Appendix

The first section contains additional analysis. The second section provides details for the numerical examples. Subsequent sections contain proofs. Throughout, I use ℓ, k, e, r, and y to denote per-capita variables, I use a ss subscript to denote a steady state, and I use g_x to indicate the instantaneous growth rate of variable x.

A Analysis of deposit-constrained degrowth, for tapped resources with $\sigma > 1$

Consider the setting of Section 5, with $\sigma > 1$. When inputs are gross substitutes, the optimal path may not be interior. Instead, it may be a corner solution in which energy inputs to deposit-tapping completely crowd out other inputs to deposit-tapping (i.e., $L_R(t), K_R(t) = 0$). I analyze these balanced growth paths under log utility:

Proposition A-1 (Deposit-Constrained Degrowth with Tapped Resources)**.** *Let Assumptions 1 and 3 hold and fix $\sigma > 1$. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, g_k as the growth rate of capital per capita, and χ as in Proposition 5. If $\rho - \Omega$ is not too large and $\Omega - g_L > \frac{\omega}{1-\omega}(\lambda + g_L)$, then there exists $X \leq \chi$ such that χ/X is strictly decreasing in σ, $\lim_{\sigma\to\infty} X = \lim_{\sigma\to\infty} \chi$, and $A_E Q_D > X$ implies the existence of a balanced growth path with $L_R(t), K_R(t) = 0$ and with $R(t)/L(t)$, $E_R(t)/L(t)$, $E_Y(t)/L(t)$, and $Z(t)/L(t)$ all growing at rate $g_e \neq 0$. Along this path:*

i $g_y = g_k = \frac{\alpha_E}{\alpha_E+\alpha_L} g_e$.

ii $g_e \in \big(-(\rho - \Omega), 0\big)$.

iii *g_e and $Z(t)/E_R(t)$ each decrease in A_E and Q_D.*

iv $EROI(t) = 1 + A_E Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} \frac{g_e+g_L}{[g_e+g_L+\lambda](\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}}}$.

Proof. See Appendix K. □

When energy resources are sufficiently productive (i.e., $A_E Q_D$ is large), the growth rates of output and energy are negative (parts i and ii). Reinvesting growing energy stocks in deposit-tapping at a constant rate eats up the available deposits, as $Z(t)/E_R(t)$ falls in the productivity of energy resources (part iii). The increasing scarcity of deposits restricts the ability to produce further energy, so the growth rate of energy must also fall until the rate of energy reinvestment comes into equilibrium with the rate of deposit finds (part iii).

This is a scenario of *deposit-constrained degrowth.* The economy accumulates a lot of spare energy. Deposits are scarce relative to energy, so their availability constrains how much of the spare energy can be devoted to tapping new deposits. As a result, a large share of the energy produced is devoted to final good production ($e_R(t)/e(t)$ is small) and thus to consumption. Nonetheless, the available deposits decline over time as they are exploited via energy inputs faster than they are found. Energy production falls with deposits, and consumption falls over time as available energy falls, albeit from a potentially high level that was enabled by the high productivity of energy deposits.

B Numerical Implementation

I implement the model by discretizing the transition equations over a one-year timestep. I determine the economy's path over 400 years by solving the social planner's problem. The control variables comprise the factor allocations and the state variables, with the transition and market-clearing equations serving as constraints. I use the Knitro solver's interior-point algorithm in Matlab. I provide an analytic gradient and an analytic Hessian.

I fix the annual growth rate of population to $g_L = 0.01$, which is roughly in keeping with the recent growth rate of world population and only a bit higher than the pre-1900 growth rate.[38] I fix the saving rate to $s = 0.25$, which is in keeping with the global rate.[39] I set the value share of energy in final good production to $\alpha_E = 0.05$, which is around the value in the U.S. over the past decades (Hassler et al., 2021; Orak and Çakır Melek, 2021). I then set $\alpha_K/(1-\alpha_E) = 0.2632$ to match the capital share relative to labor in Hassler et al. (2021), which implies $\alpha_K = 0.25$ and $\alpha_L = 0.7$. Following Hassler et al. (2021), the annual utility discount rate is $\rho = 0.0152$ and the annual depreciation rate is $\delta = 0.05$. I normalize $A(t) = 1$ and assume log utility.

For the tapped resource, I calibrate the annual discovery rate Ω to the 11% growth in global probable and proved oil reserves between 2000 and 2007 (Sorrell et al., 2012, 716). This implies $\Omega = 0.015$. I fix $\lambda = 0.041$ to the production-weighted aggregate annual decline rate of all oil fields (Sorrell et al., 2012, 718). For the manufactured resource, I set $\lambda = 0.005$ based on the median annual degradation rate for silicon panels in Jordan and Kurtz (2013) and Jordan et al. (2016).

For both the tapped and the manufactured resource, I fix $\kappa_E = 0.5$ and $\kappa_K = 0.9$ and I normalize $A_E = 1$ and $A_{LK} = 1$. For the tapped resource, I fix $\omega = 0.5$.

I calibrate Q_D and Q_S to the EROI of oil and solar resources. First, I fix the EROI of each resource to an average from Supplementary Table 1 of Slameršak et al. (2022): I set the EROI of the tapped resource to 10.65 following their analysis of gas-to-electricity, and I set the EROI of the manufactured resource to 7.75 following their analysis of solar photovoltaics. Second, I use equation (A-20) to obtain the e_{Rss}/e_{ss} that must hold in steady

[38] `https://en.wikipedia.org/wiki/Human_population_projections#/media/File:World_population_growth,_1700-2100,_2022_revision.png`

[39] `https://data.worldbank.org/indicator/NE.GDI.FTOT.ZS`

state. Finally, I back out Q_D or Q_S from that and (A-34). When exploring values of Q_D or Q_S that do not yield energy-enabled growth, I set the relevant parameter to twice or half of the boundary value χ, depending on which boundary I want to violate.

I set the initial resource stock to $R_0 = 100$ and the initial deposit stock to $Z_0 = 1000$. I follow DICE-2016R (Nordhaus, 2017) in setting the initial capital stock to $K_0 = 223$, in trillion year 2010 dollars.

C Proof of Proposition 1

Equilibrium solves the following maximization problem:

$$\begin{aligned}
\max_{L_Y(\cdot),K_Y(\cdot),L_E(\cdot),K_E(\cdot)} \int_0^\infty & e^{-\rho t} L(t) u\left(\frac{(1-s)A(t)L_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{A}(t) =& g_A A(t) \\
\dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E(t)^{\alpha_E} - \delta K(t) \\
E(t) =& Q_H L_E(t)^{\phi_{HL}} K_E(t)^{\phi_{HK}} R^{\phi_{HR}} \\
L(t) =& L_Y(t) + L_E(t) \\
K(t) =& K_Y(t) + K_E(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}
\max_{\ell_Y(\cdot),k_Y(\cdot)} \int_0^\infty & e^{-(\rho - g_L)t} u\left((1-s)A(t)\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r(t)^{\phi_{HR}}\right]^{\alpha_E}\right) \mathrm{d}t \\
\text{s.t. } \dot{A}(t) =& g_A A(t) \\
\dot{k}(t) =& sA\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r(t)^{\phi_{HR}}\right]^{\alpha_E} - (\delta + g_L)k(t) \\
\dot{r}(t) =& -g_L r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}
& u\left((1-s)A(t)\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r(t)^{\phi_{HR}}\right]^{\alpha_E}\right) \\
& + \nu(t)\left[sA\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r^{\phi_{HR}}\right]^{\alpha_E} - (\delta + g_L)k(t)\right] \\
& - \mu(t)g_L r(t) + \psi(t) g_A A(t).
\end{aligned}$$

The conditions to maximize the Hamiltonian are:

$$0 =\left[(1-s)u'((1-s)y(t))+s\nu(t)\right]\left[\frac{\alpha_L}{\ell_Y(t)}-\frac{\alpha_E\phi_{HL}}{1-\ell_Y(t)}\right]y(t),$$
$$0 =\left[(1-s)u'((1-s)y(t))+s\nu(t)\right]\left[\frac{\alpha_K}{k_Y(t)}-\frac{\alpha_E\phi_{HK}}{k(t)-k_Y(t)}\right]y(t).$$

Together, these imply:

$$\ell_Y(t) =\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}},$$
$$k_Y(t) =\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}k(t).$$

Substituting, we find

$$E(t) =Q_H\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}L(t)\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}K(t)\right)^{\phi_{HK}}R^{\phi_{HR}}$$

and

$$Y(t) =A(t)\left(\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\alpha_L}\left(\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\alpha_K}$$
$$\left[Q_H\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\phi_{HK}}R^{\phi_{HR}}\right]^{\alpha_E}L(t)^{\alpha_L+\alpha_E\phi_{HL}}K(t)^{\alpha_K+\alpha_E\phi_{HK}}.$$

Define

$$A_H(t)\triangleq A(t)^{\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}}.$$

Defining $y_H(t)\triangleq Y(t)/[A_H(t)L(t)]$ and analogously for the other variables, we find:

$$y_H(t) =\left(\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\alpha_L}\left(\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\alpha_K}$$
$$\left[Q_H\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\phi_{HK}}r_H(t)^{\phi_{HR}}\right]^{\alpha_E}k_H(t)^{\alpha_K+\alpha_E\phi_{HK}},$$
$$\dot{k}_H(t) =sy_H(t)-\left(\delta+g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)k_H(t),$$
$$\dot{r}_H(t) =-\left(g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)r_H(t).$$

The growth rate of $k_H(t)$ is:

$$\frac{\dot{k}_H(t)}{k_H(t)} =s\chi_H Q_H^{\alpha_E}r_H(t)^{\alpha_E\phi_{HR}}k_H(t)^{\alpha_K+\alpha_E\phi_{HK}-1}-\left(\delta+g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right),$$

where

$$\chi_H \triangleq \left(\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\alpha_L}\left(\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\alpha_K}\left[\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\phi_{HK}}\right]^{\alpha_E}.$$

That growth rate is constant if and only if

$$r_H(t)^{\alpha_E\phi_{HR}}k_H(t)^{\alpha_K+\alpha_E\phi_{HK}-1} = \frac{1}{s\chi_H Q_H^{\alpha_E}}\left[g_k+\delta+g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right] \tag{A-1}$$

is constant. Observe that:

$$r_H(t) = e^{-\left(g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)t}r_H(0).$$

Substitute into (A-1):

$$k_H(t) = \left[s\chi_H Q_H^{\alpha_E}\frac{e^{-\alpha_E\phi_{HR}\left(g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)t}\left(\frac{R}{L(0)A_H(0)}\right)^{\alpha_E\phi_{HR}}}{g_k+\delta+g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A}\right]^{\frac{1}{1-\alpha_K-\alpha_E\phi_{HK}}}.$$

And thus:

$$\frac{\dot{k}_H(t)}{k_H(t)} = -\frac{\alpha_E\phi_{HR}}{1-\alpha_K-\alpha_E\phi_{HK}}\left(g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right) < 0.$$

The growth rate of output per capita is:

$$\begin{aligned}\frac{\dot{y}(t)}{y(t)} =& \alpha_E\phi_{HR}\frac{\dot{r}_H(t)}{r_H(t)}+[\alpha_K+\alpha_E\phi_{HK}]\frac{\dot{k}_H(t)}{k_H(t)}+\frac{\dot{A}_H(t)}{A_H(t)}\\ =& \frac{g_A-\alpha_E\phi_{HR}g_L}{\alpha_L+\alpha_E(\phi_{HL}+\phi_{HR})}.\end{aligned}$$

The proposition follows.

D Preliminaries for Propositions 3 and 4

Equilibrium solves the following maximization problem:

$$\max_{L_Y(\cdot),K_Y(\cdot),E_Y(\cdot),L_E(\cdot),K_E(\cdot),L_R(\cdot),K_R(\cdot),E_R(\cdot)} \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s)AL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t$$

$$\begin{aligned}
\text{s.t. } \dot{L}(t) =& g_L L(t)\\
\dot{K}(t) =& sAL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E} - \delta K(t)\\
\dot{Z}(t) =& \Omega Z(t) - Z(t)^\omega F(L_R(t), K_R(t), E_R(t))^{1-\omega}\\
\dot{R}(t) =& Z(t)^\omega F(L_R(t), K_R(t), E_R(t))^{1-\omega} - \lambda R(t)\\
E(t) =& Q_M L_E(t)^{\phi_{ML}} K_E(t)^{\phi_{MK}} R(t)^{\phi_{MR}}\\
L(t) =& L_Y(t) + L_E(t) + L_R(t)\\
K(t) =& K_Y(t) + K_E(t) + K_R(t)\\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}
\max_{\ell_E(\cdot),k_E(\cdot),\ell_R(\cdot),k_R(\cdot),e_R(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} u\Bigg((1-s)A\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}&\\
\Big[Q_M \ell_E(t)^{\phi_{ML}} k_E(t)^{\phi_{MK}} r(t)^{\phi_{MR}} - e_R(t)\Big]^{\alpha_E}\Bigg) \mathrm{d}t&
\end{aligned}$$

$$\begin{aligned}
\text{s.t. } \dot{k}(t) =& sA\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\\
&\Big[Q_M \ell_E(t)^{\phi_{ML}} k_E(t)^{\phi_{MK}} r(t)^{\phi_{MR}} - e_R(t)\Big]^{\alpha_E} - (\delta+g_L)k(t)\\
\dot{z}(t) =& (\Omega - g_L)z(t) - z(t)^\omega F(\ell_R(t), k_R(t), e_R(t))^{1-\omega}\\
\dot{r}(t) =& z(t)^\omega F(\ell_R(t), k_R(t), e_R(t))^{1-\omega} - (\lambda+g_L)r(t).
\end{aligned}$$

The following lemma establishes that any balanced growth path with interior solutions must have variables growing at the rate of population:

Lemma A-2. *A balanced growth path with $L_R(t), K_R(t), E_R(t) > 0$ must have all variables growing at the rate of population.*

Proof. Use g_x to indicate the instantaneous growth rate of variable x. From the transition equation for $z(t)$, $\dot{z}(t)/z(t)$ is constant if and only if $F(\ell_R(t), k_R(t), e_R(t))/z(t)$ is constant. From the transition equation for $r(t)$, $\dot{r}(t)/r(t)$ can be constant with $F(\ell_R(t), k_R(t), e_R(t))/z(t)$ constant if and only if $F(\ell_R(t), k_R(t), e_R(t))/r(t)$ is constant. Therefore, if g_z and g_r are each constant, then $g_z = g_r = g_{F/L}$.

If $g_{F/L} \neq 0$ and e_R and k_R grow at constant rates, then ℓ_R must grow at a constant rate. The constant growth rate of ℓ_R cannot be strictly positive because ℓ_R is bounded above by 1. The constant growth rate of ℓ_R cannot be strictly negative because the (full employment) labor constraint implies that ℓ_Y and/or ℓ_E must grow at a strictly positive rate yet these variables are each bounded above by 1. Therefore a constant growth rate of ℓ_R must be zero. If $g_{F/L}$, g_{k_R}, and g_{e_R} are also to be constant, then $g_{F/L}, g_{k_R}, g_{e_R} = 0$. That implies $g_z, g_r = 0$. It follows straightforwardly that the remaining variables (in per capita form) also grow at rate zero. $\square$

So we seek a steady state in per-capita variables. The current-value Hamiltonian is:

$$
\begin{aligned}
&u\Bigg((1-s)A\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\Big[Q_M\ell_E(t)^{\phi_{ML}}k_E(t)^{\phi_{MK}}r(t)^{\phi_{MR}}-e_R(t)\Big]^{\alpha_E}\Bigg)\\
&+\nu(t)\Bigg(sA\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\Big[Q_M\ell_E(t)^{\phi_{ML}}k_E(t)^{\phi_{MK}}r(t)^{\phi_{MR}}-e_R(t)\Big]^{\alpha_E}\\
&\qquad-(\delta+g_L)k(t)\Bigg)\\
&+\gamma(t)\Big[(\Omega-g_L)z(t)-z(t)^{\omega}F(\ell_R(t),k_R(t),e_R(t))^{1-\omega}\Big]\\
&+\mu(t)\Big[z(t)^{\omega}F(\ell_R(t),k_R(t),e_R(t))^{1-\omega}-(\lambda+g_L)r(t)\Big].
\end{aligned}
$$

The costate equations are:

$$
\begin{aligned}
(\rho-g_L)\nu(t)-\dot{\nu}(t)=&\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_K y(t)}{k_Y(t)}-(\delta+g_L)\nu(t)\\
\Leftrightarrow \dot{\nu}(t)=&-\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_K y(t)}{k_Y(t)}+(\rho+\delta)\nu(t), && \text{(A-2)}\\
(\rho-g_L)\gamma(t)-\dot{\gamma}(t)=&\gamma(t)(\Omega-g_L)+\Big[\mu(t)-\gamma(t)\Big]\omega\left(\frac{F(\ell_R(t),k_R(t),e_R(t))}{z(t)}\right)^{1-\omega}\\
\Leftrightarrow \dot{\gamma}(t)=&\gamma(t)(\rho-\Omega)-\Big[\mu(t)-\gamma(t)\Big]\omega\left(\frac{F(\ell_R(t),k_R(t),e_R(t))}{z(t)}\right)^{1-\omega}, && \text{(A-3)}\\
(\rho-g_L)\mu(t)-\dot{\mu}(t)=&-\mu(t)(\lambda+g_L)+\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_E\phi_{MR}y(t)}{e_Y(t)}\frac{e(t)}{r(t)}\\
\Leftrightarrow \dot{\mu}(t)=&\mu(t)(\rho+\lambda)-\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_E\phi_{MR}y(t)}{e_Y(t)}\frac{e(t)}{r(t)}. && \text{(A-4)}
\end{aligned}
$$

The conditions to maximize the Hamiltonian are:

$$0 = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\left[\frac{\alpha_E\phi_{ML}}{e_Y(t)}\frac{e(t)}{\ell_E(t)} - \frac{\alpha_L}{\ell_Y(t)}\right]y(t), \tag{A-5}$$

$$0 = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\left[\frac{\alpha_E\phi_{MK}}{e_Y(t)}\frac{e(t)}{k_E(t)} - \frac{\alpha_K}{k_Y(t)}\right]y(t), \tag{A-6}$$

$$\begin{aligned}0 =& -\frac{\alpha_L y(t)}{\ell_Y(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] \\ &+ \left[\mu(t) - \gamma(t)\right](1-\omega)\left(\frac{z(t)}{F(\ell_R(t), k_R(t), e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t), k_R(t), e_R(t))}{\partial \ell_R(t)},\end{aligned} \tag{A-7}$$

$$\begin{aligned}0 =& -\frac{\alpha_K y(t)}{k_Y(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] \\ &+ \left[\mu(t) - \gamma(t)\right](1-\omega)\left(\frac{z(t)}{F(\ell_R(t), k_R(t), e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t), k_R(t), e_R(t))}{\partial k_R(t)},\end{aligned} \tag{A-8}$$

$$\begin{aligned}0 =& -\frac{\alpha_E y(t)}{e_Y(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] \\ &+ \left[\mu(t) - \gamma(t)\right](1-\omega)\left(\frac{z(t)}{F(\ell_R(t), k_R(t), e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t), k_R(t), e_R(t))}{\partial e_R(t)}.\end{aligned} \tag{A-9}$$

Equations (A-5) and (A-6) imply:

$$\ell_Y(t) = \frac{\alpha_L}{\alpha_E\phi_{ML}}\frac{e_Y(t)}{e(t)}\ell_E(t), \tag{A-10}$$

$$k_Y(t) = \frac{\alpha_K}{\alpha_E\phi_{MK}}\frac{e_Y(t)}{e(t)}k_E(t). \tag{A-11}$$

Take ratios of equations (A-7), (A-8), and (A-9), substitute for $\ell_Y(t)$ and $k_Y(t)$, and cancel the $e_Y(t)$:

$$1 = \frac{\kappa_L}{\phi_{ML}}\frac{1-\kappa_E}{\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{k_R(t)}{\ell_R(t)}\right)^{\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_R(t)}{\ell_R(t)}\right)^{\frac{1}{\sigma}}\frac{\ell_E(t)}{e(t)}, \tag{A-12}$$

$$1 = \frac{\kappa_K}{\phi_{MK}}\frac{1-\kappa_E}{\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{\ell_R(t)}{k_R(t)}\right)^{\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_R(t)}{k_R(t)}\right)^{\frac{1}{\sigma}}\frac{k_E(t)}{e(t)}. \tag{A-13}$$

Solve (A-12) for $k_R(t)$:

$$k_R(t) = \left[\frac{\kappa_L}{\phi_{ML}}\frac{1-\kappa_E}{\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{1}{\ell_R(t)}\right)^{\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_R(t)}{\ell_R(t)}\right)^{\frac{1}{\sigma}}\frac{\ell_E(t)}{e(t)}\right]^{-\frac{1}{\kappa_K\frac{\sigma-1}{\sigma}}}. \tag{A-14}$$

Substitute into (A-13) and solve for $\ell_R(t)$:

$$\ell_R(t) = \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}$$
$$e_R(t) \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)+1}. \tag{A-15}$$

Substitute back into (A-14):

$$k_R(t) = \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}$$
$$e_R(t) \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)+1}.$$

Substitute from (A-10) and (A-15) into the labor resource constraint and solve for $\ell_E(t)$:

$$\ell_E(t) = \Bigg[1 + \frac{\alpha_L}{\alpha_E \phi_{ML}} \frac{e_Y(t)}{e(t)}$$
$$+ \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}$$
$$\frac{e_R(t)}{e(t)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}\Bigg]^{-1}. \tag{A-16}$$

Solve for $e_R(t)/e(t)$:

$$\frac{e_R(t)}{e(t)} = \frac{[1 - \ell_E(t) - \frac{\alpha_L}{\alpha_E \phi_{ML}} \ell_E(t)]/\ell_E(t)}{-\frac{\alpha_L}{\alpha_E \phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}. \tag{A-17}$$

Substitute from (A-11) and (A-14) into the capital constraint:

$$\frac{k_E(t)}{k(t)} = \Bigg[1 + \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{e_Y(t)}{e(t)}$$
$$+ \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}$$
$$\frac{e_R(t)}{e(t)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)}\Bigg]^{-1}. \tag{A-18}$$

Substitute for $e_R(t)/e(t)$ from (A-17):

$$\begin{aligned}
&\frac{k_E(t)}{k(t)}\\
=&\Bigg[1-\frac{\alpha_K}{\alpha_E\phi_{MK}}\frac{1}{\ell_E(t)}\\
&\quad\frac{1-\ell_E(t)-\ell_E(t)\left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)}\left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1}\left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma}\left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}\left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)}\left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}{-\frac{\alpha_L}{\alpha_E\phi_{ML}}+\left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)}\left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1}\left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma}\left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}\left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)}\left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}\\
&+\left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)}\left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1}\left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma}\left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}\left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}\left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)}\\
&\quad\frac{[1-\ell_E(t)-\frac{\alpha_L}{\alpha_E\phi_{ML}}\ell_E(t)]/\ell_E(t)}{-\frac{\alpha_L}{\alpha_E\phi_{ML}}+\left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)}\left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1}\left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma}\left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}\left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)}\left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}\Bigg]^{-1}.
\end{aligned} \tag{A-19}$$

Finally, at a steady state, $\dot{e}(t)=0$ in equation (7). Therefore,

$$EROI_{ss}=1+\frac{e_{ss}}{e_{Rss}}g_L. \tag{A-20}$$

E Proof of Proposition 3

Section D contains preliminaries. By Lemma A-2, any interior balanced growth path must have per-capita variables constant. If $\sigma=1$, we have, from (A-16) and (A-18):

$$\begin{aligned}
\ell_E(t)=&\left[1+\frac{\alpha_L}{\alpha_E\phi_{ML}}\left(1-\frac{e_R(t)}{e(t)}\right)+\frac{\kappa_L}{\phi_{ML}}\frac{1-\kappa_E}{\kappa_E}\frac{e_R(t)}{e(t)}\right]^{-1},\\
\frac{k_E(t)}{k(t)}=&\left[1+\frac{\alpha_K}{\alpha_E\phi_{MK}}\left(1-\frac{e_R(t)}{e(t)}\right)+\frac{\kappa_K}{\phi_{MK}}\frac{1-\kappa_E}{\kappa_E}\frac{e_R(t)}{e(t)}\right]^{-1}.
\end{aligned} \tag{A-21}$$

These are both interior as long as $e_R(t)/e(t)<1$. With these:

$$\begin{aligned}
\ell_Y(t)=&\frac{\alpha_L}{\alpha_E\phi_{ML}}\frac{e_Y(t)}{e(t)}\ell_E(t),\\
\frac{k_Y(t)}{k(t)}=&\frac{\alpha_K}{\alpha_E\phi_{MK}}\frac{e_Y(t)}{e(t)}\frac{k_E(t)}{k(t)}.
\end{aligned} \tag{A-22}$$

And:

$$\begin{aligned}\ell_R(t) =&\frac{\kappa_L}{\phi_{ML}}\frac{1-\kappa_E}{\kappa_E}\frac{e_R(t)}{e(t)}\ell_E(t),\\ \frac{k_R(t)}{k(t)} =&\frac{\kappa_K}{\phi_{MK}}\frac{1-\kappa_E}{\kappa_E}\frac{e_R(t)}{e(t)}\frac{k_E(t)}{k(t)}.\end{aligned} \tag{A-23}$$

At a steady state, (A-4) becomes:

$$y_{ss}\Big[(1-s)u'((1-s)y_{ss})+s\nu_{ss}\Big] = \mu_{ss}(\rho+\lambda)\frac{r_{ss}}{\alpha_E\phi_{MR}}\left(1-\frac{e_{Rss}}{e_{ss}}\right).$$

Substitute into the steady-state version of (A-9):

$$0 = -\frac{1}{\phi_{MR}}\mu_{ss}(\rho+\lambda)\frac{r_{ss}}{e_{ss}} + \Big[\mu_{ss}-\gamma_{ss}\Big](1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss},k_{Rss},e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E,$$

where

$$F_{ss} \triangleq F(\ell_{Rss},k_{Rss},e_{Rss}).$$

Solve for γ_{ss}:

$$\gamma_{ss} =\mu_{ss}\left(1-\frac{\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}}{(1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss},k_{Rss},e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E}\right).$$

Substitute into (A-3):

$$\begin{aligned}0 =&(\rho-\Omega)\left((1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss},k_{Rss},e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E-\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}\right)\\ &-\omega\left(\frac{F(\ell_{Rss},k_{Rss},e_{Rss})}{z_{ss}}\right)^{1-\omega}\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}.\end{aligned}$$

From the transition equation for $z(t)$,

$$\frac{z_{ss}}{F_{ss}} =(\Omega-g_L)^{-\frac{1}{1-\omega}}. \tag{A-24}$$

Using (A-24) in the expression prior to it yields:

$$0 =(\rho-\Omega)(1-\omega)\left(\Omega-g_L\right)^{-\frac{\omega}{1-\omega}}\kappa_E-\left[(\rho-\Omega)+\omega(\Omega-g_L)\right]\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{F_{ss}}\frac{e_{Rss}}{e_{ss}}.$$

The transition equation for $r(t)$ and (A-24) yield:

$$(\lambda+g_L)\frac{r_{ss}}{F_{ss}} =(\Omega-g_L)^{-\frac{\omega}{1-\omega}}. \tag{A-25}$$

Substitute into the prior expression:

$$\frac{e_{Rss}}{e_{ss}} = \phi_{MR}(1-\omega)\kappa_E \frac{\rho-\Omega}{(\rho-\Omega)+\omega(\Omega-g_L)} \frac{g_L+\lambda}{\rho+\lambda}. \tag{A-26}$$

The right-hand side is strictly positive because $\rho > \Omega, g_L$.

It remains to verify that the steady state is feasible. Equation (A-25) shows that a steady state with other variables strictly positive has $r_{ss} > 0$ if and only if either $\omega = 0$ or $\Omega > g_L$. From (A-24), $z_{ss} > 0$ with other variables strictly positive requires $\Omega > g_L$. Because $e_{Yss} = e_{ss} - e_{Rss}$, a steady state has strictly positive final good production only if $e_{Rss}/e_{ss} < 1$. Using $\rho > g_L$ and $\kappa_E, \phi_{MR} < 1$, it is clear from (A-26) that $e_{Rss}/e_{ss} < 1$ when either $\omega = 0$ or $\Omega > g_L$. Finally, from the capital transition equation, $y_{ss}/k_{ss} = (\delta + g_L)/s$, so $k_{ss} > 0$ if $y_{ss} > 0$, and from the final good production function and (A-22), $y_{ss} > 0$ when $\ell_{Yss}, e_{Yss} > 0$ if $k_{ss} > 0$. So it is internally consistent for all variables to be strictly positive if and only if either $\omega = 0$ or $\Omega > g_L$.

The claims about prices are implied by the existence of steady states in per-capita, current value terms for $\gamma(t)$ and $\mu(t)$. The claim about $EROI_{ss}$ follows from (A-20) and (A-26).

Finally, consider the elasticity of y_{ss} with respect to Q_M. From (A-25), (A-23), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}r_{ss}}{\mathrm{d}Q_M} = (\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\left[\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M}\right]. \tag{A-27}$$

From (2), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M} = \frac{e_{ss}}{Q_M} + \phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \phi_{MR}\frac{e_{ss}}{r_{ss}}\frac{\mathrm{d}r_{ss}}{\mathrm{d}Q_M}.$$

Substitute from (A-27) and solve for $\mathrm{d}e_{ss}/\mathrm{d}Q_M$:

$$\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M} = \frac{\frac{e_{ss}}{Q_M} + \phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}. \tag{A-28}$$

From the final good production function, (A-22), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \alpha_K\frac{y_{ss}}{k_{Yss}}\frac{\mathrm{d}k_{Yss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \alpha_E\frac{y_{ss}}{e_{Yss}}\frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}}\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M}.$$

Substitute from (A-28) and rearrange to solve for $\mathrm{d}y_{ss}/\mathrm{d}Q_M$:

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \frac{\alpha_E \frac{y_{ss}}{e_{Yss}} \frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}} \frac{\frac{e_{ss}}{Q_M}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E \frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}}{1-\alpha_K \frac{y_{ss}}{k_{Yss}}\frac{\mathrm{d}k_{Yss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L} - \alpha_E \frac{y_{ss}}{e_{Yss}}\frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}} \frac{\phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}+\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}}.$$

Substitute for y_{ss}/k_{ss} from the transition equation for $k(t)$ and simplify some other terms, using constancy of shares from (A-21), (A-22), (A-23), and (A-26):

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \frac{\alpha_E y_{ss}\frac{1}{Q_M}\frac{1}{1-\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{r_{ss}}}}{1-\alpha_K-\alpha_E\frac{\phi_{MK}+\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}(1-\kappa_E)\kappa_K\frac{F_{ss}}{r_{ss}}}{1-\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{r_{ss}}}}.$$

Substitute for F_{ss}/r_{ss} from (A-25):

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}\frac{Q_M}{y_{ss}} = \frac{\alpha_E\frac{1}{1-\phi_{MR}\kappa_E}}{1-\alpha_K-\alpha_E\frac{\phi_{MK}+\phi_{MR}(1-\kappa_E)\kappa_K}{\phi_{MK}+\phi_{ML}+\phi_{MR}(1-\kappa_E)}}.$$

The left-hand side is the elasticity of y_{ss} with respect to Q_M. The right-hand side is constant in Q_M and strictly positive. We have established the claim in the proposition about the elasticity.

F Proof of Proposition 4

Section D contains preliminaries. By Lemma A-2, any interior balanced growth path must have per-capita variables constant.

Use (A-4) in (A-9) and evaluate at a steady state:

$$0 = -\frac{\alpha_E}{e_{Yss}}\mu_{ss}(\rho+\lambda)\frac{e_{Yss}}{\alpha_E\phi_{MR}}\frac{r_{ss}}{e_{ss}} + \Big[\mu_{ss}-\gamma_{ss}\Big](1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}},$$

where

$$F_{ss} \triangleq F(\ell_{Rss}, k_{Rss}, e_{Rss}).$$

Solve for μ_{ss}:

$$\mu_{ss} = \frac{(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}}}{(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}} - (\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}}\gamma_{ss}.$$

Substitute into the steady-state version of (A-3):

$$0 = (\rho - \Omega)\left[(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}} - (\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}\right] - \omega\left(\frac{F_{ss}}{z_{ss}}\right)^{1-\omega}(\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}.$$

From the transition equation for $z(t)$:

$$\frac{z_{ss}}{F_{ss}} = (\Omega - g_L)^{-\frac{1}{1-\omega}}. \tag{A-29}$$

Use (A-29) in the prior expression and substitute for the partial derivative of F_{ss}:

$$0 =(\rho-\Omega)(1-\omega)(\Omega-g_L)^{-\frac{\omega}{1-\omega}}\kappa_E A_E^{\frac{\sigma-1}{\sigma}} - \Big[(\rho-\Omega)+\omega(\Omega-g_L)\Big](\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{F_{ss}}\frac{e_{Rss}}{e_{ss}}\left(\frac{F_{ss}}{e_{Rss}}\right)^{\frac{\sigma-1}{\sigma}}.$$

And use (A-29) in the transition equation for $r(t)$, evaluated at a steady state:

$$F_{ss} =(\Omega-g_L)^{\frac{\omega}{1-\omega}}(\lambda+g_L)r_{ss}. \tag{A-30}$$

Substitute that into the prior expression and rearrange:

$$r_{ss} = \left(\frac{\phi_{MR}\kappa_E}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\frac{e_{ss}}{e_{Rss}}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}A_E e_{Rss}. \tag{A-31}$$

When $\omega > 0$, equation (A-31) shows that a steady state requires $\Omega > g_L$. Using $\sigma \neq 1$, substitute for r_{ss} in (A-31) from (A-30), substitute for F_{ss} from (5), and rearrange:

$$e_{Rss} = \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\frac{\sigma}{\sigma-1}}\frac{A_{LK}}{A_E}\left(\frac{1}{\frac{\lambda+g_L}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\phi_{MR}\frac{e_{ss}}{e_{Rss}}-1}\right)^{\frac{\sigma}{\sigma-1}}\ell_{Rss}^{\kappa_L}k_{Rss}^{\kappa_K}. \tag{A-32}$$

Rearrange (A-12) and (A-13) and evaluate at a steady state:

$$\ell_{Ess} =\frac{\phi_{ML}}{\kappa_L}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{k_{Rss}}{\ell_{Rss}}\right)^{-\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{\ell_{Rss}}\right)^{-\frac{1}{\sigma}}e_{ss},$$

$$k_{Ess} =\frac{\phi_{MK}}{\kappa_K}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{\ell_{Rss}}{k_{Rss}}\right)^{-\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{k_{Rss}}\right)^{-\frac{1}{\sigma}}e_{ss}.$$

Substitute into the per-capita version of (2) evaluated at the steady state and solve for e_{ss}:

$$e_{ss}^{\phi_{MR}} =Q_M r_{ss}^{\phi_{MR}}\left[\frac{\phi_{ML}}{\kappa_L}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{k_{Rss}}{\ell_{Rss}}\right)^{-\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{\ell_{Rss}}\right)^{-\frac{1}{\sigma}}\right]^{\phi_{ML}}$$
$$\left[\frac{\phi_{MK}}{\kappa_K}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{\ell_{Rss}}{k_{Rss}}\right)^{-\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{k_{Rss}}\right)^{-\frac{1}{\sigma}}\right]^{\phi_{MK}}.$$

Substitute for r_{ss} from (A-31):

$$\begin{aligned}\left(\frac{e_{ss}}{e_{Rss}}\right)^{\phi_{MR}} =& Q_M\left(\left(\frac{\phi_{MR}\kappa_E}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\frac{e_{ss}}{e_{Rss}}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}A_E\right)^{\phi_{MR}}\\ &\left[\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\right]^{\phi_{ML}+\phi_{MK}}\left[\frac{\phi_{ML}}{\kappa_L}\right]^{\phi_{ML}}\left[\frac{\phi_{MK}}{\kappa_K}\right]^{\phi_{MK}}\\ &\ell_{Rss}^{\frac{\kappa_K(\sigma-1)+1}{\sigma}\phi_{ML}-\kappa_L\frac{\sigma-1}{\sigma}\phi_{MK}}k_{Rss}^{\frac{\kappa_L(\sigma-1)+1}{\sigma}\phi_{MK}-\kappa_K\frac{\sigma-1}{\sigma}\phi_{ML}}e_{Rss}^{-\frac{1}{\sigma}(\phi_{ML}+\phi_{MK})}.\end{aligned}$$

Substitute for e_{Rss} from (A-32) and rearrange to obtain:

$$\begin{aligned}\frac{e_{ss}}{e_{Rss}} =& (\Omega-g_L)^{\frac{\omega}{1-\omega}(\sigma-1)}\left[(1-\kappa_E)A_{LK}^{\frac{\sigma-1}{\sigma}}\right]^{\sigma\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}\\ &\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_M\left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}}\left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}}\ell_{Rss}^{\kappa_K\phi_{ML}-\kappa_L\phi_{MK}}k_{Rss}^{\kappa_L\phi_{MK}-\kappa_K\phi_{ML}}\right]^{\frac{-(\sigma-1)}{\phi_{MR}}}\\ &\left(\frac{1}{\lambda+\rho}\phi_{MR}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\right)^{-\sigma}\\ &\left[\frac{1}{\frac{\lambda+g_L}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\phi_{MR}\frac{e_{ss}}{e_{Rss}}-1}\right]^{\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}.\end{aligned}$$

Apply Assumption 2 to eliminate ℓ_{Rss} and k_{Rss}:

$$\begin{aligned}\frac{e_{ss}}{e_{Rss}} =& (\Omega-g_L)^{\frac{\omega}{1-\omega}(\sigma-1)}\left[(1-\kappa_E)A_{LK}^{\frac{\sigma-1}{\sigma}}\right]^{\sigma\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_M\left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}}\left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}}\right]^{\frac{-(\sigma-1)}{\phi_{MR}}}\\ &\left(\frac{1}{\lambda+\rho}\phi_{MR}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\right)^{-\sigma}\\ &\left[\frac{1}{\frac{\lambda+g_L}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\phi_{MR}\frac{e_{ss}}{e_{Rss}}-1}\right]^{\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}.\end{aligned} \tag{A-33}$$

A strictly positive solution requires either $\omega=0$ or $\Omega>g_L$. If $e_{ss}/e_{Rss}\in\left(0,\frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)$, the left-hand side is strictly positive but the right-hand side is strictly negative, so the equation cannot hold. For $e_{ss}/e_{Rss}>\frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}$ and either $\omega=0$ or $\rho>\Omega$, the right-hand side of (A-33) monotonically decreases in e_{ss}/e_{Rss}, going to 0 as $e_{ss}/e_{Rss}\to\infty$ and going to infinity as $e_{ss}/e_{Rss}\to\frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}$ from above. The left-hand side monotonically increases in e_{ss}/e_{Rss} and is strictly positive. So when either $\omega = 0$ or $\Omega > g_L$, there is a unique $e_{ss}/e_{Rss} > 0$ that solves the equation. That solution has:

$$\frac{e_{ss}}{e_{Rss}} > \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}.$$

From (A-31), the steady state has $r_{ss} > 0$ if and only if $\Omega > g_L$. Because $e_{Yss} = e_{ss} - e_{Rss}$, the steady state has strictly positive final good production if and only if $e_{ss}/e_{Rss} > 1$, which holds because $\rho > \Omega, g_L$. From (A-20), $EROI(t)$ is constant along the balanced growth path and

$$EROI_{ss} > 1 + \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}g_L.$$

We have proved part i of the proposition.

The right-hand side of equation (A-33) increases in A_E and Q_M if $\sigma < 1$ and decreases in A_E and Q_M if $\sigma > 1$. So the e_{ss}/e_{Rss} that solves (A-33)—and, by (A-20), $EROI_{ss}$—increases in A_E and Q_M if $\sigma < 1$ and decreases in A_E and Q_M if $\sigma > 1$. We have proved parts ii and iii of the proposition.

The claims about prices are implied by the existence of steady states in per-capita, current value terms for $\gamma(t)$ and $\mu(t)$.

G Proof of Proposition 5

The setting of Proposition 5 matches those of Propositions 3 and 4 as $\phi_{MR} \to 1$ (which implies $\phi_{ML}, \phi_{MK} \to 0$) and with Q_D in place of Q_M. Lemma A-2 still applies here. Therefore any interior balanced growth path must have per-capita variables constant and prices growing at rate $\rho - g_L$.

The case with $\sigma = 1$ (i.e., part i of the proposition) follows from taking the limit as $\phi_{MR} \to 1$ in the proof of Proposition 3.

Now consider $\sigma \neq 1$. Using $e_{ss} = Q_D r_{ss}$ in equation (A-31) and taking $\phi_{MR} \to 1$ (and $\phi_{ML}, \phi_{MK} \to 0$) yields:

$$\frac{e_{ss}}{e_{Rss}} = \left(\frac{(\lambda+\rho)\left[(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)\right]}{(\rho-\Omega)(1-\omega)\kappa_E}\right)^{\sigma}\frac{[A_E Q_D]^{1-\sigma}}{(\lambda+g_L)(\Omega-g_L)^{\frac{\omega}{1-\omega}(1-\sigma)}}. \tag{A-34}$$

From (A-30), an interior solution requires either $\omega = 0$ or $\Omega > g_L$. In either case, e_{ss}/e_{Rss} is interior.

The steady state has strictly positive final good production if and only if $e_{ss}/e_{Rss} > 1$. From (A-34), $e_{ss}/e_{Rss} > 1$ if and only if

$$\kappa_E^{-\sigma}[A_E Q_D]^{1-\sigma} > (\lambda+g_L)^{1-\sigma}\left(\frac{\lambda+g_L}{\lambda+\rho}\right)^{\sigma}(\Omega-g_L)^{\frac{\omega}{1-\omega}(1-\sigma)}\left(\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\right)^{\sigma}. \tag{A-35}$$

For $\sigma < 1$, inequality (A-35) holds if and only if

$$A_E Q_D > (\lambda + g_L) \left(\frac{\lambda + \rho}{\kappa_E(\lambda + g_L)} \right)^{\frac{\sigma}{\sigma-1}} (\Omega - g_L)^{\frac{\omega}{1-\omega}} \left(\frac{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}{(1-\omega)(\rho - \Omega)} \right)^{\frac{\sigma}{\sigma-1}},$$

where the right-hand side is χ from the proposition. We have proved part ii of the proposition. For $\sigma > 1$, inequality (A-35) holds if and only if

$$A_E Q_D < (\lambda + g_L) \left(\frac{\lambda + \rho}{\kappa_E(\lambda + g_L)} \right)^{\frac{\sigma}{\sigma-1}} (\Omega - g_L)^{\frac{\omega}{1-\omega}} \left(\frac{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}{(1-\omega)(\rho - \Omega)} \right)^{\frac{\sigma}{\sigma-1}},$$

where the right-hand side is χ from the proposition. We have proved part iii of the proposition.

$EROI(t)$ along a balanced growth path follows from equations (A-20) and (A-34). We have proved part iv of the proposition.

H Proof of Proposition 7

Consider a case with $\ell_R(t), k_R(t) = 0$ and $e_R(t) > 0$. By labor and capital market-clearing, $\ell_Y(t) = 1$ and $k_Y(t) = k(t)$. Equilibrium then solves the following maximization problem:

$$\begin{aligned}
\max_{E_Y(\cdot), E_R(\cdot)} & \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s) A L(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)} \right) \mathrm{d}t \\
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& s A L(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{R}(t) =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t) - \lambda R(t) \\
E(t) =& Q_S R(t) \\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}
\max_{e_Y(\cdot)} & \int_0^\infty e^{-(\rho - g_L)t} u\Big((1-s) A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} \Big) \mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& s A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} - (\delta + g_L) k(t) \\
\dot{r}(t) =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_S r(t) - e_Y(t)] - (\lambda + g_L) r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}
& u\Big((1-s) A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} \Big) + \nu(t) \Big(s A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} - (\delta + g_L) k(t) \Big) \\
& + \mu(t) \Big[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_S r(t) - e_Y(t)] - (\lambda + g_L) r(t) \Big].
\end{aligned}$$

The costate equations are:

$$
\begin{aligned}
(\rho - g_L)\nu(t) - \dot{\nu}(t) =& \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_K y(t)}{k(t)} - (\delta + g_L)\nu(t) \\
\Leftrightarrow \dot{\nu}(t) =& -\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_K y(t)}{k(t)} + (\rho+\delta)\nu(t), \qquad \text{(A-36)} \\
(\rho - g_L)\mu(t) - \dot{\mu}(t) =& \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - \mu(t)(\lambda + g_L) \\
\Leftrightarrow \dot{\mu}(t) =& \mu(t)\left[\rho + \lambda - \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S\right]. \qquad \text{(A-37)}
\end{aligned}
$$

The condition to maximize the Hamiltonian is:

$$
0 = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{e_Y(t)} - \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E. \qquad \text{(A-38)}
$$

Substituting into (A-36), we obtain:

$$
\dot{\nu}(t) = -\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\alpha_K e_Y(t)}{\alpha_E k(t)} + (\rho+\delta)\nu(t). \qquad \text{(A-39)}
$$

Time-differentiating (A-38), we obtain:

$$
\begin{aligned}
0 =& \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\alpha_E \left[\frac{\dot{y}(t)}{e_Y(t)} - \frac{y(t)}{e_Y(t)}\frac{\dot{e}_Y(t)}{e_Y(t)}\right] \\
&+ (1-s)^2 u''((1-s)y(t))\frac{\alpha_E y(t)}{e_Y(t)}\dot{y}(t) + s\dot{\nu}(t)\frac{\alpha_E y(t)}{e_Y(t)} - \dot{\mu}(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E.
\end{aligned}
$$

Substitute from (A-37), (A-38), and (A-39):

$$
\begin{aligned}
0 =& \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[\frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}\right] + \alpha_E(1-s)^2 u''((1-s)y(t))\frac{y(t)}{e_Y(t)}\dot{y}(t) \\
&+ s y(t)\left[-\alpha_K \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\mu(t)}{k(t)} + \alpha_E(\rho+\delta)\frac{\nu(t)}{e_Y(t)}\right] - \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[(\rho+\lambda) - \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S\right].
\end{aligned}
$$

Time-differentiate the final good production function and substitute for $\dot{y}(t)$ in the previous equation:

$$
\begin{aligned}
0 =& \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[\frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}\right] + \alpha_E(1-s)^2 u''((1-s)y(t))\frac{y(t)}{e_Y(t)} y(t)\left[\alpha_K \frac{\dot{k}(t)}{k(t)} + \alpha_E \frac{\dot{e}_Y(t)}{e_Y(t)}\right] \\
&+ s y(t)\left[-\alpha_K \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\mu(t)}{k(t)} + \alpha_E(\rho+\delta)\frac{\nu(t)}{e_Y(t)}\right] - \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[(\rho+\lambda) - \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S\right].
\end{aligned}
$$

Use Assumption 3 and rearrange:

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}-(\rho+\lambda)+\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-s\alpha_K\frac{y(t)}{k(t)}\right]\\&-\alpha_E\frac{1}{e_Y(t)}\left[\alpha_K\frac{\dot{k}(t)}{k(t)}+\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}\right]+\nu(t)s\alpha_E(\rho+\delta)\frac{y(t)}{e_Y(t)}.\end{aligned} \tag{A-40}$$

Now consider a balanced growth path. $\dot{k}(t)/k(t)$ is constant over time if and only if

$$\frac{y(t)}{k(t)} =\frac{g_k+\delta+g_L}{s} \tag{A-41}$$

Therefore $g_k = g_y$, which, in the final good production function, implies that

$$\begin{aligned}g_y =&\alpha_K g_k+\alpha_E g_{e_Y}\\ \Leftrightarrow g_y =&\frac{\alpha_E}{\alpha_E+\alpha_L}g_{e_Y}.\end{aligned}$$

The resource transition equation implies:

$$\frac{\dot{r}(t)}{r(t)} =\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[Q_S-\frac{e_Y(t)}{r(t)}\right]-(\lambda+g_L).$$

The growth rate of r is constant if and only if $e_Y(t)$ grows at g_r. The growth rate of r is the same as the growth rate of e. Therefore $e_R(t)$ grows at rate g_e and

$$g_y =\frac{\alpha_E}{\alpha_E+\alpha_L}g_e. \tag{A-42}$$

Using this and $g_k = g_y$ in (A-41),

$$\frac{y(t)}{k(t)} =\frac{\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\delta+g_L}{s} \tag{A-43}$$

Use $g_k = g_y$ in (A-40) and substitute from (A-42) and (A-43):

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[-\frac{\alpha_L}{\alpha_E+\alpha_L}g_e-(\rho+\lambda)+\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-\alpha_K\left(\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\delta+g_L\right)\right]\\&-\alpha_E\frac{1}{e_Y(t)}\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\nu(t)s\alpha_E(\rho+\delta)\frac{y(t)}{e_Y(t)}.\end{aligned}$$

Multiply through by $e_Y(t)$ and multiply the final term by $k(t)/k(t)$, again using (A-43):

$$\begin{aligned}0 =&\mu(t)e_Y(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[-\frac{\alpha_L}{\alpha_E+\alpha_L}g_e-(\rho+\lambda)+\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-\alpha_K\left(\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\delta+g_L\right)\right]\\&-\alpha_E\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\nu(t)k(t)\alpha_E(\rho+\delta)\left(\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\delta+g_L\right).\end{aligned} \tag{A-44}$$

$\nu(t)k(t)$ is constant if and only if $\dot{\nu}(t)/\nu(t) = -g_k$. In that case, (A-39) implies

$$\nu(t)k(t) = \mu(t)e_Y(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\alpha_K}{\alpha_E} \frac{1}{g_k + \rho + \delta}. \tag{A-45}$$

Substitute into (A-44) and use (A-42):

$$0 = \mu(t)e_Y(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[-\frac{\alpha_L}{\alpha_E + \alpha_L} g_e - (\rho + \lambda) + \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L} g_e + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L} g_e + \rho + \delta} \frac{\alpha_E}{\alpha_E + \alpha_L} g_e \right] - \alpha_E \frac{\alpha_E}{\alpha_E + \alpha_L} g_e. \tag{A-46}$$

If g_e is constant, so too is $\mu(t)e_Y(t)$. From (A-37), $\mu(t)$ grows at rate $\rho + \lambda - \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S$, so $\mu(t)e_Y(t)$ constant implies:

$$g_e = \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho + \lambda). \tag{A-47}$$

Substituting into (A-46), we find:

$$0 = \frac{\alpha_E}{\alpha_E + \alpha_L} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho + \lambda) \right) \left\{ \mu(t)e_Y(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[1 - \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho + \lambda) \right] + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho + \lambda) \right] + \rho + \delta} \right] - \alpha_E \right\}.$$

This holds if either $g_e = 0$ or

$$\mu(t)e_Y(t) = \frac{\alpha_E}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[1 - \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho+\lambda) \right] + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho+\lambda) \right] + \rho + \delta} \right]}.$$

The right-hand side is strictly positive for $g_e \neq 0$ if and only if:

$$1 > \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho + \lambda) \right] + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho + \lambda) \right] + \rho + \delta}. \tag{A-48}$$

Observe that $y(t)/k(t) > 0$ requires, from (A-43), that the numerator on the right-hand side of (A-48) be strictly positive and that $\nu(t)k(t) > 0$ requires, from (A-45), that the denominator on the right-hand side of (A-48) be strictly positive. Using $\rho > g_L$, these two conditions are jointly satisfied if and only if the numerator in (A-48) is strictly positive, so if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S > \rho + \lambda - \frac{\alpha_E + \alpha_L}{\alpha_E}(g_L + \delta). \tag{A-49}$$

If this last inequality holds, then inequality (A-48) also holds, because $\alpha_K < 1$. If $\mu(t)e_Y(t)$ is weakly negative, then (A-48) would not hold and so (A-49) would not hold, which means that the path is not feasible. So $\mu(t)e_Y(t) > 0$ on a feasible path, as would be expected. Finally, observe that $r(t)$ grows at a constant rate if and only if

$$\frac{e_R(t)}{e(t)} = \frac{g_e + \lambda + g_L}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S}. \tag{A-50}$$

Substituting from (A-47), we have $e_R(t)/e(t) \in (0, 1)$ if and only if

$$\frac{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho - g_L)}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S} \in (0, 1).$$

Because $\rho > g_L$, this condition holds if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S > \rho - g_L. \tag{A-51}$$

We have found a feasible path along which $\ell_R(t), k_R(t) = 0$ with $e_R(t) > 0$ and all variables growing at a constant rate. If $A_E Q_S > \chi$, then $\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S > \rho + \lambda$ (observing that $(\lambda + g_L)[(\lambda + \rho)/(\lambda + g_L)]^{\frac{\sigma}{\sigma-1}}$ decreases in σ for $\sigma > 1$) and thus inequalities (A-49) and (A-51) hold. Part i of the proposition follows from $g_y = g_k$ and (A-42). Part ii follows from (A-47). Part iii follows from (7), (A-47), and (A-50).

I Proof of Proposition 8

Equilibrium solves the following maximization problem:

$$\begin{aligned}
\max_{L_Y(\cdot),K_Y(\cdot),E_Y(\cdot),E_R(\cdot),E_B(\cdot),B_R(\cdot),B_B(\cdot)} &\int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s)AL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t\\
\text{s.t. } \dot{L}(t) =& g_L L(t)\\
\dot{K}(t) =& sAL(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E} - \delta K(t)\\
\dot{R}(t) =& A_E B_R(t)^{\kappa_B} E_R(t)^{\kappa_E} - \lambda R(t)\\
\dot{B}(t) =& A_B B_B(t)^{\beta_B} E_B(t)^{\beta_E} - \Psi B(t)\\
E(t) =& Q_S R(t)\\
L(t) =& L_Y(t)\\
K(t) =& K_Y(t)\\
E(t) =& E_Y(t) + E_R(t) + E_B(t)\\
B(t) =& B_B(t) + B_R(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}
\max_{e_Y(\cdot),e_R(\cdot),b_R(\cdot)} &\int_0^\infty e^{-(\rho-g_L)t} u\left((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\right) \mathrm{d}t\\
\text{s.t. } \dot{k}(t) =& sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta+g_L)k(t)\\
\dot{r}(t) =& A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - (\lambda+g_L)r(t)\\
\dot{b}(t) =& A_B[b(t)-b_R(t)]^{\beta_B}[Q_S r(t) - e_Y(t) - e_R(t)]^{\beta_E} - (\Psi+g_L)b(t).
\end{aligned}$$

$\dot{k}(t)/k(t)$ is constant over time if and only if $y(t)/K(t)$ is constant over time. Therefore $g_k = g_y$, which, in the final good production function, implies that

$$\begin{aligned}
g_y =& \alpha_K g_k + \alpha_E g_{e_Y}\\
\Leftrightarrow g_y =& \frac{\alpha_E}{\alpha_E+\alpha_L} g_{e_Y}.
\end{aligned}$$

We have established part i of the proposition.

The current-value Hamiltonian is:

$$\begin{aligned}
&u\left((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\right) + \nu(t)\left(sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta+g_L)k(t)\right)\\
&+\mu(t)\left[A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - (\lambda+g_L)r(t)\right]\\
&+\Upsilon(t)\left[A_B[b(t)-b_R(t)]^{\beta_B}[Q_S r(t) - e_Y(t) - e_R(t)]^{\beta_E} - (\Psi+g_L)b(t)\right].
\end{aligned}$$

The costate equations are:

$$
\begin{aligned}
(\rho - g_L)\nu(t) - \dot{\nu}(t) =& \alpha_K(1-s)u'((1-s)y(t))y(t)/k(t) + \alpha_K s\nu(t)y(t)/k(t) - (\delta + g_L)\nu(t) \\
\Leftrightarrow \dot{\nu}(t) =& -\frac{\alpha_K y(t)}{k(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] + (\rho+\delta)\nu(t), \qquad \text{(A-52)} \\
(\rho - g_L)\mu(t) - \dot{\mu}(t) =& Q_S\beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} - \mu(t)(\lambda + g_L) \\
\Leftrightarrow \dot{\mu}(t) =& -Q_S\beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} + \mu(t)[\rho+\lambda], \qquad \text{(A-53)} \\
(\rho - g_L)\Upsilon(t) - \dot{\Upsilon}(t) =& \beta_B\frac{\Upsilon(t)}{b_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} - \Upsilon(t)(\Psi + g_L) \\
\Leftrightarrow \dot{\Upsilon}(t) =& -\beta_B\frac{\Upsilon(t)}{b_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} + \Upsilon(t)[\rho+\Psi]. \qquad \text{(A-54)}
\end{aligned}
$$

The conditions to maximize the Hamiltonian are:

$$
0 = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{e_Y(t)} - \beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E}, \qquad \text{(A-55)}
$$

$$
0 = \kappa_E\frac{\mu(t)}{e_R(t)}A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - \beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E}, \qquad \text{(A-56)}
$$

$$
0 = \kappa_B\frac{\mu(t)}{b_R(t)}A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - \beta_B\frac{\Upsilon(t)}{b_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E}. \qquad \text{(A-57)}
$$

Equations (A-56) and (A-57) imply:

$$
\frac{\kappa_E b_R(t)}{\kappa_B e_R(t)} = \frac{\beta_E b_B(t)}{\beta_B e_B(t)}. \qquad \text{(A-58)}
$$

Rearrange equation (A-55):

$$
\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{\beta_E e_Y(t)}. \qquad \text{(A-59)}
$$

Time-differentiating yields:

$$
\begin{aligned}
&\left[\frac{\dot{\Upsilon}(t)}{\Upsilon(t)} - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B\frac{\dot{b}_B(t)}{b_B(t)} + \beta_E\frac{\dot{e}_B(t)}{e_B(t)}\right]\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} \\
=& \left[(1-s)^2\dot{y}(t)u''((1-s)y(t)) + s\dot{\nu}(t)\right]\frac{\alpha_E y(t)}{\beta_E e_Y(t)} \\
&+ \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{\beta_E e_Y(t)}\left[\frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}\right].
\end{aligned}
$$

Substitute from (A-52), (A-54), and (A-59):

$$
\begin{aligned}
&-\beta_B A_B \left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E} + \rho + \Psi - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B \frac{\dot{b}_B(t)}{b_B(t)} + \beta_E \frac{\dot{e}_B(t)}{e_B(t)} \\
=&\frac{(1-s)^2 \dot{y}(t) u''((1-s)y(t)) + s(\rho+\delta)\nu(t)}{(1-s)u'((1-s)y(t)) + s\nu(t)} - s\frac{\alpha_K y(t)}{k(t)} + \frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}.
\end{aligned}
$$

Substitute from the transition equation for $b(t)$, for $\nu(t)$ from (A-55), and for $\Upsilon(t)$ from (A-56):

$$
\begin{aligned}
&-\left[\frac{\dot{b}(t)}{b(t)} + \Psi + g_L\right] \beta_B \frac{b(t)}{b_B(t)} + \rho + \Psi - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B \frac{\dot{b}_B(t)}{b_B(t)} + \beta_E \frac{\dot{e}_B(t)}{e_B(t)} \\
=&\frac{(1-s)^2 \dot{y}(t) u''((1-s)y(t)) + (\rho+\delta)\frac{e_Y(t)}{\alpha_E y(t)} \kappa_E \mu(t) A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} - (\rho+\delta)(1-s)u'((1-s)y(t))}{\frac{e_Y(t)}{\alpha_E y(t)} \kappa_E \mu(t) A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}} \\
&- s\frac{\alpha_K y(t)}{k(t)} + \frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}.
\end{aligned}
$$

Use Assumption 3 and rearrange:

$$
\begin{aligned}
&\frac{\kappa_E}{\alpha_E}\mu(t) e_Y(t) A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} \\
&\left[-\left(\frac{\dot{b}(t)}{b(t)} + \Psi + g_L\right)\beta_B \frac{b(t)}{b_B(t)} + \Psi - \delta - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B \frac{\dot{b}_B(t)}{b_B(t)} + \beta_E \frac{\dot{e}_B(t)}{e_B(t)} + s\frac{\alpha_K y(t)}{k(t)} - \frac{\dot{y}(t)}{y(t)} + \frac{\dot{e}_Y(t)}{e_Y(t)}\right] \\
=&-\left(\frac{\dot{y}(t)}{y(t)} + \rho + \delta\right).
\end{aligned}
$$

Substitute constant growth rates, recognizing that $e(t)$ and $b(t)$ must grow at the same rate g_e:

$$
\begin{aligned}
&\frac{\kappa_E}{\alpha_E}\mu(t) e_Y(t) A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} \left[-\left(g_e + \Psi + g_L\right)\beta_B \frac{b(t)}{b_B(t)} + \Psi - \delta + s\frac{\alpha_K y(t)}{k(t)} - g_y + g_e\right] \\
&= -[g_y + \rho + \delta].
\end{aligned}
$$

Substitute for $y(t)/k(t)$ from the transition equation for $k(t)$ and use $g_y = g_k = \frac{\alpha_E}{1-\alpha_K} g_e$:

$$
\begin{aligned}
&\frac{\kappa_E}{\alpha_E}\mu(t) e_Y(t) A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} \left[-\left(g_e + \Psi + g_L\right)\beta_B \frac{b(t)}{b_B(t)} + \Psi - \delta + \alpha_K\left(\delta + g_L\right) + (1-\alpha_E) g_E\right] \\
=&-\left[\frac{\alpha_E}{1-\alpha_K} g_e + \rho + \delta\right].
\end{aligned}
$$

Substitute from (A-58):

$$\frac{\kappa_E}{\alpha_E}\mu(t)e_Y(t)A_E\left(\frac{\kappa_B\beta_E b_B(t)}{\kappa_E\beta_B e_B(t)}\right)^{\kappa_B}$$
$$\left[-\left(g_e+\Psi+g_L\right)\beta_B\frac{b(t)}{b_B(t)}+\Psi-\delta+\alpha_K\left(\delta+g_L\right)+(1-\alpha_E)g_E\right]$$
$$=-\left[\frac{\alpha_E}{1-\alpha_K}g_e+\rho+\delta\right]. \tag{A-60}$$

For this to hold, $\mu(t)e_Y(t)$ must be constant. Therefore $g_\mu=-g_e$. Use this in (A-53):

$$g_e+\rho+\lambda = Q_S\beta_E\frac{\Upsilon(t)}{\mu(t)}A_B\left(\frac{b_B(t)}{e_B(t)}\right)^{\beta_B}. \tag{A-61}$$

For this to hold, $\Upsilon(t)$ must grow at the same rate as $\mu(t)$. Use that in (A-54) and rearrange:

$$\frac{e_B(t)}{b_B(t)}=\left(\frac{g_e+\rho+\Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}. \tag{A-62}$$

Equation (A-56) implies:

$$\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E} = \frac{\kappa_E}{\beta_E}\frac{\mu(t)}{e_R(t)}A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E}.$$

Substitute from (A-61) and (A-58), and then substitute from (A-62):

$$g_e+\rho+\lambda = \kappa_E A_E Q_S\left(\frac{\kappa_B\beta_E}{\kappa_E\beta_B}\left(\frac{g_e+\rho+\Psi}{\beta_B A_B}\right)^{-\frac{1}{\beta_E}}\right)^{\kappa_B}. \tag{A-63}$$

The left-hand side monotonically increases in g_e, and the right-hand side monotonically decreases in g_e. As $g_e\to\infty$, the left-hand side goes to ∞ and the right-hand side goes to zero. As $g_e\to-(\rho+\Psi)$ from above, the left-hand side goes to a finite value and the right-hand side goes to $-\infty$. So there is exactly one intersection at some $g_e\in\big(-[\rho+\Psi],\infty\big)$. That intersection has $g_e>0$ if and only if the right-hand side is greater than the left-hand side at $g_e=0$, so if and only if

$$\kappa_E A_E Q_S > (\rho+\lambda)\left(\frac{\kappa_E\beta_B}{\kappa_B\beta_E}\left(\frac{\rho+\Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}\right)^{\kappa_B}.$$

This is equivalent to the condition on χ_0 given in the proposition. Because the right-hand side of (A-63) increases in Q_S while the left-hand side of (A-63) is independent of Q_S, g_e

increases in Q_S. Similar analysis holds that g_e increases in A_E and A_B and decreases in Ψ, ρ, and λ. We have established part ii of the proposition.

We established that $\mu(t)$ and $\Upsilon(t)$ grow at rate $-g_e$. Therefore the current-value prices of $R(t)$ and $B(t)$ each grow at rate $\rho - (g_e + g_L)$. These prices increase over time if and only if $g_e < \rho - g_L$. Part iii of the proposition follows from that observation and, from (A-63), that g_e increases in $A_E Q_S$.

Now consider the feasibility of the solution. From the transition equation for $r(t)$,

$$g_e = Q_S A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} \frac{e_R(t)}{e(t)} - (\lambda + g_L).$$

Substitute from (A-58):

$$g_e = Q_S A_E \left(\frac{\kappa_B \beta_E b_B(t)}{\kappa_E \beta_B e_B(t)}\right)^{\kappa_B} \frac{e_R(t)}{e(t)} - (\lambda + g_L).$$

Rearrange:

$$\frac{e(t)}{e_R(t)} = Q_S A_E \left(\frac{\kappa_B \beta_E b_B(t)}{\kappa_E \beta_B e_B(t)}\right)^{\kappa_B} \frac{1}{g_e + \lambda + g_L}.$$

And substitute from (A-62):

$$\frac{e(t)}{e_R(t)} = Q_S A_E \left(\frac{\kappa_B \beta_E}{\kappa_E \beta_B} \left(\frac{g_e + \rho + \Psi}{\beta_B A_B}\right)^{\frac{-1}{\beta_E}}\right)^{\kappa_B} \frac{1}{g_e + \lambda + g_L}. \tag{A-64}$$

Feasibility requires $g_e > -(\lambda + g_L)$ and

$$g_e + \lambda + g_L < Q_S A_E \left(\frac{\kappa_B \beta_E}{\kappa_E \beta_B} \left(\frac{g_e + \rho + \Psi}{\beta_B A_B}\right)^{\frac{-1}{\beta_E}}\right)^{\kappa_B}.$$

Substituting from (A-63), this is equivalent to:

$$g_e > -\frac{1}{1-\kappa_E}(\rho - \kappa_E g_L) - \lambda.$$

This condition is satisfied when $g_e > 0$.

From the transition equation for $b(t)$,

$$g_e = A_B \left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E} \frac{b_B(t)}{b(t)} - (\Psi + g_L).$$

Rearrange:

$$\frac{b(t)}{b_B(t)} = A_B \left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E} \frac{1}{g_e + \Psi + g_L}.$$

And substitute from (A-62):

$$\frac{b(t)}{b_B(t)} = \frac{g_e + \rho + \Psi}{\beta_B} \frac{1}{g_e + \Psi + g_L}. \tag{A-65}$$

Feasibility on the robot side requires

$$1 < \frac{1}{\beta_B} \frac{g_e + \Psi + \rho}{g_e + \Psi + g_L},$$

while holds for all $g_e > 0$.

Substitute (A-64) into (7):

$$EROI(t) = 1 + A_E Q_S \left(\frac{\kappa_B \beta_E}{\kappa_E \beta_B} \left(\frac{g_e + \rho + \Psi}{\beta_B A_B} \right)^{\frac{-1}{\beta_E}} \right)^{\kappa_B} \frac{g_e + g_L}{g_e + \lambda + g_L}.$$

Substitute from (A-63):

$$EROI(t) = 1 + \frac{1}{\kappa_E} \frac{g_e + g_L}{g_e + \lambda + g_L} (g_e + \rho + \lambda). \tag{A-66}$$

$EROI(t)$ depends on Q_S, A_E, and A_B only through g_e in (A-66). $EROI(t)$ increases in g_e in (A-66) because the fraction and the final term both increase in g_e. Part iv of the proposition follows from this observation and part ii. Part v also follows from using $g_e = 0$ in (A-66) and recalling that $EROI(t)$ increases in g_e in (A-66).

J Proof of Proposition 9

In a biomass economy in which Assumption 1 holds, Proposition 1 showed that there is no balanced growth path with output per capita growing forever at a strictly positive rate. Therefore output per capita in any economy using the biomass resource is bounded above. Define $\bar{Y}_H$ as the least upper bound on output in the biomass economy. Assume $\xi_D \geq \bar{Y}_H$. In that case, an economy that uses the biomass resource cannot begin accessing the oil resource or, because $\xi_S > \xi_D$, the solar resource. We have established part i of the proposition.

If $\xi_M < \bar{Y}_H$, then it is feasible to develop the coal resource. From Proposition 3, there exists a balanced growth path in the economy with the coal resource in which all variables grow at the rate of population. Denote output per capita along this path as y_{ss}^M. Welfare on the balanced growth path is $u(s\, y_{ss}^M)/(\rho - g_L)$. Using Assumption 3, welfare can be made arbitrarily large by making y_{ss}^M arbitrarily large. From Proposition 3, y_{ss}^M increases with Q_M in constant elasticity fashion, so y_{ss}^M and $u(s\, y_{ss}^M)$ can be made arbitrarily large by making Q_M arbitrarily large. Welfare at the time of first developing a coal resource may be lower than welfare along the balanced growth path but can also be made arbitrarily large by making Q_M arbitrarily large. In contrast, from Proposition 1, welfare in the biomass economy is bounded from above by $u(s\,\bar{Y})/(\rho - g_L)$, which is independent of Q_M. Therefore, the economy finds it optimal and feasible to eventually develop the coal resource if Q_M is sufficiently large and ξ_M is sufficiently small. We have established part ii of the proposition.

Now consider a case in which $y_{ss}^M \in (\xi_D, \xi_S)$, so that it is feasible to develop the oil resource from an economy that uses the coal resource. From Proposition 5, there exists a balanced growth path in the economy with the oil resource in which all variables grow at the rate of population. Denote output per capita along this path as y_{ss}^D. Following the argument of part ii, Q_D can be made large enough to make it optimal to develop the oil resource from an economy that uses the coal resource. Therefore, an economy that uses the coal resource finds it optimal and feasible to eventually develop the oil resource if Q_D is sufficiently large and ξ_D is not too large. We have established part iii of the proposition.

Now consider a case in which $y_{ss}^D > \xi_S$, so that it is feasible to develop the solar resource from an economy that uses the oil resource. Consider first a manufactured solar resource. From Corollary 6, there exists a balanced growth path in the economy with the solar resource in which all variables grow at the rate of population. Denote output per capita along this path as y_{ss}^S. Following the argument of part ii, Q_S can be made large enough to make it optimal to develop the solar resource from an economy that uses the oil resource. Therefore, an economy that uses the oil resource finds it optimal and feasible to eventually develop the solar resource if Q_S is sufficiently large and ξ_S is not too large. We have established part iv of the proposition for a manufactured solar resource.

For a self-replicating solar resource, observe from Proposition 8 that the growth rate increases in Q_S and, from (A-63), that it does so without bound. Welfare at the time of first developing a solar resource may be lower but can be made arbitrarily large by making Q_S arbitrarily large. Therefore, an economy that uses the oil resource finds it optimal and feasible to eventually develop the solar resource if Q_S is sufficiently large and ξ_S is not too large. We have established part iv of the proposition for a self-replicating solar resource.

K Proof of Proposition A-1

Consider a case with $\ell_R(t), k_R(t) = 0$ and $e_R(t) > 0$. By labor and capital market-clearing, $\ell_Y(t) = 1$ and $k_Y(t) = k(t)$. Equilibrium solves the following maximization problem:

$$\begin{aligned}
&\max_{E_Y(\cdot)} \int_0^\infty e^{-\rho t} L(t) u\left((1-s)\frac{AL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{Z}(t) =& \Omega Z(t) - Z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t)\right)^{1-\omega} \\
\dot{R}(t) =& Z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t)\right)^{1-\omega} - \lambda R(t) \\
E(t) =& Q_D R(t) \\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita, substituting, and applying Assumption 3, this is equivalent to:

$$\begin{aligned}
&\max_{e_Y(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} \ln\left((1-s)Ak(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right) \mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} - (\delta + g_L)k(t) \\
\dot{z}(t) =& (\Omega - g_L)z(t) - z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \\
\dot{r}(t) =& z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} - (\lambda + g_L) r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}
&\ln\left((1-s)Ak(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right) + \nu(t)\left[sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} - (\delta + g_L)k(t)\right] \\
&+ \gamma(t)\left[(\Omega - g_L)z(t) - z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega}\right] \\
&+ \mu(t)\left[z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} - (\lambda + g_L) r(t)\right].
\end{aligned}$$

The costate equations are:

$$\begin{aligned}(\rho - g_L)\nu(t) - \dot{\nu}(t) =& \alpha_K k(t)^{-1} + \alpha_K \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E} k(t)^{-1} - \nu(t)[\delta + g_L] \\ \Leftrightarrow \dot{\nu}(t) =& (\rho + \delta)\nu(t) - \alpha_K k(t)^{-1} - \alpha_K \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E} k(t)^{-1} \qquad \text{(A-67)} \\ (\rho - g_L)\gamma(t) - \dot{\gamma}(t) =& \gamma(t)(\Omega - g_L) + [\mu(t) - \gamma(t)]\omega z(t)^{\omega-1} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \\ \Leftrightarrow \dot{\gamma}(t) =& \gamma(t)(\rho - \Omega) - [\mu(t) - \gamma(t)]\omega z(t)^{\omega-1} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \\ & \qquad \text{(A-68)} \\ (\rho - g_L)\mu(t) - \dot{\mu}(t) =& -\mu(t)(\lambda + g_L) + [\mu(t) - \gamma(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D \\ \Leftrightarrow \dot{\mu}(t) =& \mu(t)(\lambda + \rho) - [\mu(t) - \gamma(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D. \\ & \qquad \text{(A-69)}\end{aligned}$$

The condition to maximize the Hamiltonian is:

$$\begin{aligned}0 =& \alpha_E e_Y(t)^{-1} - [\mu(t) - \gamma(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \\ &+ \alpha_E \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E} e_Y(t)^{-1}. \qquad \text{(A-70)}\end{aligned}$$

Time-differentiate (A-70):

$$\begin{aligned}0 =& -\alpha_E e_Y(t)^{-1} \frac{\dot{e}_Y(t)}{e_Y(t)} - [\dot{\mu}(t) - \dot{\gamma}(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \\ &- [\mu(t) - \gamma(t)]\omega(1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \\ &\quad \left\{\frac{\dot{z}(t)}{z(t)} - \frac{Q_D \dot{r}(t) - \dot{e}_Y(t)}{Q_D r(t) - e_Y(t)}\right\} \\ &+ \alpha_E \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E} e_Y(t)^{-1} \left\{\frac{\dot{\nu}(t)}{\nu(t)} + \alpha_K \frac{\dot{k}(t)}{k(t)} - (1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right\}\end{aligned}$$

Substitute from the transition equation for $k(t)$ and from equations (A-67) and (A-70) and rearrange:

$$\begin{aligned}&\alpha_E e_Y(t)^{-1} \frac{1}{1-\omega} \frac{\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{\omega}}{z(t)^{\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E} \left\{\alpha_E \frac{\dot{e}_Y(t)}{e_Y(t)} + (\rho - g_L) - \alpha_K k(t)^{-1}\nu(t)^{-1}\right\} \\ =& -[\dot{\mu}(t) - \dot{\gamma}(t)] \\ &- [\mu(t) - \gamma(t)]\left\{\omega \frac{\dot{z}(t)}{z(t)} - \omega \frac{Q_D \dot{r}(t) - \dot{e}_Y(t)}{Q_D r(t) - e_Y(t)}\right. \\ &\qquad \left. - (\rho - g_L) - (1-\alpha_K)(\delta + g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1} + (1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right\}.\end{aligned}$$

Substitute from the transition equations for $r(t)$ and $z(t)$ and solve for $\dot{\gamma}(t)$: for $\dot{\gamma}(t)$:

$$\begin{aligned}\dot{\gamma}(t) =&\dot{\mu}(t)\\ &+\alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^\omega\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}}\left\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\right\}\\ &+[\mu(t)-\gamma(t)]\Bigg\{\omega(\Omega-g_L)-\omega z(t)^\omega\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\left(\frac{1}{z(t)}+\frac{Q_D}{e_R(t)}\right)\\ &\qquad+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)}+\omega\frac{\dot{e}_Y(t)}{e_R(t)}-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}\\ &\qquad+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\Bigg\}.\end{aligned}$$

Substitute for $\gamma(t)$ from (A-70):

$$\begin{aligned}\dot{\gamma}(t) =&\dot{\mu}(t)\\ &+\alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^\omega\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}}\\ &\Bigg\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\\ &\quad+\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]\\ &\quad\Bigg[\omega(\Omega-g_L)-\omega z(t)^\omega\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\left(\frac{1}{z(t)}+\frac{Q_D}{e_R(t)}\right)+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)}\\ &\qquad+\omega\frac{\dot{e}_Y(t)}{e_R(t)}-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\Bigg]\Bigg\}.\end{aligned} \tag{A-71}$$

Substitute for $\gamma(t)$ from (A-70) into (A-69):

$$\dot{\mu}(t) =\mu(t)(\lambda+\rho)-\alpha_E e_Y(t)^{-1}Q_D\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]. \tag{A-72}$$

Substitute $\dot{\mu}(t)$ into (A-71) and rearrange:

$$\begin{aligned}
\dot{\gamma}(t) =& \mu(t)(\lambda+\rho) - \alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\left(\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]Q_D + \omega\frac{e_R(t)}{z(t)}\right) \\
&+ \alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega}} \\
&\left\{\omega(\Omega-g_L) + \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} + \omega\frac{\dot{e}_Y(t)}{e_R(t)} + \frac{\dot{e}_Y(t)}{e_Y(t)}\right. \\
&\quad + \nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E} \\
&\qquad \left[\omega(\Omega-g_L) - \omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\frac{1}{z(t)}\right. \\
&\qquad + \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} + \omega\frac{\dot{e}_Y(t)}{e_R(t)} - (\rho-g_L) - (1-\alpha_K)(\delta+g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1} \\
&\qquad \left.\left. + (1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\right\}.
\end{aligned} \tag{A-73}$$

Solve (A-70) for $\gamma(t)$, substitute that and $\dot{\gamma}(t)$ from (A-73) into (A-68), and rearrange:

$$\begin{aligned}
&\frac{\dot{e}_Y(t)}{e_Y(t)}\left\{1+\omega\frac{e_Y(t)}{e_R(t)} + \nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\left[(1-\alpha_E)+\omega\frac{e_Y(t)}{e_R(t)}\right]\right\} \\
=& -\frac{1}{\alpha_E}e_Y(t)\mu(t)\,(1-\omega)(\lambda+\Omega)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left(\frac{z(t)}{\kappa_E^{\frac{\sigma}{\sigma-1}}A_E e_R(t)}\right)^{\omega} \\
&+ \left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]Q_D\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left(\frac{z(t)}{\kappa_E^{\frac{\sigma}{\sigma-1}}A_E e_R(t)}\right)^{\omega} \\
&- (\rho-\Omega) - \omega(\Omega-g_L) - \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} \\
&- \nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E} \\
&\quad \left[\rho-\Omega+\omega(\Omega-g_L)+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)} - (\rho-g_L) - (1-\alpha_K)(\delta+g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1}\right]
\end{aligned} \tag{A-74}$$

Now consider a balanced growth path. From its transition equation, $k(t)$ grows at a constant rate if and only if $y(t)/k(t)$ is constant. So $g_k = g_y$. Substituting into the final good production function,

$$g_y = \frac{\alpha_E}{\alpha_E+\alpha_L}g_e. \tag{A-75}$$

From its transition equation, $z(t)$ grows at constant rate g_e if and only if:

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{e_R(t)}{z(t)} = (\Omega - g_L - g_e)^{\frac{1}{1-\omega}} . \tag{A-76}$$

Using this and the transition equation for $r(t)$, $r(t)$ grows at constant rate g_e if and only if:

$$\begin{aligned} g_e + \lambda + g_L &= \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right)^{1-\omega} \left(\frac{e_R(t)}{z(t)}\right)^{-\omega} \frac{e_R(t)}{r(t)} \\ &= \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\Omega - g_L - g_e\right)^{\frac{-\omega}{1-\omega}} \frac{e_R(t)}{r(t)} \\ \Leftrightarrow \frac{e_R(t)}{r(t)} &= [g_e + \lambda + g_L] \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right)^{-1} (\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}} . \end{aligned} \tag{A-77}$$

Substituting for $y(t)/k(t)$ from the capital transition equation into (A-67), we find:

$$\frac{\dot{\nu}(t)}{\nu(t)} = (\rho + \delta) - \alpha_K k(t)^{-1} \nu(t)^{-1} - \alpha_K [g_k + \delta + g_L].$$

$\nu(t)k(t)$ is constant if and only if $\dot{\nu}(t)/\nu(t) = -g_k$. Using that condition, $g_k = g_y$, and (A-75) in the foregoing equation yields

$$\nu(t)k(t) = \frac{\alpha_K}{\alpha_E g_e + (\delta + \rho) - \alpha_K(\delta + g_L)}. \tag{A-78}$$

Substitute $y(t)/k(t)$ from the capital transition equation, $g_k = g_y$, g_y from (A-75), $\nu(t)k(t)$ from (A-78), and $\dot{e}_Y(t)/e_Y(t) = g_e$ into (A-74):

$$\begin{aligned} & g_e \left[1 + \omega \frac{e_Y(t)}{e_R(t)}\right] \frac{1}{1-\alpha_K} \frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)} \\ = & -\frac{1}{\alpha_E} e_Y(t) \mu(t) \, (1-\omega)(\lambda + \Omega) \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right]^{1-\omega} \left(\frac{z(t)}{e_R(t)}\right)^{\omega} \\ & - \frac{1}{1-\alpha_K} \frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta+g_L)} \\ & \quad \left[(\rho - \Omega) + \omega(\Omega - g_L) + \omega(\lambda + g_L)\frac{e(t)}{e_R(t)} - Q_D \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right]^{1-\omega} \left(\frac{z(t)}{e_R(t)}\right)^{\omega}\right]. \end{aligned} \tag{A-79}$$

Observe that

$$\frac{e_R(t)}{r(t)} = \frac{e_R(t)}{e_Y(t)} \frac{e_Y(t)}{r(t)} = \frac{e_R(t)}{e_Y(t)} Q_D \frac{e_Y(t)}{e(t)} = \frac{e_R(t)}{e_Y(t)} Q_D \left(1 - \frac{e_R(t)}{e(t)}\right) = \frac{e_R(t)}{e_Y(t)} \left(Q_D - \frac{e_R(t)}{r(t)}\right)$$

.

Therefore:

$$\frac{e_Y(t)}{e_R(t)} = \frac{Q_D - \frac{e_R(t)}{r(t)}}{\frac{e_R(t)}{r(t)}}.$$

Using that and (A-77),

$$\frac{e_Y(t)}{e_R(t)} + \frac{1}{\omega} = \frac{Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E + \frac{1-\omega}{\omega}\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}{\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}.$$

Substitute into (A-79), and also use $e_R(t)/r(t)$ from (A-77) and $e_R(t)/z(t)$ from (A-76):

$$\begin{aligned}
&\omega g_e \frac{Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E + \frac{1-\omega}{\omega}\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}{\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}} \Bigg[\alpha_E g_e + (1-\alpha_K)(\delta+\rho)\Bigg] \\
&= -\frac{1-\alpha_K}{\alpha_E} e_Y(t)\mu(t)\,(1-\omega)(\lambda+\Omega)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \Bigg[\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)\Bigg](\Omega - g_L - g_e)^{\frac{-\omega}{1-\omega}} \\
&\quad - \Bigg[\alpha_E g_e + (1-\alpha_K)(\delta+\rho)\Bigg] \\
&\qquad \Bigg[(\rho - \Omega) + \omega(\Omega - g_L) - Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E (\Omega - g_L - g_e)^{\frac{-\omega}{1-\omega}} \frac{g_e + (1-\omega)(\lambda + g_L)}{g_e + \lambda + g_L}\Bigg].
\end{aligned} \tag{A-80}$$

This last equation requires $\mu(t)e_Y(t)$ be constant, which in turn requires that $g_\mu = -g_e$. Substitute that into the left-hand side of (A-72) and, in the right-hand side of (A-72), substitute $\dot{k}(t)$ from its transition equation, $g_k = g_y$ and g_y from (A-75), and $\nu(t)k(t)$ from (A-78), and then solve for $\mu(t)e_Y(t)$:

$$\mu(t)e_Y(t) = \alpha_E \frac{Q_D}{g_e + \lambda + \rho} \frac{1}{1-\alpha_K} \frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)}. \tag{A-81}$$

Substitute into (A-80):

$$(g_e + \rho - \Omega) + \omega(\Omega - g_L - g_e) = (1-\omega) Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E (\Omega - g_L - g_e)^{\frac{-\omega}{1-\omega}} \frac{g_e + \rho - \Omega}{g_e + \rho + \lambda} \tag{A-82}$$

The left-hand side of (A-82) increases in g_e. The right-hand side of (A-82) is real-valued for $g_e < \Omega - g_L$ and, in those case, increases in g_e for $g_e > -(\rho + \lambda)$. As g_e approaches $-(\rho - \Omega)$ from above, the left-hand side of (A-82) approaches $\omega(\rho - g_L) > 0$ and the right-hand side of (A-82) approaches zero. As g_e approaches $\Omega - g_L$ from below, the left-hand side of (A-82) approaches $\rho - g_L > 0$ and the right-hand side of (A-82) approaches positive infinity. Therefore there exists $g_e \in \big(-(\rho - \Omega), \Omega - g_L\big)$ such that (A-82) holds. An intersection occurs at strictly negative g_e if the left-hand side of (A-82) is strictly less than the right-hand side of (A-82) when each is evaluated at $g_e = 0$. That sufficient condition is:

$$\begin{aligned} 0 <&(1-\omega)Q_D\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\,(\Omega-g_L)^{\frac{-\omega}{1-\omega}}\,\frac{\rho-\Omega}{\rho+\lambda}-(\rho-\Omega)-\omega(\Omega-g_L)\\ \Leftrightarrow Q_DA_E >&\frac{1}{1-\omega}\,(\Omega-g_L)^{\frac{\omega}{1-\omega}}\,\frac{\rho+\lambda}{\kappa_E^{\frac{\sigma}{\sigma-1}}(\rho-\Omega)}\Big\{(\rho-\Omega)+\omega(\Omega-g_L)\Big\}. \end{aligned} \tag{A-83}$$

Defining χ as in Proposition 5, this last inequality is equivalent to

$$Q_DA_E > \left[\frac{\lambda+\rho}{\lambda+g_L}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right]^{\frac{-1}{\sigma-1}}\chi. \tag{A-84}$$

Define X as the right-hand side of (A-84). If $Q_DA_E > X$, then the g_e that solves (A-82) is strictly negative. Observing that the terms inside square brackets in (A-84) are strictly greater than 1 and recalling that $\sigma > 1$ under the conditions of the proposition, we find: (i) $X \leq \chi$; (ii) χ/X is strictly decreasing in σ; and (iii) $\lim_{\sigma\to\infty} X = \lim_{\sigma\to\infty}\chi$.

Increasing Q_DA_E increases the right-hand side of (A-82) when it is strictly positive, without affecting the left-hand side of (A-82). Because the right-hand side of (A-82) cuts the left-hand side of (A-82) from below around the first intersection in g_e (which is the one whose existence is implied by (A-83)), an increase in Q_DA_E moves that intersection to smaller g_e.

Now consider the feasibility of a balanced growth path. From the capital transition equation, $g_k = g_y$, and (A-75), $y(t)/k(t) > 0$ if and only if

$$0 <\frac{1}{s}\left[\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\delta+g_L\right].$$

This is satisfied if and only if

$$g_e > -\frac{\alpha_E+\alpha_L}{\alpha_E}(\delta+g_L). \tag{A-85}$$

From (A-78), $\nu(t)k(t) > 0$ if and only if

$$g_e > -\frac{1}{\alpha_E}\left[(1-\alpha_K)\delta+\rho-\alpha_K g_L\right].$$

Using that $\rho > g_L$, this inequality holds whenever inequality (A-85) holds. From (A-77), $e_R(t)/e(t) > 0$ if and only if

$$0 < \left[g_e+\lambda+g_L\right](\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}}, \tag{A-86}$$

which holds if and only if $g_e \in (-(\lambda+g_L), \Omega-g_L)$. From (A-81), $\mu(t)e_Y(t) > 0$ if and only if

$$0 < \frac{1}{g_e+\lambda+\rho}\frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta+g_L)}.$$

The first fraction is strictly positive for $g_e > -(\rho-\Omega)$, which is met by the analyzed solution to (A-82). Both the denominator and the numerator in the second fraction are strictly positive when inequality (A-85) holds. From (A-76), $e_R(t)/z(t) > 0$ if and only if $g_e < \Omega - g_L$, which is met by the analyzed solution to (A-82). Finally, from (A-77), $e_R(t)/e(t) < 1$ if and only if

$$Q_D A_E > \kappa_E^{-\frac{\sigma}{\sigma-1}} \left[g_e+\lambda+g_L\right] (\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}}. \tag{A-87}$$

At $g_e = 0$, the right-hand side of (A-87) is strictly less than the right-hand side of (A-83), so that inequality (A-87) is implied by inequality (A-83) when $g_e = 0$. The derivative of the right-hand side of (A-87) with respect to g_e is

$$\kappa_E^{-\frac{\sigma}{\sigma-1}} (\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}} \left[1 - \frac{\omega}{1-\omega}\frac{g_e+\lambda+g_L}{\Omega-g_L-g_e}\right].$$

The right-hand side of (A-87) is zero at $g_e = -(\lambda+g_L)$ (which is a lower bound on g_e from inequality (A-86)) and increases until it reaches a maximum and then decreases to 0 at $g_e = \Omega - g_L$. That maximum is at strictly positive g_e if and only if the above derivative is strictly positive when evaluated at $g_e = 0$, and thus if and only if

$$\Omega - g_L > \frac{\omega}{1-\omega}(\lambda+g_L). \tag{A-88}$$

When inequality (A-88) holds, we know that inequality (A-87) holds at all $g_e < 0$ because its right-hand side increases in g_e up to at least $g_e = 0$ and we showed that the inequality does hold at $g_e = 0$.

From inequalities (A-85) and (A-86), a balanced growth path with $g_e \in \big(-(\rho-\Omega), \Omega-g_L\big)$ is feasible if

$$g_e > \max\left\{-\frac{\alpha_E+\alpha_L}{\alpha_E}(\delta+g_L), -(\lambda+g_L)\right\}. \tag{A-89}$$

If $\rho - \Omega$ is not too large, then this inequality holds for all $g_e > -(\rho-\Omega)$.

We have found a path along which $\ell_R(t), k_R(t) = 0$ with $e_R(t) > 0$ and all variables grow at a constant rate. The conditions of the proposition ensure that the path is feasible, since inequalities (A-84), (A-87), and (A-89) hold. Part i of the proposition follows from (A-75). Part ii of the proposition follows from the analysis of (A-82). Part iiii of the proposition follows from that same analysis and from (A-76).

Using (A-77), (7) becomes:

$$EROI(t) = 1 + Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} \frac{g_e+g_L}{\left[g_e+\lambda+g_L\right] (\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}}}.$$

We have proved part iv of the proposition.

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