)$$

$$m_1'' - m_1 = c_1^j(m_1, p_0) - c_2^j(m_1, p_0)$$

In Figure 3, the y-axis corresponds to variations in the price of good k (rather than variations in the price of good j, as in Figure 1), holding other prices fixed at p_0 levels. The price vector p differs from p_0 only in the price of good k. In this figure, we have that

$$e_1(1, p, p_0) = m_2'' \text{ and } e_2(1, p, p_0) = m_2$$

so that

$$\int_{e_1(1,p,p_0)}^{e_2(1,p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{m_2''}^{m_2} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = 1$$

At the same time, we have that

$$e_1(m,p,p_0) = m_1 \text{ and } e_2(m,p,p_0) = \tilde{m}_1 > m_1''$$

So that

$$\int_{e_1(m,p,p_0)}^{e_2(m,p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = \int_{m_1}^{\tilde{m}_1} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'$$

$$> \int_{m_1}^{m_1''} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' = 1$$

Thus, we have that, in this figure,

$$\int_{e_1(m,p,p_0)}^{e_2(m,p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm' \neq \int_{e_1(1,p,p_0)}^{e_2(1,p,p_0)} \frac{1}{c_1^j(m',p_0) - c_2^j(m',p_0)} dm'$$

which is a violation of condition (b.ii) for $i = 2$.

To see how this violation leads to the inconsistency of our axioms, note that, following the steps laid out in (9), we have that

$$((m_1, m_2), p_0) \sim ((m_1'', m_2''), p_0)$$

Following similar steps but using an auxiliary price p that differs from p_0 in the price of good k rather than in the price of good j, we obtain

$$((m_1, m_2), p_0) \sim ((\tilde{m}_1, m_2''), p_0)$$

and hence

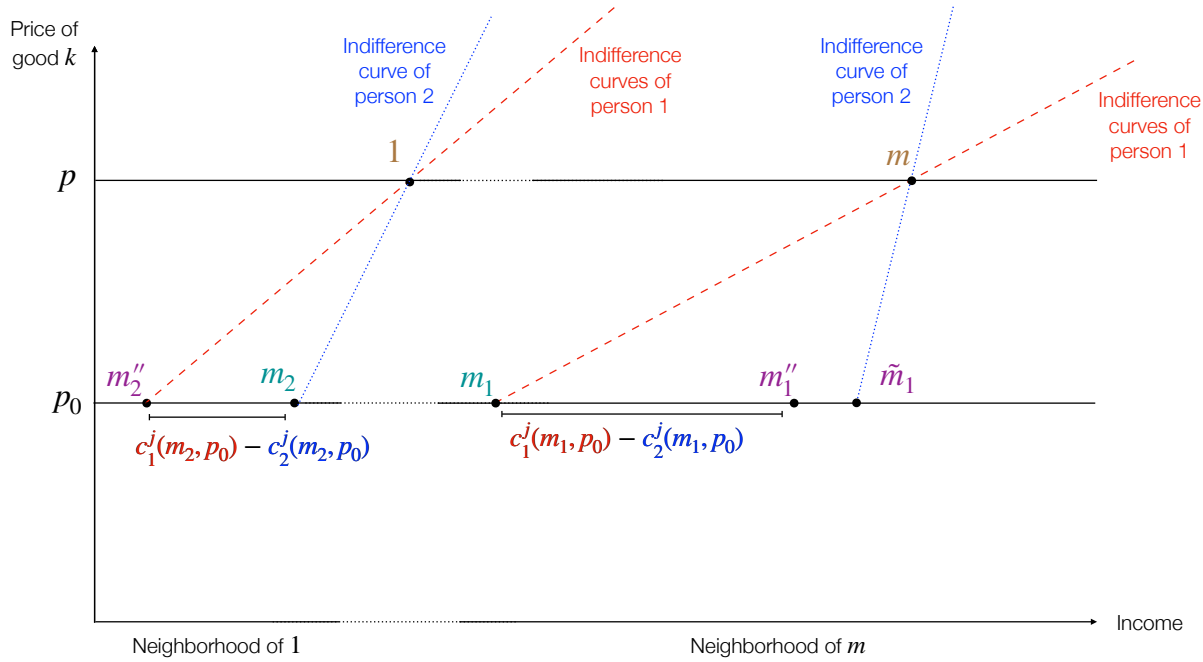
$$((\tilde{m}_1, m_2''), p_0) \sim ((m_1, m_2), p_0) \sim ((m_1'', m_2''), p_0)$$

This is a violation of the Pareto condition, as $\tilde{m}_1 > m_1''$.

4.2 Practical implications

The analysis in the previous section suggests that the two axioms are unlikely to be consistent in practice. To see this, note that, in Figure 3, consistency requires $\tilde{m}_1 = m_1''$. Even if we happen to have a preference profile for which this holds, a small perturbation of person 2's indifference curves can result in a situation in which $\tilde{m}_1 \neq m_1''$.

Figure 3: A violation of condition (b.ii)



This fragility prompts the question: is Theorem 1 of any use, given that the two axioms are consistent only in knife-edge cases? One way to think about this more formally is to consider a multi-profile setting, in which the social preference relation is a function, F, of the individual preferences,

$$(\preceq_1, ..., \preceq_I) \overset{F}{\longmapsto} \preceq$$

In this setting, we might require the social preference relation to be consistent with both axioms whenever this is possible; that is, the function F should map each $(\preceq_1, ..., \preceq_I)$ that satisfies the conditions of Theorem 2 to a social preference ranking that satisfies both axioms. Our uniqueness result can then be used to characterize the function F for a "measure-zero" set of individual preference profiles (we put "measure-zero" in quotation marks because we did not formally define a topology on the set of preference profiles). If we are willing to make some continuity assumptions on F, then this also tells us something about what the social preference relation looks like for individual preference profiles that are in the "neighborhoods" of these preference profiles.

5 Application

We now turn to a calibration of the social welfare function identified in (2) based on consumption expenditure data. This exercise serves two purposes. Our main aim is to provide a practical theory-grounded approach for decisionmaking based on our results. We present a calibration procedure that can be used to estimate the social welfare function, and illustrate this procedure using US consumer expenditure data. We find that, for US consumers, our social welfare function ranks income distributions approximately according to the sum of log incomes.

A secondary goal is to assess the degree to which individual preferences are well-approximated by a preference profile that satisfies our existence condition. As we explain below, our existence condition implies certain regularities in consumption patterns. In our application, we find that the consumption patterns in the US are roughly consistent with these regularities.

5.1 Calibration procedure

In what follows, we focus on the calibration of $W(\cdot, p)$, which represents the social ranking of income distributions conditional on prevailing prices. For the purpose of this calibration, we set $p_0 = p$, where p are the prices that the surveyed consumers face. Given that $e_i(m, p, p) = m$, the social welfare function can be rewritten as in expression (3).

To calculate this expression, we must know the demand functions, $c_1^j(\cdot, p)$ and $c_2^j(\cdot, p)$. Unfortunately, surveys typically tell us only what people consume given their actual budgets, and not what they would consume given hypothetical budgets. To proceed, we need to relate demand functions to the cross-sectional distribution of expenditures.

Let $C_x^j(m, p)$ denote the x-th percentile of the distribution of expenditures on good j, given income m and prices p. Consider the following assumption.

Assumption 1. *If, for some individual i with income m, it holds that $c_i^j(m, p) = C_x^j(m, p)$, then, for every m', it holds that $c_i^j(m', p) = C_x^j(m', p)$.*

This assumption says that if Anne's consumption of asparagus is at the 75th percentile among all people of her income level, then this would be true regardless of what her income level happens to be. This assumption attributes any differences between consumption patterns between rich and poor people to differences in their incomes, rather than differences in their preferences. It does not allow for the possibility that historical or other factors affect both income and tastes.

Using this assumption, we can identify "individual 1" as an individual whose consumption of good j is at the $\bar{x}$-th percentile among others with the same income level, and "individual

2" as an individual whose consumption is at the $\underline{x}$-th percentile among others with the same income level. We can empirically estimate the functions $c_2^j(\cdot, p) = C_{\underline{x}}^j(\cdot, p)$ and $c_1^j(\cdot, p) = C_{\bar{x}}^j(\cdot, p)$, based on how the relevant percentiles of the consumption of good j change with income. We can then define the function μ as

$$\mu(m) := \int_{\hat{m}}^{m} \frac{1}{C_{\bar{x}}^j(m', p) - C_{\underline{x}}^j(m', p)} dm',$$

for some $\hat{m}$ in our sample.[10] Our estimated social welfare function is then

$$W(\mathbf{m}) = \sum_{i=1}^{I} \mu(m_i)$$

where $W(\mathbf{m}) = W(\mathbf{m}, p)$ and $\mu(m) = \mu(m, p)$ (as in expression 8).

Note that, in this procedure, we make two arbitrary choices. First, we need to choose the expenditure category j (food, housing, etc). Second, we need to choose which two percentiles to compare ($\underline{x}$ and $\bar{x}$). In principle, different choices may yield very different estimates of μ. In this case, we have a violation of our existence condition: because the social ranking of income distributions is unique, it must be the same regardless of how we make these choices. However, if different specifications result in similar estimates of μ, then, under Assumption 1, preferences are "close" to satisfying our existence condition.

In our framework, Assumption 1 effectively rules out the problem illustrated in Figure 2. Hence, varying the expenditure category j and the percentiles $\underline{x}$ and $\bar{x}$ allows us to assess the extent to which the problem outlined in Figure 3 arises.

5.2 Empirical implementation

Our data source is the United States Consumer Expenditure Survey's Quarterly Interview. It has been conducted quarterly since 1980, and each survey contains a (weighted) representative sample of approximately 6,000 US households. Households are asked to report their expenditures on 14 expenditure categories over the previous quarter.

We use data for the four most recent quarters available for this survey (2021-Q4 to 2022-Q3). We perform two adjustments to the sample. First, we restrict our sample to single consumer units. This sample restriction allows us to circumvent well-known difficulties with comparing consumption across different family structures. Second, we drop the bottom 5% and the top 10% of expenditure levels.[11] The purpose of this sample restriction is to

[10] Note that the choice of $\hat{m}$ is inconsequential, as it only changes the estimated social welfare function by a constant.

[11] Dropping the bottom 5% is sufficient for removing all negative and zero values.

get rid of atypical consumption quarters. If a household happens to be interviewed in a quarter in which it purchased a house or a car, its reported expenditure will vastly exceed its typical consumption expenditure. Similarly, there may be some quarters with unusually low expenditures.[12] Figure 6 in the appendix shows the distribution of total consumption expenditure without removing the extreme values.

The number of observations in our final sample is 5549 consumption units. Descriptive statistics for each expenditure category for this sample are provided in Table 1 in the appendix.

In our theoretical framework, m represents both income and consumption expenditure (equation (1)). In practice, income varies over the lifecycle, and people can smooth these fluctuations through borrowing and saving. It is therefore more appropriate to identify m with lifetime income. Here, we estimate m using quarterly consumption expenditure, echoing the view that people smooth consumption over time.[13]

We chose not to supplement consumption expenditure with an imputed value of leisure or home production. A person's leisure at a given quarter is a poor measure of his lifetime leisure consumption. As income varies over the lifecycle, so does leisure. Most notably, people consume vastly more leisure after retirement. From a theoretical perspective, we can ignore the value of leisure if we are willing to assume that labor supply is inelastic. This assumption is very problematic for married couples that may choose to have one of the spouses specialize in home production (for example, a stay-at-home parent). However, as our sample contains only single consumer units, the avenues for substituting home production for market production seem more limited.

To test the consistency of the data with our axioms, we estimate μ for multiple consumption categories, as well as multiple combinations of high and low consumption percentiles. We use five different consumption categories: housing, food, transport, health, and entertainment. These are the consumption categories with the largest means in the dataset, and the only ones with a majority of nonzero observations.[14] Our choice of consumption percentiles is arbitrary, and aimed to reflect a range of possibilities. The (low/high) percentile pairs we chose for our estimations were: 25/75, 10/90, 10/60, and 40/90.

For estimating the expenditure functions $C_x^j(\cdot, p)$, we proceed as follows. Recall that

[12]For example, because insurance reimbursements are recorded as negative expenses, total expenditures may be abnormally small or even negative in quarters in which the household is reimbursed for an expensive medical procedure.

[13]Household expenditure includes some categories which are not obviously "consumption" categories, such as education, cash contributions and insurance and pension contributions. Our results are robust to the exclusion of these categories from our measure of m.

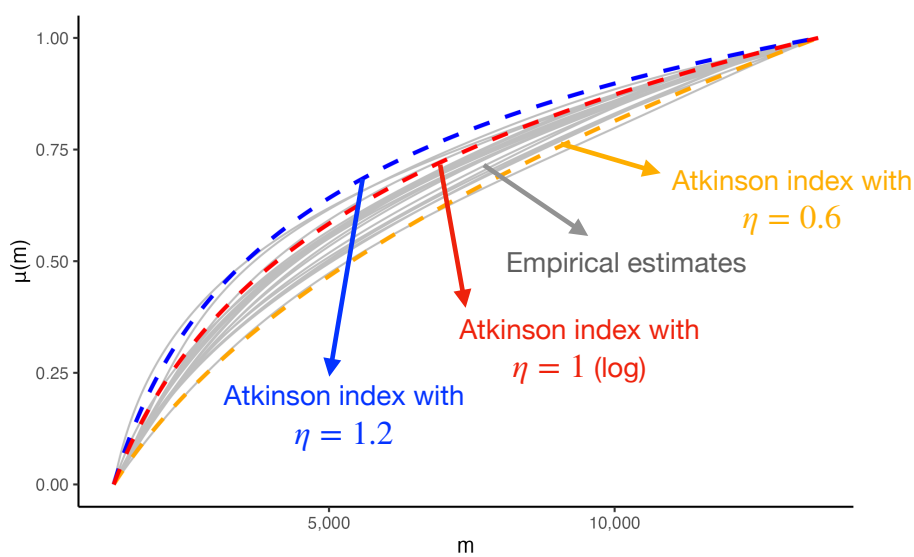
[14]Even though "insurance and pensions" is a larger category than entertainment, we exclude it because it is more appropriately classified as savings than consumption.

$C_x^j(m,p)$ is the x-th percentile of the expenditure on good j, conditional on a budget (m,p). To estimate this function, we re-weigh our sample so that weights are distributed equally across the distribution of m, and estimate the coefficients of the polynomial approximation $c_x^j = \sum_{k=0}^{5} \beta_k m_i^k$, where c_x^j is the consumption of an individual at the xth percentile on good j. This estimation is done through a quantile regression for each of the percentiles we use. We choose a polynomial of degree 5, which provides a good fit for the data (see Figure 7 in the appendix). However, our results are robust to variations in the polynomial degree.

5.3 Results

Figure 4 presents the estimates obtained from our various specifications. As expected, different specifications of $\underline{x}$, $\bar{x}$ and j yield different estimates of μ. Nevertheless, the results illustrate a surprising uniformity of the estimates across the various specifications. This uniformity lends support to the hypothesis that the empirical distribution of preferences is "close" to a preference profile for which our axioms are consistent with one another.

Figure 4: Empirical estimates of μ



Note: Each of the 20 gray lines represents the estimated μ for a combination of expenditure category and a pair of expenditure quantiles. Thicker lines represent the Atkinson index $(m^{1-\eta}/(1-\eta))$ with different inequality aversion coefficients. All lines are normalized so that their range is between zero and one in the given domain of total expenditure values.

The concavity of μ reflects the degree of inequality aversion. A logarithmic μ, as in the homothetic case, is a good approximation for most estimated curves. Across the different specifications, the curvature of μ lies between the curvature of an Atkinson index with $\eta = 0.6$

and the curvature of an Atkinson index with $\eta = 1.2$. This places some bounds on the degree of inequality aversion implied by our welfare criterion.

Our finding is based on the empirical regularity that the distribution of consumption expenditures is roughly proportional to income. We would obtain lower degrees of inequality aversion if we found that consumption patterns become more similar at higher income levels, and higher degrees of inequality aversion if we were to find that consumption is more heterogeneous at higher income levels.

6 Conclusion

Income Anonymity reflects the idea that we should be able to rank income distributions without knowing which income belongs to which person. In certain circumstances, this principle is inconsistent with Pareto. However, when it is not, the combination with Pareto implies concrete guidelines for how income distributions should be ranked.

The implied welfare criterion can be evaluated based on the ordinal properties of individuals' preferences. In fact, under certain assumptions, data about the cross-sectional distribution of consumption expenditures suffices for calibrating the social welfare function. We find that, in the United States, the two axioms imply a social objective that is approximately the maximization of the sum of log incomes.

However, it is important to emphasize that the social welfare criterion may vary with the distribution of individual preferences, and may depend on the particular prices that consumers face. It would be erroneous to conclude, based on our analysis, that the social welfare criterion should always and everywhere be the maximization of the sum of log incomes. Instead, different circumstances may warrant a re-calibration of the social welfare criterion.

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A Proof of Theorem 1

It will be convenient to use the notation $\preceq$ to denote various partial orderings of $\mathbb{R}^I_{++} \times \mathbb{R}^J_{++}$. Fix some p_0. For two income distributions $\mathbf{m}, \mathbf{m}'$, define

$$\mathbf{m} \preceq \mathbf{m}' \Leftrightarrow (\mathbf{m}, p_0) \preceq (\mathbf{m}', p_0)$$

For each $1 \leq i < I$, define the partial ordering $\preceq$ on $\mathbb{R}^i_{++}$ as

$$(m_1, ..., m_i) \preceq (m'_1, ..., m'_i) \Leftrightarrow (\forall m_{i+1}, ..., m_I)(m_1, ..., m_I) \preceq (m'_1, ..., m'_i, m_{i+1}, ..., m_I)$$

A.1 Defining S_m

We begin by defining a sequence of "jumps". There are two kinds of jumps, which we will call "jumps to the right" and "jumps to the left" (we will draw pictures consistent with this terminology, but it is in principle possible that a jump to the right brings us somewhere to the left of where we started, and a jump to the left brings us to the right of where we started; this is not going to matter for our construction).

Without loss of generality, assume that individuals 1 and 2 have different preferences. Fix some p_0. For a price vector, p, a *jump to the right through* p is a function denoted $[p] : \mathbb{R}_{++} \mapsto \mathbb{R}_{++}$, defined as

$$[p](\cdot) = e_2(e_1(\cdot, p_0, p), p, p_0)$$

Figure 1 illustrates this construction. Starting from the point (m, p_0), we follow the indifference curve of individual 1 up until the point of intersection with the price vector p. At that point, we switch to the indifference curve of person 2, and travel back to the point in which it intersects with the price vector p. The resulting income level is m_1. We say that m_1 is obtained from m through jumping to the right through $[p]$.

A *jump to the left through* p is defined as the inverse of a jump to the right through p, as

$$[p]^{-1}(\cdot) = e_1(e_2(\cdot, p_0, p), p, p_0)$$

A sequence of n jumps to the right through p is denoted $[p]^n$ (which is the function $[p]$ composed n times with itself). A sequence of n jumps to the left through p is denoted $[p]^{-n}$, which is the composition of the function $[p]^{-1}$ n times with itself. For completeness of notation, let $[p]^0$ denote the identity function ($[p]^0(m) = m$ for all m).

A sequence of jumps is a composition of the form

$$S = [p_k] \circ \cdots \circ [p_1]$$

Note that, as a composition of strictly increasing and invertible functions, any sequence of jumps is strictly increasing and invertible.

To proceed, consider the following claims.

Claim 1. *For any two incomes m, m', and a jump, $[p]$, it holds that*

$$(m, m') \sim ([p]^{-1}(m), [p](m')).$$

Proof. Consider a permutation σ such that $\sigma(1) = 2$ and $\sigma(2) = 1$. By applying Income

Anoymity with permutation σ and Pareto indifference, we get

$$\begin{aligned}
((m,m'),p_0) &\sim ((m',m),p_0) && \text{(Income Anonymity)}\\
&\sim ((e_1(m',p_0,p),e_2(m,p_0,p)),p) && \text{(Pareto)}\\
&\sim ((e_2(m,p_0,p),e_1(m',p_0,p)),p) && \text{(Income Anonymity)}\\
&\sim ((e_1(e_2(m,p_0,p),p,p_0),e_2(e_1(m',p_0,p),p,p_0)),p_0) && \text{(Pareto)}\\
&= ([p]^{-1}(m),[p](m')),p_0).
\end{aligned}$$

□

Claim 2. *Let m, m' and m'' be such that $(m,m') \preceq (m,m'')$. Then, $m' \leq m''$.*

Proof. Note that person 1 is always indifferent between $((m,m'),p_0)$ and $((m,m''),p_0)$. If $m' > m''$, then person 2 strictly prefers the former. In this case, the Pareto condition demands that $((m,m'),p_0) \succ ((m,m''),p_0)$, in contradiction to the assumption that $((m,m'),p_0) \preceq ((m,m''),p_0)$. □

Claim 3. *For every two sequences of jumps, S and S', it holds that $S \circ S' = S' \circ S$.*

Proof. It suffices to show that, for any p, p', it holds that $[p] \circ [p'] = [p'] \circ [p]$, since it is then possible to apply pairwise permutations to jumps in a sequence $S \circ S'$ of arbitrary length and obtain that $S \circ S' = S' \circ S$. We thus show that $[p] \circ [p'] = [p'] \circ [p]$. We have

$$(([p](m),[p'](m)),p_0) \sim ((m,[p]([p'](m))),p_0). \qquad \text{(Claim 1)}$$

Similarly,

$$\begin{aligned}
(([p](m),[p'](m)),p_0) &\sim (([p'](m),[p](m)),m),p_0) && \text{(Claim 1)}\\
&\sim ((m,[p'](m))),p_0). && \text{(Income Anonymity)}
\end{aligned}$$

Thus, by transitivity,

$$((m,[p]([p'](m))),p_0) \sim ((m,[p'](m))),p_0)$$

Hence, by Claim 2, it holds that $[p]([p'](m)) = [p'](m))$.

□

Claim 4. *For any two incomes, m, m', and a sequence of jumps, S, it holds that*

$$(m,m') \sim (S^{-1}(m),S(m')).$$

Proof. We use induction on the number of jumps in the sequence S, which we denote by n. For the case $n = 0$, the claim follows from the reflexivity of the indifference relation.

Assume that the claim holds for a sequence S with n jumps. For an arbitrary p, we have

$$\begin{aligned}
(m, m') &\sim ((S^{-1}(m), S(m')) && \text{(Inductive hypothesis)} \\
&\sim ([p]^{-1}(S^{-1}(m)), [p](S(m'))) && \text{(Claim 1)} \\
&= (S^{-1}([p]^{-1}(m)), [p](S(m'))) && \text{(Claim 3)} \\
&= (([p] \circ S)^{-1}(m), ([p] \circ S)(m')).
\end{aligned}$$

This completes the proof. □

Because individuals 1 and 2 have different preferences, we can choose p_0 such that there exists some p_1 satisfying

$$m_0 < [p_1](m_0).$$

Claim 5. *The sequence* $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ *is strictly increasing.*

Proof. First, note that for any p and any k, $[p]^k$ is strictly increasing in m, since equivalent income functions are strictly increasing in income.

Suppose that $[p_1]^k(m_0) < [p_1]^{k+1}(m_0)$ for some k (we assumed this for $k = 0$). We now show that this implies that $[p_1]^{k+1}(m_0) < [p_1]^{k+2}(m_0)$, and that $[p_1]^{k-1}(m_0) < [p]^k(m_0)$, from which we can conclude that $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is strictly increasing. We have that

$$\begin{aligned}
[p_1]^{k+1}(m_0) &= [p_1]([p_1]^k(m_0)) \\
&< [p_1]([p_1]^{k+1}(m_0)) \\
&= [p_1]^{k+2}(m_0),
\end{aligned}$$

where the inequality follows from the hypothesis and fact that $[p_1]$ is strictly increasing in m. Similarly,

$$\begin{aligned}
[p_1]^k(m_0) &= [p_1]^{-1}([p]^{k+1}(m_0)) \\
&> [p_1]^{-1}([p]^k(m_0)) \\
&= [p_1]^{k-1}(m_0).
\end{aligned}$$

We have thus established that $[p_1]^k(m_0) < [p_1]^{k+1}(m_0)$ for all k.

□

Claim 6. *The sequence* $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ *is unbounded in* $\mathbb{R}_{++}$.

Proof. Assume by way of contradiction that there exists $m' > 0$ for which either $[p_1]^k(m_0) < m'$ for every k, or $[p_1]^k(m_0) > m'$ for every k. Consider the case where $[p_1]^k(m_0) < m'$ for all k (the proof for the other case is similar, and hence omitted). This implies that $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is bounded from above. As $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is an increasing and bounded sequence, it converges—let m^* be its limit. By continuity of $[p_1]$, it follows that

$$\begin{aligned} m^* &= \lim_{k\to\infty} [p_1]^k(m_0) \\ &= \lim_{k\to\infty} [p_1]\left([p_1]^{k-1}(m_0)\right) \\ &= [p_1]\left(\lim_{k\to\infty} [p_1]^{k-1}(m_0)\right) \\ &= [p_1](m^*), \end{aligned}$$

(where the third equality uses the continuity of $[p_1]$).

By Claim 4, we have that

$$([p](m_0), m^*) \sim ([p]^{-1}([p](m_0)), [p](m^*)) = (m_0, m^*)$$

But this is a contradiction to Claim 2, as $[p](m_0) > m_0$.

□

Claim 7. *For every m, there exists a sequence of jumps, S, for which $m = S(m_0)$.*

Proof. Fix some m. Note that, by Claims 5 and 6, the sequence $\{[p_1]^k(m_0)\}_{k=-\infty}^{\infty}$ is unbounded and strictly increasing. Hence, there exists some k such that

$$[p_1]^k(m_0) \leq m < [p_1]^{k+1}(m_0)$$

We show that there exists a price p for which

$$[p]([p_1]^k(m_0)) = m$$

To see this, note that

1. For $p = p_0$, the function $[p]$ is the identity function, and hence, $[p]([p_1]^k(m_0)) = [p_1]^k(m_0) \leq m$.

2. For $p = p_1$, we have $[p]([p_1]^k(m_0)) = [p_1]^{k+1}(m_0) > m$.

Further, note that $[p](m_0)$ is a continuous function of p. Using an intermediate value argument, it follows that there exists a linear combination of p_0 and p_1 which satisfies the above condition. □

We thus define S_m as a sequence of jumps for which $S_m(m_0) = m$. Note that, as a sequence of jumps, S_m is strictly increasing and invertible.

A.2 Uniqueness

We start with the current proof of uniqueness (Theorem 1). By construction of S_m, we have that

$$(m_1, ..., m_I) = (S_{m_1}(m_0), ..., S_{m_I}(m_0))$$

By Claim 4,

$$(S_{m_1}(m_0), ..., S_{m_I}(m_0)) \sim (S_{m_2}(S_{m_1}(m_0)), S_{m_2}^{-1}(S_{m_2}(m_0)), S_{m_3}(m_0)..., S_{m_I}(m_0))$$

$$= ((S_{m_2} \circ S_{m_1})(m_0), m_0, S_{m_3}(m_0)..., S_{m_I}(m_0))$$

Using similar steps, we obtain that

$$(m_1, ..., m_I) \sim ((S_{m_I} \circ \cdots \circ S_{m_1})(m_0), m_0, ..., m_0)$$

Thus, we have that

$$(m_1, ..., m_I) \preceq (m'_1, ..., m'_I) \Leftrightarrow$$

$$((S_{m_I} \circ \cdots \circ S_{m_1})(m_0), m_0, ..., m_0) \preceq ((S_{m'_I} \circ \cdots \circ S_{m'_1})(m_0), m_0, ..., m_0)$$

which, by Claim 2, holds if and only if

$$(S_{m_I} \circ \cdots \circ S_{m_1})(m_0) \leq (S_{m'_I} \circ \cdots \circ S_{m'_1})(m_0)$$

We have thus established that our preference relation must be representable by $(S_{m_I} \circ \cdots \circ S_{m_1})(m_0)$, and hence it is unique.

A.3 Proof of Corollary 2

Wakker [1989, p. 70] says that $\succeq$ satisfies *generalized triple cancellation* if, for all i and all income vectors m^1, m^2, m^3, m^4,

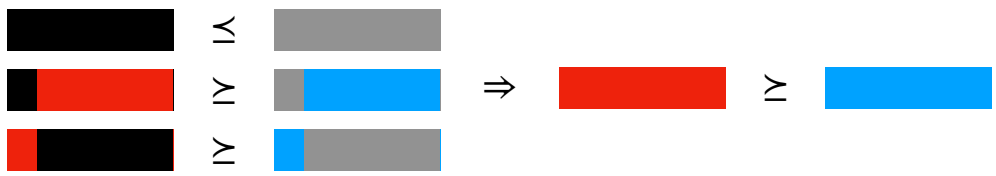
$$m^1 \preceq m^2,$$
$$(m_i^3, m_{-i}^1) \succeq (m_i^4, m_{-i}^2), \text{ and}$$
$$(m_i^1, m_{-i}^3) \succeq (m_i^2, m_{-i}^4)$$

imply that

$$m^3 \succeq m^4.$$

This condition is illustrated in Figure 5.

Figure 5: Illustration of Wakker's condition



Note: Consider a situation in which (a) the gray vector is better than the black vector, and (b) there are mixtures of the red vector and the black vector and mixtures of the gray vector and the blue vector such that the black-red mixtures are better than the corresponding gray-blue mixtures. Wakker's condition requires that, in this case, the red vector must be better than the blue vector.

Theorem 3 (Wakker, 1989, p. 70)**.** *Suppose a relation $\succeq$ on $\mathbb{R}^I_{++}$ is a continuous weak order satisfying generalized triple cancellation. Then, there exists a continuous additive representation for $\succeq$.*

We now show that preferences $\succeq$ satisfy the necessary conditions for this theorem.

Claim 8. *Preferences $\succeq$ satisfy generalized triple cancellation.*

Proof. For an income vector m_{-i}, let $S_{m_{-i}}$ denote a composition of the sequences of jumps S_{m_j}, for all $j \neq i$. By the representation shown in the previous section, we have

$$\begin{aligned} m^1 \preceq m^2 &\Rightarrow S_{m^1}(m_0) \leq S_{m^2}(m_0) \\ (m_i^3, m_{-i}^1) \succeq (m_i^4, m_{-i}^2) &\Rightarrow S_{m_i^3} \circ S_{m_{-i}^1}(m_0) \geq S_{m_i^4} \circ S_{m_{-i}^2}(m_0) \\ (m_i^1, m_{-i}^3) \succeq (m_i^2, m_{-i}^4) &\Rightarrow S_{m_i^1} \circ S_{m_{-i}^3}(m_0) \geq S_{m_i^2} \circ S_{m_{-i}^4}(m_0) \end{aligned}$$

Given that sequences of jumps are strictly increasing in their initial arguments, the inequalities above imply that

$$S_{m^2} \circ S_{m_i^3} \circ S_{m_{-i}^1} \circ S_{m_i^1} \circ S_{m_{-i}^3}(m_0) \geq S_{m^1} \circ S_{m_i^4} \circ S_{m_{-i}^2} \circ S_{m_i^2} \circ S_{m_{-i}^4}(m_0).$$

Combining Claim 3 and the fact that sequences of jumps are strictly increasing in their initial arguments, we can then cancel out sequences of jumps that appear on both sides of the inequality. It then follows that

$$S_{m^3}(m_0) \geq S_{m^4}(m_0).$$

This implies that $m^3 \succeq m^4$, as we wanted to show.

□

Claim 9. *The preference relation represented by* $(m_1, ..., m_I) \mapsto S_{m_I} \circ \cdots \circ S_{m_1}(m_0)$ *is continuous.*

Proof. It is sufficient to show that $S_{m_I} \circ \cdots \circ S_{m_1}(m_0)$ is continuous in $m_1, ..., m_I$ (because a preference relation that is represented by a continuous function is continuous). We begin by showing that $S_m(\tilde{m})$ is continuous in m. Note that

$$\begin{aligned} S_m(\tilde{m}) &= S_m(S_{\tilde{m}}(m_0)) && \text{(Definition of } S_{\tilde{m}}) \\ &= S_{\tilde{m}}(S_m(m_0)) && \text{(Claim 3)} \\ &= S_{\tilde{m}}(m). && \text{(Definition of } S_m) \end{aligned}$$

As $S_{\tilde{m}}$ is a composition of continuous functions, it is continuous. It follows that, for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $|m' - m| < \delta$, then $|S_{\tilde{m}}(m') - S_{\tilde{m}}(m)| < \epsilon$, and hence

$$|S_{m'}(\tilde{m}) - S_m(\tilde{m})| < \epsilon,$$

establishing the continuity of $S_m(\tilde{m})$ with respect to m. Because $S_m(\tilde{m})$ is also continuous in $\tilde{m}$ (as a composition of continuous functions), the mapping $f(m, \tilde{m}) = S_m(\tilde{m})$ is continuous in both arguments. Note that

$$S_{m_1}(m_0) = f(m_1, m_0)$$

$$S_{m_2}(S_{m_1}(m_0)) = f(m_2, S_{m_1}(m_0)) = f(m_2, f(m_1, m_0)),$$

and so on. Hence, by induction, it holds that $S_{m_I} \circ \cdots \circ S_{m_1}(m_0)$ is a composition of continuous functions, and is therefore continuous. □

Theorem 3 and Claims 8 and 9 allow us to conclude that social preferences are represented by

$$\sum_{i=1}^{I} \phi_i(m_i).$$

Because these preferences satisfy Income Anonymity, it follows that $\phi := \phi_1 = \phi_i$ for all i. Hence, our social preference relation is represented simply by

$$\sum_{i=1}^{I} \phi(m_i).$$

We now go back to characterizing our social preference relation over the entire domain of income distributions and prices (not just incomes given p_0). As the social preference relation

must satisfy the Pareto indifference condition, it must be represented by

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi(e_i(m_i, p_i, p_0)).$$

By Income Anonymity, for any m, p and i, it holds that

$$\sum_{k \neq 1,i} \phi(e_k(m_0, p, p_0)) + \phi(e_1(m_0, p, p_0)) + \phi(e_i(m, p, p_0)) =$$
$$\sum_{k \neq 1,i} \phi(e_k(m_0, p, p_0)) + \phi(e_1(m, p, p_0)) + \phi(e_i(m_0, p, p_0))$$

$$\Rightarrow \phi(e_i(m, p, p_0)) = \underbrace{\phi(e_1(m, p, p_0))}_{\mu(m,p)} + \underbrace{(\phi(e_i(m_0, p, p_0)) - \phi(e_1(m_0, p, p_0)))}_{\gamma_i(p)}, \tag{10}$$

and thus

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \mu(m_i, p_i) + \gamma_i(p_i). \tag{11}$$

A.4 Establishing the representation (2)

To establish the representation in (2), note that, by Pareto, ϕ must be strictly increasing. By Lebesgue's Theorem for the differentiability of monotone functions, it follows that ϕ is differentiable almost everywhere.[15]

Claim 10. *The functions $\{e_i(\cdot, \cdot, p_0)\}$ are differentiable with respect to m and $\{p^j\}_{j=1}^{J}$.*

Proof. To show that the functions $\{e_i\}$ are differentiable, define the function

$$F(m, p, m', p') = v_i(m, p) - v_i(m', p').$$

Let m' be such that $v_i(m, p) - v_i(m', p') = 0$. Since the functions $\{v_i\}$ are differentiable and strictly increasing in income, it follows that $\frac{\partial F}{\partial m'}(m, p, m', p') \neq 0$. Hence, since the $\{v_i\}$ are continuously differentiable, by the implicit function theorem, we can define continuously differentiable functions $\tilde{e}_i(m, p, p')$ around each m' satisfying $F(m, p, \tilde{e}_i(m, p, p'), p') = 0$.

By definition of the equivalent income function, we have that for each m', it holds that $\tilde{e}_i(m, p, p') = e_i(m, p, p')$. But since e_i is continuous, this equality and the continuous differentiability of each $\tilde{e}_i$ imply the continuous differentiability of e_i. Since the choice of i was arbitrary, it follows that $\{e_i\}$ are continuously differentiable. □

[15] See http://mathonline.wikidot.com/lebesgue-s-theorem-for-the-differentiability-of-monotone-fun

Hence, by (10), the functions μ and γ_i are differentiable almost everywhere (as compositions of $\phi(\cdot)$ which is differentiable almost everywhere and $\{e_i\}$ which are differentiable). It follows that γ_i is differentiable: to see this, observe that the set $\{e_i(m_0, \lambda p, p_0)|\lambda \in \mathbb{R}_{++}\}$ is open (as $e_i(m_0, \cdot, p_0)$ is continuous). Hence, we can choose λ so that $\phi(e_i(m_0, \lambda p, p_0))$ and $\phi(e_1(m_0, \lambda p, p_0)$ are both differentiable at λp. By (10), it follows that γ_i is a linear combination of two functions that are differentiable at λp, and is thus differentiable at λp. But, as the indirect utility function is homogeneous of degree 0, it holds that

$$\mu(\lambda m, \lambda p) + \gamma_i(\lambda p) = \mu(m, p) + \gamma_i(p)$$

as $\gamma_1 \equiv 0$ (by 10), it holds that $\mu(\lambda m, \lambda p) = \mu(m, p)$, and hence the above implies that $\gamma_i(\lambda p) = \gamma_i(p)$. As γ_i is differentiable at λp, it follows that it is also differentiable at p.

In addition, note that $\mu(\cdot, p_0)$ is differentiable almost everywhere, as, by (10),

$$\mu(m, p_0) = \phi(e_1(m, p_0, p_0)) = \phi(m)$$

which is differentiable almost everywhere (as ϕ is continuous and strictly monotone).

By Roy's identity, for each (m, p) such that $\mu(m, p)$ is differentiable, it holds that

$$-\frac{\frac{\partial \mu(m,p)}{\partial p^j}}{\frac{\partial \mu(m,p)}{\partial m}} = c_1^j(m, p) \text{ and } -\frac{\frac{\partial \mu(m,p)}{\partial p^j} + \frac{\partial \gamma_2(p)}{\partial p^j}}{\frac{\partial \mu(m,p)}{\partial m}} = c_2^j(m, p)$$

Hence,

$$\frac{\frac{\partial \gamma_2(p)}{\partial p^j}}{\frac{\partial \mu(m,p)}{\partial m}} = c_1^j(m, p) - c_2^j(m, p)$$

$$\Rightarrow \frac{\partial \mu(m, p)}{\partial m} = \frac{\frac{\partial \gamma_2(p)}{\partial p^j}}{c_1^j(m, p) - c_2^j(m, p)}$$

Because $\mu(\cdot, p_0)$ is differentiable almost everywhere, it holds that

$$\mu(m, p_0) = \mu(1, p_0) + \int_1^m \frac{\partial \mu(m', p_0)}{\partial m'} dm' = \mu(1, p_0) + \int_1^m \frac{\frac{\partial \gamma_2(p_0)}{\partial p^j}}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

$$\Rightarrow \mu(m, p_0) = \mu(1, p_0) + \frac{\partial \gamma_2(p_0)}{\partial p^j} \int_1^m \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

As $\{\mu + \gamma_i\}$ represent individual preferences, it holds that $\mu(m_i, p_i) + \gamma_i(p_i) = \mu(e_i(m_i, p_i, p_0), p_0)$. Thus, (11) can be rewritten as

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \mu(e_i(m_i, p_i, p_0), p_0) =$$

$$= \sum_{i=1}^{I} \left(\mu(1, p_0) + \frac{\partial \gamma_2(p_0)}{\partial p^j} \int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' \right)$$

$$= I\mu(1, p_0) + \frac{\partial \gamma_2(p_0)}{\partial p^j} \sum_{i=1}^{I} \int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

As $c_1^j(m_0, p_0) > c_2^j(m_0, p_0)$ and $c_i^j(\cdot, p_0)$ is continuous, it must hold that $c_1^j(m', p_0) > c_2^j(m', p_0)$ for every m' in an open neighborhood of m_0. Because $\mu(\cdot, p_0)$ is strictly increasing at m_0, it must hold that $\frac{\partial \gamma_2(p_0)}{\partial p^j} > 0$. Hence, this social welfare function is an affine transformation of (2), and hence these social preferences are also represented by (2).

B Proof of Theorem 2

We begin by establishing that (a) implies (b). Assume that $c_1^j(m_0, p_0) > c_2^j(m_0, p_0)$. By Theorem 1, when a social preference relation that satisfies the axioms exists, then it must be represented by (2). For this social preference relation to satisfy Pareto, its ranking of individual i's budget sets, (m_i, p_i) (holding the other individuals' budgets fixed), must coincide with the preferences of individual i. Note that $e_i(\cdot, \cdot, p_0)$ is a representation of individual i's preferences (it is the money-metric utility function, with reference prices p_0). Thus, for our social welfare function to be Paretian, it must hold that the integral

$$\int_1^{e_i(m_i, p_i, p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

is a strictly monotone transformation of $e_i(m_i, p_i, p_0)$. This holds if and only if condition (b.i) holds for $i = 1$ and $i' = 2$. Note that there is nothing in this argument that relies on individuals 1 and 2, the good j or the budget (m_0, p_0) specifically: the only requirement is that they jointly satisfy $c_1^j(m_0, p_0) > c_2^j(m_0, p_0)$. A similar argument thus establishes that, when our axioms are consistent, then, for every i, i', j, m_0 and p_0,

$$c_i^j(m_0, p_0) > c_{i'}^j(m_0, p_0) \Rightarrow c_i^j(m, p_0) > c_{i'}^j(m, p_0) \ \forall m$$

We have thus established that (2) represents a Paretian social preference relation only if condition (b.i) holds.

To satisfy Income Anonymity, the social welfare function in (2) must be symmetric in $m_1, ..., m_I$ whenever $p = p_1 = ... = p_I$. To see when this is the case, it is useful to rewrite

the social welfare function as

$$W(\mathbf{m}, p) = \left(\sum_{i=1}^{I} \int_{1}^{e_1(m_i,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' \right)$$

$$+ \left(\sum_{i=1}^{I} \int_{e_1(m_i,p,p_0)}^{e_i(m_i,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' \right) \tag{12}$$

The first component is symmetric in $\{m_i\}$. Thus, for the function to be symmetric in $\{m_i\}$, the second component must be symmetric in $\{m_i\}$ as well. In particular, the value of the function should remain the same if we switch m_1 and m_i. This requires that

$$\int_{e_1(m_1,p,p_0)}^{e_1(m_1,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' + \int_{e_1(m_i,p,p_0)}^{e_i(m_i,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' =$$

$$\int_{e_1(m_i,p,p_0)}^{e_1(m_i,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' + \int_{e_1(m_1,p,p_0)}^{e_i(m_1,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

As the first terms in the summations are both zero, this condition holds only if

$$\int_{e_1(m_i,p,p_0)}^{e_i(m_i,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' = \int_{e_1(m_1,p,p_0)}^{e_i(m_1,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

As this must hold for any m_i, it follows that the term on the left hand side must be independent of m_i. This implies condition (b.ii).

We now show that (b) implies (c). Observe that, when (b.i) and (b.ii) hold, individual i's preferences are represented by

$$\int_{1}^{e_i(m,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' =$$

$$\int_{1}^{e_1(m,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm' + \int_{e_1(m,p,p_0)}^{e_i(m,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'$$

$$= \underbrace{\int_{1}^{e_1(m,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'}_{\mu(m,p)} + \underbrace{\int_{e_1(1,p,p_0)}^{e_i(1,p,p_0)} \frac{1}{c_1^j(m', p_0) - c_2^j(m', p_0)} dm'}_{\gamma_i(p)}$$

Finally, observe that (c) implies (a): when individual preferences are represented by $\mu(m_i, p_i) + \gamma_i(p_i)$, then it is straightforward to verify that the social preference relation represented by $\sum_{i=1}^{I} \mu(m_i, p_i) + \gamma_i(p_i)$ satisfies Pareto and Income Anonymity, and hence such a social preference relation exists.

C Proof of Corollary 1

Assume by way of contradiction that $\eta > 1$. By Theorem 1, the social preferences over income distributions conditional on prices p are represented by (3).

The social marginal rate of substitution between m_i and $m_{i'}$ is therefore

$$\frac{\frac{\partial W(\mathbf{m},p)}{\partial m_i}}{\frac{\partial W(\mathbf{m},p)}{\partial m_{i'}}} = \frac{c_1^j(m_{i'},p) - c_2^j(m_{i'},p)}{c_1^j(m_i,p) - c_2^j(m_i,p)}$$

Using the budget constraint from the consumer's optimization problem, it holds that

$$c_1^j(m_{i'},p) - c_2^j(m_{i'},p) \leq c_1^j(m_{i'},p) \leq \frac{m_{i'}}{p^j}$$

Hence, for every $m_{i'}$, it holds that

$$\frac{\frac{\partial W(\mathbf{m},p)}{\partial m_i}}{\frac{\partial W(\mathbf{m},p)}{\partial m_{i'}}} \leq \frac{m_{i'}}{p^j(c_1^j(m_i,p) - c_2^j(m_i,p))} \tag{13}$$

However, as these social preferences are also represented by the Atkinson index with $\eta > 1$, and as the social marginal rate of substitution is a property of the ordinal preference relation that does not depend on its particular representation, it holds that

$$\frac{\frac{\partial W(\mathbf{m},p)}{\partial m_i}}{\frac{\partial W(\mathbf{m},p)}{\partial m_{i'}}} = \frac{\frac{\partial W^{Atkinson}(\mathbf{m})}{\partial m_i}}{\frac{\partial W^{Atkinson}(\mathbf{m})}{\partial m_{i'}}} = \left(\frac{m_{i'}}{m_i}\right)^\eta$$

By (13), it follows that, for every $m_{i'}$,

$$\left(\frac{m_{i'}}{m_i}\right)^\eta \leq \frac{m_{i'}}{p^j(c_1^j(m_i,p) - c_2^j(m_i,p))}$$

$$\Rightarrow m_{i'}^{\eta-1} \leq \frac{m_i^\eta}{p^j(c_1^j(m_i,p) - c_2^j(m_i,p))}$$

but this is a contradiction: given that $\eta > 1$, the left hand side is larger than the right hand side for $m_{i'}$ sufficiently large (holding m_i fixed).

D The consistency of the axioms in the examples of section 3.2

The proof of existence uses the Gorman polar form for the indirect utility functions. For homothetic preferences, the Gorman form is $v_i(m,p) = f_i(p)m$, where $f_i > 0$. Applying the

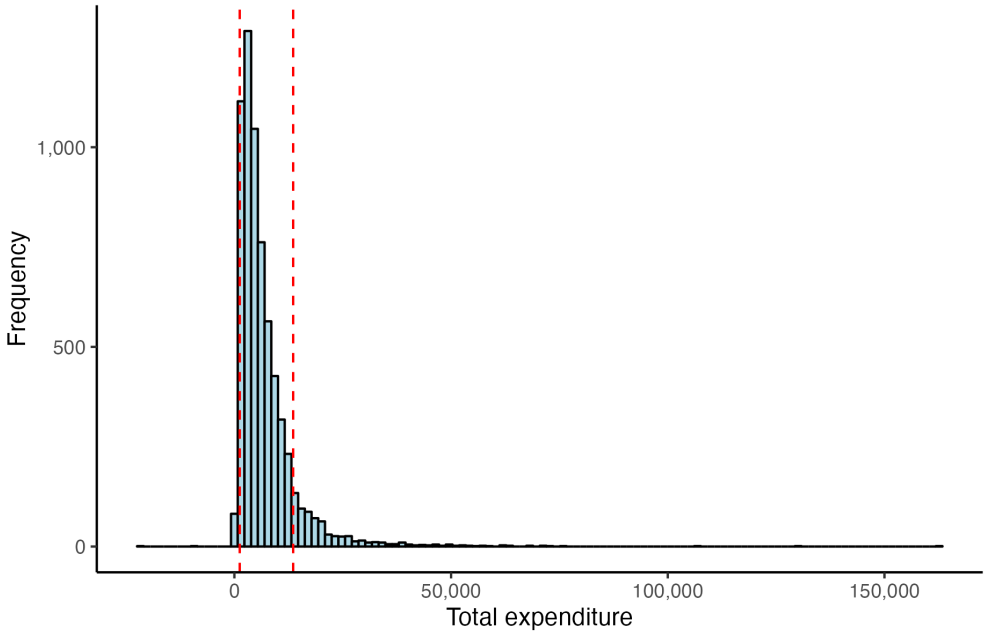
log transformation, these preferences are also represented by $\ln(v_i(m,p)) = \ln(m) + \ln(f_i(p))$. Specifying $\mu(m,p) := \ln(m)$ and $\gamma_i(p) = \ln(f_i(p))$ provides a representation that is consistent with the third clause of Theorem 2, thus proving existence.

For the Stone-Geary case, the indirect utility function is represented by $v_i(m,p) = f_i(p)(m - \sum_{j=1}^{J} p^j \underline{c}^j)$ (to see this, note that these preferences are homothetic after the cost of the subsistence bundle is deducted from income). Applying the log transformation, these preferences are also represented by $\ln(f_i(p)) + \ln(m - \sum_{j=1}^{J} p^j \underline{c}^j)$. Specifying $\mu(m,p) := \ln(m - \sum_{j=1}^{J} p^j \underline{c}^j)$ and $\gamma_i(p) = \ln(f_i(p))$ provides a representation consistent with the third clause of Theorem 2.

Finally, for quasilinear preferences, the Gorman form is $v_i(m,p) = m/p^1 + f_i(p)$. In this case, specifying $\mu(m,p) = m/p^1$ and $\gamma_i(p) = f_i(p)$ provides a representation consistent with the third clause of Theorem 2.

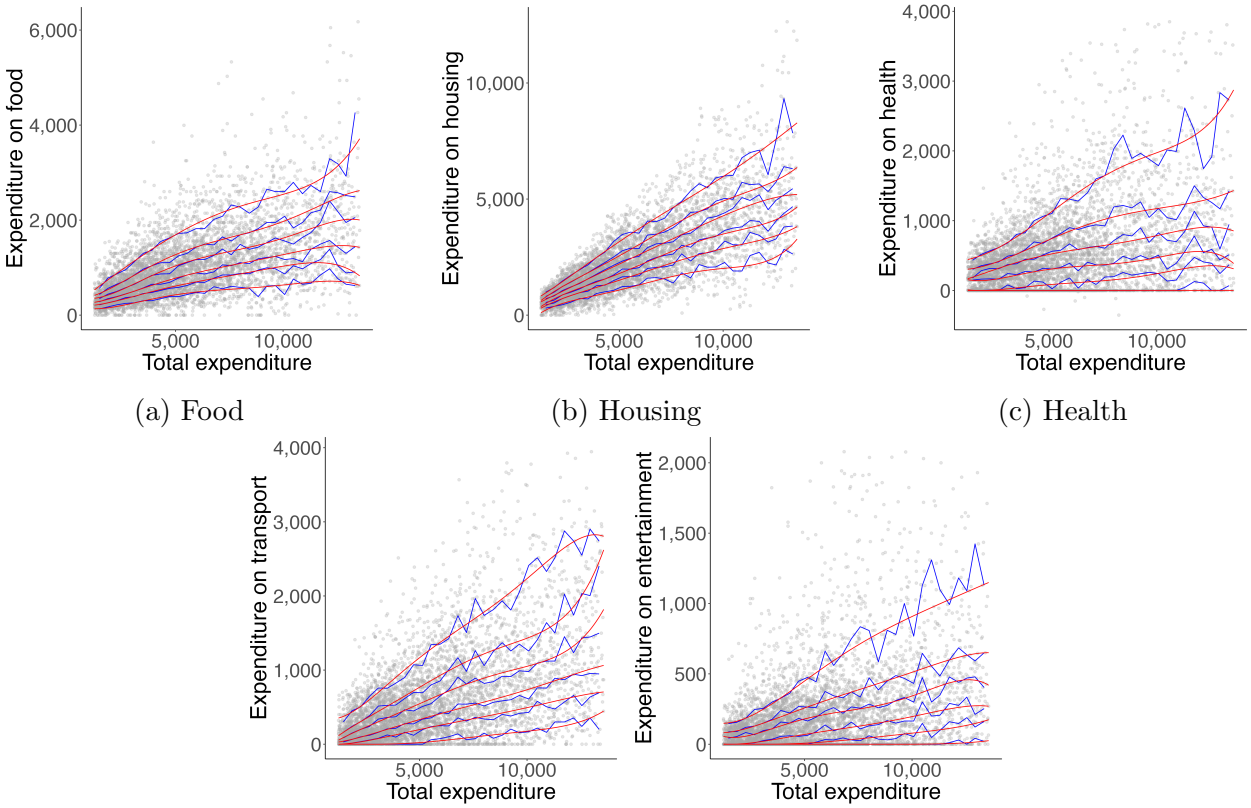
E Empirical estimates

Figure 6: Distribution of total expenditures before removing extreme values



Note: Observations are restricted to single consumers. Dashed lines represent expenditure cutoffs in our data.

Figure 7: Percentiles of the distributions of expenditures on different categories as a function of income



(a) Food (b) Housing (c) Health

(d) Transportation (e) Entertainment

Note: Gray dots represent observations in our dataset. Red lines represent the estimated housing expenditure as a polynomial function of total expenditure for the 90th, 75th, 60th, 40th, 25th, and 10th percentiles. Red lines correspond to 5th order polynomial approximations, and are estimated through quantile regressions. Blue lines represent the empirically observed expenditure quantiles for each of 30 total expenditure groups, for the percentiles indicated above. These empirical quantiles are estimated by a polynomial regression of housing consumption on a quantile, for each of the 30 bins. For ease of visualization, observations in the highest .5% expenditure levels in transport, health, and entertainment were dropped from the scatterplot.

Table 1: Descriptive statistics for expenditure categories in the final sample

Expenditure category	Mean	Median	Min	Max
Housing	2,240	1,840	0	12,638
Food	968	780	0	6,175
Transport	577	378	0	10,085
Health	524	352	-353	8,988
Insurance and pensions	490	153	0	9,697
Entertainment	211	105	0	5,282
Cash contributions	158	0	0	7,910
Alcohol	57	0	0	3,500
Apparel	54	0	0	3,000
Tobacco	43	0	0	3,900
Personal care	36	0	0	1,175
Education	16	0	0	7,778
Reading	9	0	0	900
Misc.	56	0	0	8,000