Appendix to "Informationally Efficient Climate Policy"

This appendix contains numerical details, proofs, and additional formal analysis.

A Numerical Details

There is not broad agreement on a distribution to use for climate change impacts. I calibrate the distribution for aggregate impacts $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ to Pindyck (2019). In 2016, he asked around 1,000 climate scientists and economists to report their subjective percentiles for the percentage reduction in GDP that climate change will cause in fifty years, assuming that no additional emission controls are enacted before then. He fit four distributions to the results and found that a lognormal distribution produced the highest corrected R^2. The location parameter for his fitted lognormal distribution is -2.446 and the scale parameter is 1.476. His distribution describes a parameter (which he labels ϕ) that is equal to $T_{2065} \sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$. Using this estimate and treating T_{2065} as known, $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ is lognormally distributed with location parameter $-2.446 - \ln(T_{2065})$ and scale parameter 1.476.

The value for T_{2065} should be the temperature that experts would have expected to hold based on their information in 2016. The IPCC's AR5 summarizes knowledge around that time. Hausfather and Peters (2020) suggest that a no-additional-emission-controls scenario is consistent with SSP4–6.0 from the IPCC's AR6. So I consider RCP 6.0 in the IPCC's AR5. There, the mean of the CMIP5 models is for 2.2 °C of warming in 2046–2065 relative to a 1986–2005 reference period, which in turn is on average 0.61 °C warmer than over 1850–1900 (Collins et al., 2013, Table 12.2). I therefore fix $T_{2065} = 2.2 + 0.61 = 2.81$ °C. This implies a location parameter for $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ of -3.48.

In order to determine either the emission tax or damage charges, it remains to calibrate r, α, and C_0. I take r to be the policymaker's consumption discount rate. According to World Bank date, average growth in global output per capita was 1.85% per year over 2000-2019. Choosing an annual utility discount rate of 1.5% and a coefficient of relative risk aversion of 1 to match the log utility specification, the Ramsey rule implies that $r = 0.015 + 1 * 0.0185 = 0.0335$, or 3.4% per year.

The parameter α, or the "transient climate response to cumulative carbon emissions", is 1.6/1000 °C/Gt C, from the central value in Matthews et al. (2009). This value is consistent with Collins et al. (2013) and Dietz and Venmans (2019) and is the same as used in Rudik (2020).

I calibrate initial consumption C_0 to World Bank data. In 2021, global output was \$86.7 trillion in year 2015 US dollars. Converting to year 2021 US dollars using World Bank deflators, I set $C_0 = 97,975$ billion dollars.

Now consider the calculations underlying Figure 2. Substituting (5) into (6) yields the probability that the damage charge Δ_t is constrained by the deposit D from reaching its first-best value of $C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]$:

$$Pr(\Delta_t < C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]) = 1 - F(rD/[C_0\alpha]),$$

where $F(\cdot)$ is the cumulative density function for $\tilde{\zeta}_t + \tilde{\lambda}_t$. In the absence of either measurement error in the aggregate signal or stochasticity in climate impacts (i.e., if $\tilde{\omega}^2 = 0$ and $\sigma^2 = 0$), $F(\cdot)$ is completely determined by the lognormal distribution of $\sum_{i=1}^N \kappa_i[\bar{\zeta}_i + \zeta_i]$ defined above.

A.1 Example: Evolution of Beliefs

Consider a normal distribution that has the same mean and, when $\Gamma = 0$, variance as the lognormal distribution for $\sum_{i=1}^N \kappa_i[\bar{\zeta}_i + \zeta_i]$ described above:

$$\sum_{i=1}^N \kappa_i \bar{\zeta}_i = \exp(-3.48 + 1.48^2/2) = 0.0916,$$

and

$$\tau_0^2 \sum_{i=1}^N \kappa_i^2 = [\exp(1.48^2) - 1] \exp(-2 * 3.48 + 1.48^2) = 0.0658.$$

The κ_i are drawn from a symmetric Dirichlet distribution with concentration parameter 1. The regulator and all agents know the values of the κ_i. I fix $N = 10$ in the base specification

Now consider the sources of noise. First, I assume that $\sigma^2 \sum_{i=1}^N \kappa_i^2$ is the same as the prior variance $\tau_0^2 \sum_{i=1}^N \kappa_i^2$, so that

$$\sigma^2 = \frac{0.0658}{\sum_{i=1}^N \kappa_i^2}.$$

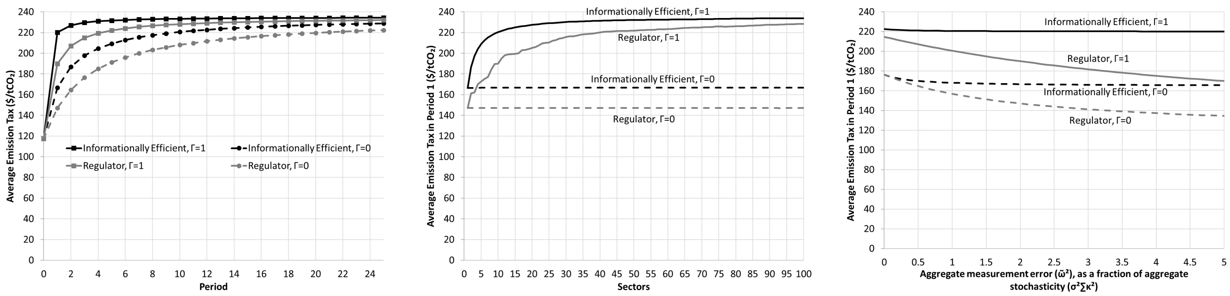
Second, in the base specification, I assume that aggregate measurement error has twice the variance as aggregate stochasticity:

$$\tilde{\omega}^2 = 2 * \sigma^2 \sum_{i=1}^N \kappa_i^2.$$

Third, I assume that sectoral measurement errors have half the variance as sectoral stochasticity:

$$\omega^2 = 0.5 * \sigma^2.$$

For the purposes of simulating beliefs, I assume that the true value of $\sum_{i=1}^N \kappa_i \zeta_i$ is the same as $\sum_{i=1}^N \kappa_i \bar{\zeta}_i$, so that the full-information optimal tax is twice the prior tax. I draw the ζ_i from a multivariate normal distribution with mean vector equal to $\sum_{i=1}^N \kappa_i \bar{\zeta}_i$ and I then adjust the draws ex post by adding a constant to ensure that $\sum_{i=1}^N \kappa_i \zeta_i = \sum_{i=1}^N \kappa_i \bar{\zeta}_i$. I draw 1 million trajectories for the random variables conditional on these ζ_i. These trajectories



(a) Evolution of beliefs (b) Influence of N in period 1 (c) Influence of $\tilde{\omega}^2$ in period 1

Figure A-1: An example of how informationally efficient beliefs and the regulator's beliefs would each evolve on average (i.e., of $E_0[\hat{\mu}_t|\boldsymbol{\zeta}]$ and $E_0[\tilde{\mu}_t|\boldsymbol{\zeta}]$). Appendix A details the calibration. The emission tax optimal at prior beliefs is \$118 per tCO_2, and simulations assume that the emission tax conditional on true knowledge of the ζ_i would be twice as large (\$236 per tCO_2). Left: Evolution of average beliefs over the first 25 periods, with $N = 10$ and $\tilde{\omega}^2/[\sigma^2 \sum_{i=1}^{N} \kappa_i^2] = 2$. Middle: The effect of the number of sectors (N) on period 1 beliefs. Right: The effect of aggregate measurement error ($\tilde{\omega}^2$) on period 1 beliefs.

yield 1 million trajectories for $\hat{\mu}_t$ and $\tilde{\mu}_t$ from equations (3) and (4), respectively. Averaging over these trajectories yields trajectories $E_0[\hat{\mu}_t|\boldsymbol{\zeta}]$ and $E_0[\tilde{\mu}_t|\boldsymbol{\zeta}]$.

Figure A-1 provides an example that illustrates how the regulator's beliefs (gray) differ from informationally efficient beliefs (black), for the extreme cases when unknown sectoral effects are independent of each other ($\Gamma = 0$, dashed) and are perfectly correlated with each other ($\Gamma = 1$, solid). The emission tax that would be optimal at time 0 beliefs is \$118 per tCO_2. The depicted simulations assume that the initial tax that would be optimal with perfect information about the ζ_i would be twice as large.

The left panel assesses how beliefs converge to the truth.[40] In these cases, informationally efficient beliefs converge to the truth faster than do the regulator's beliefs. Both types of beliefs converge faster when sectors are perfectly correlated with each other than when sectors are independent of each other: informationally efficient beliefs infer more from each observation in the presence of correlation, and the regulator places more weight on the data when correlation increases the prior variance.

The middle and right panels plot the emission tax chosen, on average, after observing signals from time 0. The middle panel shows that the number of sectors N does not affect beliefs when sectoral effects are independent (the calibration scales σ^2 so that aggregate stochasticity is independent of N). However, correlation makes the average speed of learning

[40]The average speed of convergence is used here for illustration, but it is not a measure of the quality of beliefs. Such a measure would also account for the standard deviation of beliefs. For instance, when $\Gamma = 0.5$, the regulator's beliefs converge towards the true value on average faster than do informationally efficient beliefs, but they are more sensitive to randomness in any particular trajectory.

increase in the number of sectors N, and informationally efficient beliefs in particular update much faster when the economy has multiple sectors that provide information about each other. The right panel shows that aggregate measurement error $\tilde{\omega}^2$ (which increases to the right) markedly slows learning by the regulator. In contrast, informationally efficient beliefs are less sensitive to aggregate measurement error because they can use the disentangled sectoral signals directly and thereby mitigate that source of error.[41]

B Taxing the Stock of Carbon

Assume informational efficiency, so that all agents observe all $\zeta_{it} + \lambda_{it}$. Denote firm i's cumulative emissions from time 0 to time $t-1$ as M_{it}. In period t, the regulator taxes M_{it} at rate $\tilde{\nu}_t$. Firms discount the future at per-period rate r. Final-good firms' problem is as in Appendix G.

The representative intermediate-good producer in sector i solves the following Bellman equation:

$$J_{it}(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = \max_{L_{it}, e_{it}, R_{it}} \Bigg\{ \hat{E}_t \left[p_{it} \exp[-\zeta_{it} T_t] \right] L_{it} Y^{it}(e_{it}) - w_{it} L_{it} - \tilde{\nu}_t M_{it} - p_t^R R_{it} + \frac{1}{1+r} \hat{E}_t [J_{i(t+1)}(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] \Bigg\},$$

with each control weakly positive. At an interior solution, the first-order condition for emissions is

$$\hat{E}_t \left[p_{it} \exp[-\zeta_{it} T_t] \right] L_{it} Y^{it\prime}(e_{it}) = -\frac{1}{1+r} \hat{E}_t \left[\frac{\partial J_{i(t+1)}(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})}{\partial e_{it}} \right],$$

and the first order condition for carbon removal is

$$p_t^R = \frac{1}{1+r} \hat{E}_t \left[\frac{\partial J_{i(t+1)}(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})}{\partial R_{it}} \right].$$

Substitute for p_{it} and p_t^R and then for $p_t C_t$ in the first-order conditions as in Appendix G:

$$-\frac{1}{1+r} \hat{E}_t \left[\frac{\partial J_{i(t+1)}(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})}{\partial e_{it}} \right] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0, \tag{A-1}$$

$$\frac{1}{1+r} \hat{E}_t \left[\frac{\partial J_{i(t+1)}(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})}{\partial R_{it}} \right] = \frac{c_t'(R_t)}{1 - c_t(R_t)} C_0. \tag{A-2}$$

[41] It is hard to detect visually in the figure, but aggregate measurement error does slow learning even for informationally efficient beliefs with $\Gamma = 1$. The convergence of informationally efficient beliefs and the regulator's beliefs when $\Gamma = 0$ and $\tilde{\omega}^2 = 0$ illustrates Corollary 2.

The envelope theorem yields:

$$\frac{\partial J_{it}(M_{it},\hat{\mu}_t,\hat{\Omega}_t)}{\partial M_{it}} = -\,\tilde{\nu}_t + \frac{1}{1+r}\hat{E}_t\left[\frac{\partial J_{i(t+1)}(M_{i(t+1)},\hat{\mu}_{t+1},\hat{\Omega}_{t+1})}{\partial M_{i(t+1)}}\right].$$

Recursively substituting, we find:

$$\frac{\partial J_{it}(M_{it},\hat{\mu}_t,\hat{\Omega}_t)}{\partial M_{it}} = -\sum_{s=0}^{\infty}\frac{1}{(1+r)^s}\hat{E}_t[\tilde{\nu}_{t+s}].$$

Advancing by one timestep, substituting into (A-1) and (A-2), and applying the law of iterated expectations, an interior solution must satisfy:

$$\sum_{s=1}^{\infty}\frac{1}{(1+r)^s}\hat{E}_t[\tilde{\nu}_{t+s}] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})}C_0,$$

$$\sum_{s=1}^{\infty}\frac{1}{(1+r)^s}\hat{E}_t[\tilde{\nu}_{t+s}] = \frac{c_t'(R_t)}{1-c_t(R_t)}C_0.$$

Now set $\tilde{\nu}_t = C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]$. The foregoing conditions become:

$$\alpha\sum_{s=1}^{\infty}\frac{1}{(1+r)^s}\hat{E}_t[\tilde{\zeta}_{t+s}+\tilde{\lambda}_{t+s}] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})}, \tag{A-3}$$

$$\alpha\sum_{s=1}^{\infty}\frac{1}{(1+r)^s}\hat{E}_t[\tilde{\zeta}_{t+s}+\tilde{\lambda}_{t+s}] = \frac{c_t'(R_t)}{1-c_t(R_t)}, \tag{A-4}$$

which in turn are equivalent to

$$\alpha\frac{1}{r}\left[\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k + \hat{\mu}_t\right] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})},$$

$$\alpha\frac{1}{r}\left[\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k + \hat{\mu}_t\right] = \frac{c_t'(R_t)}{1-c_t(R_t)}.$$

Once we adjust for the possibility of corner solutions, these conditions are the same as the conditions for welfare maximization in (1) and (2). Therefore this choice of $\tilde{\nu}_t$ must be the optimal choice for a regulator who can commit to a choice rule at time 0.

Finally, consider what would happen if we did not assume informational efficiency. In that case, the expectations in (A-3) and (A-4) vary with the index i when different firms have different information. Firms will expect different stock taxes and thus arrive at different estimates of the marginal cost of emissions and the marginal benefit of carbon removal. In the absence of a mechanism by which firms can coordinate expectations, a stock tax may fail to equalize marginal abatement costs across sectors—and thus violate a basic principle of efficiency.

C Carbon Shares When the Regulator and Market Apply Different Discount Rates

Extend the analysis of Section 5.1 to a case in which the market interest rate is $\delta > 0$ but the regulator discounts future welfare at rate $\rho > 0$. Equations (1) and (2) hold with ρ in place of r. Denote damage charges and refunds in this environment as Δ_t^* and d_t^*. Define damage charges as

$$\Delta_t^* \triangleq \frac{\delta}{\rho}\Delta_t,$$

with Δ_t as in (6). Define refunds as

$$\begin{aligned} d_t^* \triangleq& \delta D^* - \Delta_t^* \\ =& \delta \left(D^* - \frac{1}{\rho}\Delta_t \right), \end{aligned}$$

for some deposit $D^* \geq 0$. In the special case that $\delta = \rho$ and $D^* = D$, we have $\Delta_t^* = \Delta_t$ and $d_t^* = d_t$ and so are back to the carbon share policy analyzed in the main text. Analogously to the main text,

$$\begin{aligned} \sum_{s=1}^{\infty} \frac{1}{(1+\delta)^s}[d_{t+s}^* + \Delta_{t+s}^*] =& \sum_{s=1}^{\infty} \frac{1}{(1+\delta)^s}\delta D^* \\ =& D^*, \end{aligned}$$

so that the policy is again either revenue-neutral or revenue-positive when deposits are invested at the market interest rate.

Trivially adjusting the proof of Lemma 1 to use δ in place of ρ and d_{t+j}^* in place of d_{t+j}, equation (8) becomes

$$\hat{q}_t^* = \sum_{j=0}^{\infty} \frac{1}{(1+\delta)^j}\hat{E}_t[d_{t+j}^*].$$

Adjusting the proof of Proposition 4, the cost of emitting in period t becomes

$$\begin{aligned} D^* - (\hat{q}_t^* - \hat{E}[d_t^*]) =& D^* - \sum_{j=1}^{\infty} \frac{1}{(1+\delta)^j}\hat{E}_t[\delta\, D^* - \Delta_{t+j}^*] \\ =& \frac{\delta}{\rho}\sum_{j=1}^{\infty} \frac{1}{(1+\delta)^j}\hat{E}_t[\Delta_{t+j}]. \end{aligned}$$

Equation (A-22) then becomes

$$\begin{aligned} D^* - (\hat{q}_t^* - \hat{E}[d_t^*]) =& \frac{\delta}{\rho} C_0\, \alpha \left[\frac{1}{\delta} \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \frac{1}{\delta} \hat{\mu}_t - \sum_{j=1}^{\infty} \frac{1}{(1+\delta)^j} \chi_{t,t+j} \right] \\ =& C_0\, \alpha \left[\frac{1}{\rho} \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \frac{1}{\rho} \hat{\mu}_t - \sum_{j=1}^{\infty} \frac{1}{(1+\delta)^j} \frac{\delta}{\rho} \chi_{t,t+j} \right]. \end{aligned}$$

The rest of the proof of Proposition 4 then goes through straightforwardly. We have established that Proposition 4 holds with damage charges Δ_t^* and refunds d_t^*. Therefore, again using $\breve{L}$ to represent the expected loss relative to the welfare-maximizing benchmark from using a carbon share policy and stipulating Assumption 1, we have shown that $\breve{L} \to 0$ as $D^* \to \infty$ when $\delta \neq \rho$, damage charges are Δ_t^*, and refunds are d_t^*.

D Analyzing Emission Taxes with a Revenue Lockbox Used to Fund Carbon Removal

Consider the implications of a dynamic revenue constraint in a world in which the regulator does not learn about damages but in which technological progress in carbon removal can eventually make negative net emissions optimal even under the prior belief. Proposition 2 established that the regulator's unconstrained-optimal tax would be

$$\nu_t = C_0 \frac{\alpha}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t \right].$$

This tax is unaffected by the possibility of technological progress in carbon removal technologies and is constant over time in the absence of learning about damages (i.e., when $\tilde{\mu}_t$ is constant over time). Because this tax is also the subsidy a regulator would like to offer for carbon removal in a negative net emission scenario, the tax that the regulator collects at the time of emission is exactly equal to the subsidy the regulator would subsequently offer to remove that same unit of emission from the atmosphere. The dynamic revenue constraint would therefore never bind unless it became optimal to bring future carbon stocks below their initial level M_0.

Now let the regulator learn about damages. In that case, observing unfavorable information about climate damages could increase $\tilde{\mu}_t$ by enough to make negative net emissions optimal whether or not there is progress in carbon removal technology. The tax the regulator collects at the time of emission is then strictly less than the subsidy the regulator would subsequently offer to remove that same unit of emission from the atmosphere. The regulator has sufficient tax revenue in its lockbox to bring carbon only part of the way back to

M_0. The more pessimistic damage estimates become, the more likely this constraint on the regulator's ability to fund negative net emissions becomes binding. And if carbon removal technology simultaneously progresses quickly, then this constraint again becomes more likely to bind because a given subsidy will procure even more removal. The regulator might again be unable to procure the optimal level of negative net emissions without raising funds from taxpayers.

Further, when the lockbox might fail to incentivize optimal removal, the regulator distorts emission decisions in anticipation that the dynamic revenue constraint might bind:

Corollary 5 (Emission Tax With A Lockbox)**.**

1. *Consider a period in which net emissions are strictly positive. The tax with a lockbox is strictly greater (strictly less) and emissions are strictly less (strictly greater) than given in Proposition 2 if marginally raising that tax increases (decreases) tax revenue.*
2. *Consider a period in which net emissions are weakly (strictly) negative. The optimal payment from a lockbox that the regulator offers for carbon removal is weakly (strictly) less, and net emissions are weakly (strictly) greater, than the welfare-maximizing benchmark.*

Proof. See Appendix O. □

The marginal value of a higher emission tax is comprised of its marginal value in a setting without revenue constraints and its effect on future negative emission constraints via its effect on the revenue that will be stored in the lockbox. The latter effect distorts the emission tax away from its unconstrained-optimal level in order to prepare for the possibility that sufficiently negative net emissions become optimal. When emissions are strictly positive, raising an emission tax increases revenue by charging more per unit of net emissions but reduces revenue by reducing net emissions. The first part of the corollary establishes that the optimal tax in the presence of a lockbox is higher (lower) when the former (latter) dominates. Small distortions in the emission tax do not impose first-order costs today, but they can raise extra revenue that provides first-order benefits by weakening a potential future constraint on negative net emissions.[42]

Once the regulator is already paying for negative emissions out of the lockbox, reducing the emission tax clearly leaves more revenue in the lockbox: a lower price requires the regulator to pay less per unit of carbon removal and also procures less carbon removal. The second part of the corollary establishes that a regulator already paying for negative emissions prepares for the possibility of future binding constraints by reducing the price offered for carbon removal.

[42] The possibility of hitting the constraint at some future time did not distort the optimal tax in Proposition 2 because the combination of logarithmic utility and multiplicative-exponential damages makes the optimal tax independent of future emission and removal trajectories. This combination of assumptions also makes the lockbox work perfectly in the absence of learning about damages.

E Proof of Proposition 1

Consider the zero-mean random vector

$$s_t \triangleq \begin{bmatrix} \sum_{k=1}^N \kappa_k \zeta_k \\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j} + \lambda_{1j}] - \bar{\zeta}_1 \\ \vdots \\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj} + \lambda_{Nj}] - \bar{\zeta}_N \\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \end{bmatrix}.$$

Observe that

$$\hat{\mu}_t \triangleq E\left[\sum_{k=1}^N \kappa_k \zeta_k \,\middle|\, \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j} + \lambda_{1j}] - \bar{\zeta}_1, ..., \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj} + \lambda_{Nj}] - \bar{\zeta}_N, \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k\right].$$

Let Ψ_t indicate the $(N+1)\times(N+1)$ covariance matrix of the final $(N+1)\times 1$ elements of s_t and Σ_t indicate the $1\times(N+1)$ vector of covariances between $\sum_{k=1}^N \kappa_k \zeta_k$ and the other elements of s_t, so that

$$\Sigma_t \triangleq \left[\kappa_1 \tau_0^2 + (1-\kappa_1)\Gamma\tau_0^2, \quad \ldots, \quad \kappa_N \tau_0^2 + (1-\kappa_N)\Gamma\tau_0^2, \quad \tau_0^2 \sum_{k=1}^N \kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^N \sum_{j=k+1}^N \kappa_k \kappa_j\right].$$

From the projection theorem,

$$\hat{\mu}_t = \Sigma_t \Psi_t^{-1} \begin{bmatrix} \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j} + \lambda_{1j}] - \bar{\zeta}_1 \\ \vdots \\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj} + \lambda_{Nj}] - \bar{\zeta}_N \\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \end{bmatrix}. \tag{A-5}$$

Consider Ψ_t^{-1}. Label the $N\times N$ upper-left block of Ψ_t as Ψ_A, the $N\times 1$ upper right block as Ψ_B, the $1\times N$ lower left block as Ψ_C, and the 1×1 lower right block as Ψ_D. From familiar results for block matrix inversion,

$$\Psi_t^{-1} = \begin{bmatrix} \Psi_A^{-1} + \Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1} & -\Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1} \\ -(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1} & (\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1} \end{bmatrix}. \tag{A-6}$$

Element (i,i) of Ψ_A is $\tau_0^2 + \sigma^2/t + \omega^2/t$, element (i,j) of Ψ_A is $\Gamma\tau_0^2$ for $i \neq j$, the ith element of Ψ_B and Ψ_C is $\kappa_i(\tau_0^2 + \sigma^2/t) + (1-\kappa_i)\Gamma\tau_0^2$, and Ψ_D equals $(\tau_0^2 + \sigma^2/t)\sum_{k=1}^N \kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^N \sum_{j=k+1}^N \kappa_k\kappa_j + \tilde{\omega}^2/t$.

Conjecture that each diagonal of Ψ_A^{-1} is

$$\frac{\tau_0^2 + \sigma^2/t + \omega^2/t + (N-2)\Gamma\tau_0^2}{(\tau_0^2 + \sigma^2/t + \omega^2/t)^2 + (\tau_0^2 + \sigma^2/t + \omega^2/t)(N-2)\Gamma\tau_0^2 - (N-1)[\Gamma\tau_0^2]^2}$$

and each off-diagonal of Ψ_A^{-1} is

$$\frac{-\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}.$$

Under the conjecture for Ψ_A^{-1}, each diagonal element in $\Psi_A\Psi_A^{-1}$ is

$$\begin{aligned}&[\tau_0^2+\sigma^2/t+\omega^2/t]\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\&-\sum_{k=1}^{N-1}\Gamma\tau_0^2\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2-(N-1)[\Gamma\tau_0^2]^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2}\\=&1,\end{aligned}$$

and each off-diagonal element in $\Psi_A\Psi_A^{-1}$ is

$$\begin{aligned}&\Gamma\tau_0^2\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\&-[\tau_0^2+\sigma^2/t+\omega^2/t]\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\&-\sum_{k=1}^{N-2}\Gamma\tau_0^2\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2-(N-1)[\Gamma\tau_0^2]^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2}\\=&0.\end{aligned}$$

We have shown that $\Psi_A\Psi_A^{-1}$ is the identity matrix under the conjecture for Ψ_A^{-1} and thus have confirmed that the conjecture is correct. Observe that the denominator of each element simplifies to

$$\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big].$$

With Ψ_A^{-1} in hand, we now calculate Ψ_t^{-1} from (A-6). Element i of $\Psi_C\Psi_A^{-1}$ and also of $\Psi_A^{-1}\Psi_B$ is

$$\begin{aligned}&\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}[\kappa_i(\tau_0^2+\sigma^2/t)+(1-\kappa_i)\Gamma\tau_0^2]\\&+\sum_{k=1,\neq i}^{N}\frac{-\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\&\qquad[\kappa_k(\tau_0^2+\sigma^2/t)+(1-\kappa_k)\Gamma\tau_0^2]\\=&\frac{(1-\Gamma)\tau_0^2+\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_i+\frac{\Gamma\tau_0^2\omega^2/t}{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]}.\end{aligned}$$

We then have:

$$\Psi_C\Psi_A^{-1}\Psi_B = \sum_{i=1}^{N} \frac{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\kappa_i+\Gamma\tau_0^2\omega^2/t}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}$$
$$\Big[\kappa_i((1-\Gamma)\tau_0^2+\sigma^2/t)+\Gamma\tau_0^2\Big]$$
$$=\frac{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]^2\sum_{i=1}^N\kappa_i^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}$$
$$+\frac{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]+\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\omega^2/t}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\Gamma\tau_0^2.$$

And

$$(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}$$
$$=\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]$$
$$\Bigg\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]$$
$$\Big[(\tau_0^2+\sigma^2/t)\sum_{k=1}^N\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^N\sum_{j=k+1}^N\kappa_k\kappa_j+\tilde{\omega}^2/t\Big]$$
$$-\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]^2\sum_{i=1}^N\kappa_i^2$$
$$-\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2-\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\Gamma\tau_0^2\omega^2/t\Bigg\}^{-1}$$
$$=\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]$$
$$\Bigg\{\omega^2/t\Bigg(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\sum_{k=1}^N\kappa_k^2+\Gamma\tau_0^2\omega^2/t\Bigg)$$
$$+\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\Bigg\}^{-1},$$

where the final line uses $\sum_{k=1}^N\kappa_k=1$. Element i of $\Psi_A^{-1}\Psi_B(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}$ and also of

$(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is

$$\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_i+\Gamma\tau_0^2\omega^2/t\right\}$$
$$\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}.$$

Element (i,i) of $\Psi_A^{-1}+\Psi_A^{-1}\Psi_B(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is

$$\frac{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+(N-1)\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]}$$
$$+\frac{\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_i+\Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]}$$
$$\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_i+\Gamma\tau_0^2\omega^2/t\right\}$$
$$\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1},$$

and element (m, n) of $\Psi_A^{-1} + \Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is, for $m \neq n$,

$$\frac{-\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$+ \frac{\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_m + \Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_n + \Gamma\tau_0^2\omega^2/t\right\}$$
$$\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left. + \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}.$$

The foregoing pieces define Ψ_t^{-1} from (A-6). $\Sigma_t\Psi_t^{-1}$ is $1 \times N + 1$ and, from (A-5), determines how $\hat{\mu}_t$ uses the signals in s_t. Element $k \in \{1, ..., N\}$ of $\Sigma_t \Psi_t^{-1}$ is

$$
\begin{aligned}
&[\kappa_k \tau_0^2 + (1-\kappa_k)\Gamma\tau_0^2] \frac{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + (N-1)\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]} \\
&+ \frac{\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_k + \Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]} \\
&\quad \sum_{i=1}^{N} [\kappa_i \tau_0^2 + (1-\kappa_i)\Gamma\tau_0^2] \left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\} \\
&\quad \left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad \left. + \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1} \\
&- [(1-\kappa_k)(1-\Gamma)\tau_0^2 + (N-1)\Gamma\tau_0^2] \frac{\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]} \\
&- \left[\tau_0^2 \sum_{k=1}^{N}\kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j\right] \\
&\quad \left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_k + \Gamma\tau_0^2\omega^2/t\right\} \\
&\quad \left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad \left. + \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{(1-\Gamma)\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_k \\
&+\frac{\sigma^2/t+\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2} \\
&-[\omega^2/t]\left(\frac{(1-\Gamma)\tau_0^2+\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_k+\frac{\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}\right) \\
&\quad\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{i=1}^{N}\kappa_i^2+\left[\sigma^2/t+\omega^2/t\right]\Gamma\tau_0^2\right\} \\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}. \quad \text{(A-7)}
\end{aligned}
$$

Element $N+1$ of $\Sigma_t\Psi_t^{-1}$ is

$$
\begin{aligned}
&-\sum_{k=1}^{N}[\kappa_k\tau_0^2+(1-\kappa_k)\Gamma\tau_0^2]\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_k+\Gamma\tau_0^2\omega^2/t\right\} \\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1} \\
&+\left[\tau_0^2\sum_{k=1}^{N}\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j\right] \\
&\quad\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right] \\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
=&[\omega^2/t]\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\left[\sigma^2/t+\omega^2/t\right]\Gamma\tau_0^2\right\}\\
&\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}. \quad \text{(A-8)}
\end{aligned}
$$

With these elements, we can determine $\hat{\mu}_t$ from (A-5).

Define

$$
\begin{aligned}
\hat{Z}_t \triangleq&[\omega^2/t]\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\left[\sigma^2/t+\omega^2/t\right]\Gamma\tau_0^2\right\}\\
&\left\{[\omega^2/t]\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\left.+[\tilde{\omega}^2/t]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\right\}^{-1}. \quad \text{(A-9)}
\end{aligned}
$$

Clearly $\hat{Z}\geq 0$. Observe that $\sum_{k=1}^{N}\kappa_k^2$ is minimized when each $\kappa_k=1/N$ and thus

$$
\begin{aligned}
\hat{Z}_t <&[\omega^2/t]\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\left[\sigma^2/t+\omega^2/t\right]\Gamma\tau_0^2\right\}\\
&\left\{[\omega^2/t]\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2[\sigma^2/t+\omega^2/t]\right)\right.\\
&\left.+[\tilde{\omega}^2/t]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\right\}^{-1}\\
<&1.
\end{aligned}
$$

Therefore $\hat{Z}_t\in[0,1)$. By inspection, $\hat{Z}_t\to 0$ as $\tilde{\omega}/\omega\to\infty$. The expression in the proposition follows from (A-5), (A-7), (A-8), and the definition (A-9).

F Proof of Corollary 1

From Proposition 1, $\hat{Z}_t \to 0$ as $\omega^2 \to 0$ if $\tilde{\omega}^2 > 0$. In that case, the expression in part (i) follows from (3), $\lambda_{kj} = 0$, and the definition of $\tilde{\zeta}_j$. Now consider the case in which $\tilde{\omega}^2 = 0$ and $\omega^2 \to 0$. From (A-9),

$$\hat{Z}_t = \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + \left[\sigma^2/t + \omega^2/t\right]\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2 + \Gamma\tau_0^2\omega^2/t}$$

if $\tilde{\omega}^2 = 0$, which goes to

$$\frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + [\sigma^2/t]\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2}$$

as $\omega^2 \to 0$. Substituting into (3), using $\lambda_{kj} = \tilde{\lambda}_j = 0$, and applying the definition of $\tilde{\zeta}_j$ yields the expression in part (i). We have established the first part of the corollary.

From (A-9),

$$\lim_{\sigma^2, \tilde{\omega}^2 \to 0} \hat{Z}_t = 1.$$

Substituting into (3), setting $\tilde{\lambda}_j = 0$, and setting $\sigma^2 = 0$ yields the expression in part (ii) of the corollary.

From (A-9),

$$\lim_{\Gamma, \tilde{\omega}^2 \to 0} \hat{Z}_t = \frac{\left[\tau_0^2 + \sigma^2/t + \omega^2/t\right]\tau_0^2 \sum_{k=1}^N \kappa_k^2}{\left[\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2} = \frac{\tau_0^2}{\tau_0^2 + \sigma^2/t}.$$

Substituting into (3), setting $\tilde{\lambda}_j = 0$, and setting $\Gamma = 0$ yields the expression in part (iii) of the corollary.

From (A-9),

$$\begin{aligned}\lim_{\tilde{\omega}^2 \to 0, \kappa_i \to 1/N\, \forall i} \hat{Z}_t &= \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 + \left[\sigma^2/t + \omega^2/t\right]N\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right] + N\Gamma\tau_0^2\omega^2/t} \\ &= \frac{(1-\Gamma)\tau_0^2 + N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2}.\end{aligned}$$

Using this,

$$\begin{aligned}
&\lim_{\tilde{\omega}^2\to 0,\kappa_i\to 1/N\,\forall i} \frac{(1-\hat{Z}_t)(1-\Gamma)\tau_0^2-\hat{Z}_t\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\\
&\qquad=-\frac{N\Gamma\tau_0^2\sigma^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]+N\Gamma\tau_0^2\omega^2/t},\\
&\lim_{\tilde{\omega}^2\to 0,\kappa_i\to 1/N\,\forall i} \frac{\sigma^2/t+(1-\hat{Z}_t)\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}\\
&\qquad=\frac{N\Gamma\tau_0^2\sigma^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]+N\Gamma\tau_0^2\omega^2/t}.
\end{aligned}$$

Substituting into (3) and setting $\tilde{\lambda}_j = 0$ yields the expression in part (iv) of the corollary.

G Proof of Proposition 2

I first solve for market equilibrium conditional on the choice of tax and then consider how the regulator would design the tax to maximize welfare.

Let p_t be the price of consumption, p_{it} be the time t price of intermediate i, and p_t^R be the price paid for removal. Final-good firms solve:

$$\max_{R_t,\{Y_{it}\}_{i=1}^N} \left\{ p_t\left(1-c_t(R_t)\right)Y_t - \sum_{i=1}^N p_{it}Y_{it} + p_t^R R_t \right\}.$$

The first-order conditions are

$$\begin{aligned}
p_{it} =& p_t \kappa_i \frac{C_t}{Y_{it}},\\
p_t^R =& p_t \frac{c_t'(R_t)}{1-c_t(R_t)}\, C_t.
\end{aligned} \tag{A-10}$$

The latter condition becomes a $\leq$ when $R_t = 0$.

The representative intermediate-good producer in sector i solves

$$\max_{L_{it},e_{it},R_{it}\geq 0} \left\{ E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] L_{it} Y^{it}(e_{it}) - w_{it}L_{it} - \nu_t \max\{e_{it}-R_{it},0\} - p_t^R R_{it} \right\}.$$

The $\max\{e_{it} - R_{it}, 0\}$ reflects that, unless the regulator could discriminate subsidies by sectors, the regulator cannot pay firms in sector i for negative emissions or else it would violate its revenue constraint if emissions in other sectors net out to zero or less. A maximum clearly has $e_{it} - R_{it} \geq 0$.

The first-order condition for L_{it} is

$$w_{it} = E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] Y^{it}(e_{it}).$$

If $e_{it} - R_{it} > 0$, then the first-order condition for e_{it} is

$$\nu_t = E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] L_{it} Y^{it\prime}(e_{it}),$$

and the first-order condition for a solution with $R_{it} > 0$ is

$$\nu_t = E_{it}[p_t^R].$$

Substitute for p_{it} and p_t^R in the first-order conditions:

$$w_{it} = \kappa_i \frac{1}{L_{it}} E_{it}[p_t C_t], \tag{A-11}$$

$$\nu_t = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} E_{it}[p_t C_t], \tag{A-12}$$

$$\nu_t = E_{it}\left[\frac{c_t'(R_t)}{1 - c_t(R_t)} p_t C_t\right]. \tag{A-13}$$

At a corner solution with $e_{it} = 0$, the second condition's equality would become a $\geq$, and at a corner solution with $R_t = 0$, the third condition's equality would become a $\leq$.

Household maximization implies $p_t = u'(C_t)$. Therefore, using $p_0 = 1$,

$$p_t = \frac{u'(C_t)}{u'(C_0)} = \frac{C_0}{C_t}.$$

Households' budget constraint $\sum_{i=1}^N w_{it} L_{it} = p_t C_t$ and the first-order condition imply $1 = \sum_{i=1}^N \kappa_i$, which does hold. Substitute $p_t C_t = C_0$ in (A-11) through (A-13):

$$w_{it} = \kappa_i \frac{1}{L_{it}} C_0, \tag{A-14}$$

$$\nu_t = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0, \tag{A-15}$$

$$\nu_t = \frac{c_t'(R_t)}{1 - c_t(R_t)} C_0. \tag{A-16}$$

The last equation reocgnizes that R_t is nt uncertain if each R_{it} is not. The wage must be equal in sectors with nonzero employment, so $w_{it} = w_t$ for some $w_t > 0$. Equation (A-14) becomes:

$$L_{it} = \kappa_i \frac{1}{w_t} C_0.$$

From the budget constraint, $w_t = C_0$ and thus $L_{it} = \kappa_i$. Therefore equilibrium L_{it} is independent of e_{it}, ν_t, and T_t.

Conjecture that ν_t/C_0 is independent of T_t (we will confirm this conjecture when studying the regulator's problem). Then from (A-15) and (A-16), the time t market equilibrium is also independent of T_t. It is also independent of the random shocks ϵ_{it} and the unknown damage parameters ζ_{it} and thus is not stochastic.

As ν_t increases, e_{it} strictly decreases while $e_{it} > 0$. Aggregating over all i, $\sum_{i=1}^N e_{it}$ strictly decreases in ν_t and R_t weakly increases in ν_t. There exists some $\bar{\nu}_t > 0$ such that $\sum_{i=1}^N e_{it} - R_t = 0$ if $\nu_t = \bar{\nu}_t$ and $\sum_{i=1}^N e_{it} - R_t > 0$ if $\nu_t < \bar{\nu}_t$.

The regulator solves the following Bellman equation:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = \max_{\nu_t} \tilde{E}_t\Bigg[u\Bigg((1 - c_t(R_t^*)) \prod_{i=1}^N [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i}\Bigg) + \frac{1}{1+r}\tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\Bigg],$$

where $\tilde{E}_t$ denotes expectations over the regulator's information set at the beginning of period t and where a star represents market equilibrium. Market outcomes will be insensitive to ν_t for $\nu_t > \bar{\nu}_t$ and thus the regulator's objective will be constant in ν_t for $\nu_t > \bar{\nu}_t$. Maximized welfare is therefore equivalent for a regulator who solves the following problem in which ν_t is constrained to be less than or equal to $\bar{\nu}_t$:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = \max_{\nu_t \le \bar{\nu}_t} \tilde{E}_t\Bigg[u\Bigg((1 - c_t(R_t^*)) \prod_{i=1}^N [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i}\Bigg) + \frac{1}{1+r}\tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\Bigg].$$

At an interior solution, the regulator's first-order condition is

$$0 = \sum_{i=1}^N \frac{\kappa_i Y_e^{i\prime}(e_{it}^*)}{Y^{it}(e_{it}^*)} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r}\tilde{E}_t\Big[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\Big]\alpha \sum_{i=1}^N \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{c_t'(R_t^*)}{1 - c_t(R_t^*)}\frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r}\tilde{E}_t\Big[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})\Big]\alpha\frac{\partial R_t^*}{\partial \nu_t}. \tag{A-17}$$

Substitute from (A-15) and (A-16):[43]

$$0 = \frac{\nu_t}{C_0^*}\sum_{i=1}^{N}\frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r}\tilde{E}_t\left[\tilde{W}_T(T_{t+1},\tilde{\mu}_{t+1},\tilde{\Omega}_{t+1})\right]\alpha\sum_{i=1}^{N}\frac{\partial e_{it}^*}{\partial \nu_t}$$
$$- \frac{\nu_t}{C_0^*}\frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r}\tilde{E}_t\left[\tilde{W}_T(T_{t+1},\tilde{\mu}_{t+1},\tilde{\Omega}_{t+1})\right]\alpha\frac{\partial R_t^*}{\partial \nu_t}.$$

Factor out common terms:

$$0 = \left(\frac{\nu_t}{C_0} + \frac{1}{1+r}\alpha\tilde{E}_t\left[\tilde{W}_T(T_{t+1},\tilde{\mu}_{t+1},\tilde{\Omega}_{t+1})\right]\right)\left(\sum_{i=1}^{N}\frac{\partial e_{it}^*}{\partial \nu_t} - \frac{\partial R_t^*}{\partial \nu_t}\right).$$

Applying the Implicit Function Theorem to equations (A-15) and (A-16), we find:

$$\frac{\partial e_{it}^*}{\partial \nu_t} = \frac{-1}{\frac{-Y_{ee}^{it}}{Y^{it\prime}} + \frac{Y^{it\prime}}{Y^{it}}}\frac{1}{\nu_t} < 0 \quad \text{if } e_{it}^* > 0, \tag{A-18}$$

$$\frac{\partial R_t^*}{\partial \nu_t} = \frac{1}{\frac{c_t''(R_t^*)}{1-c_t(R_t^*)}C_0^* + \frac{[c_t'(R_t^*)]^2}{[1-c_t(R_t^*)]^2}C_0^*} > 0 \quad \text{if } R_{it}^* > 0. \tag{A-19}$$

Therefore, if some firm is at an interior solution for either emissions or removal:

$$0 = \frac{\nu_t}{C_0^*} + \frac{1}{1+r}\alpha\tilde{E}_t\left[\tilde{W}_T(T_{t+1},\tilde{\mu}_{t+1},\tilde{\Omega}_{t+1})\right]. \tag{A-20}$$

From the envelope theorem:

$$\tilde{W}_T(T_t,\tilde{\mu}_t,\tilde{\Omega}_t) = -\tilde{\zeta}_t - \tilde{\lambda}_t + \frac{1}{1+r}\tilde{E}_t[\tilde{W}_T(T_{t+1},\tilde{\mu}_{t+1},\tilde{\Omega}_{t+1})],$$

where we recognize that the market equilibrium is independent of temperature under the conjecture that ν_t/C_0 is independent of temperature. Recursively substitute into (A-20) and rearrange:

$$\nu_t = C_0^*\,\alpha\frac{1}{r}\left[\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k + \tilde{\mu}_t\right].$$

We have confirmed the conjecture that ν_t/C_0^* is independent of temperature when $\sum_{i=1}^{N} e_{it} - R_t > 0$. We have established Proposition 2.

[43] At a corner with either $e_{it} = 0$ or $R_{it} = 0$, we would have $\partial e_{it}^*/\partial \nu_t = 0$ or $\partial R_t^*/\partial \nu_t = 0$, respectively, and so would end up at the same optimal tax as below as long as at least one is interior.

H Proof of Lemma 1

Conjecture that the value of the carbon share depends linearly on each $\hat{E}_t[d_{t+j}]$ and $\hat{E}_t[\Delta_{t+j}]$ for $j \geq 0$:

$$\hat{q}_t = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \hat{E}_t[d_{t+j}] + \sum_{j=0}^{\infty} \Psi_{t,t+j} \hat{E}_t[\Delta_{t+j}],$$

with $\{\Lambda_{t,t+j}\}_{j=0}^{\infty}$ and $\{\Psi_{t,t+j}\}_{j=0}^{\infty}$ sequences to be determined.

First consider a case in which $R_t = 0$. The value of a carbon share in period t is $\hat{E}_t[d_t] + \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}]$.

Next consider a case in which $R_t > 0$. The payoff of a shareholder who removes carbon in period t is

$$(1+r)D - \hat{E}_t[\Delta_t] - p_t^R,$$

and the payoff of a shareholder who does not remove carbon in period t is

$$\hat{E}_t[d_t] + \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}].$$

In a competitive equilibrium with abundant carbon shares, shareholders must be indifferent between the two options, implying that

$$p_t^R = (1+r)D - \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}] - \hat{E}_t[d_t] - \hat{E}_t[\Delta_t]. \tag{A-21}$$

Equilibrium payoffs are then identical whether $R_t = 0$ or $R_t > 0$. By absence of arbitrage, the value of the carbon share is:

$$\hat{q}_t = (1+r)D - \hat{E}_t[\Delta_t] - p_t^R.$$

Substitute for $\hat{q}_{t+1}$ from the guess:

$$\hat{q}_t = \hat{E}_t[d_t] + \frac{1}{1+r}\sum_{j=1}^{\infty} \Lambda_{t+1,t+j} \hat{E}_t[d_{t+j}] + \frac{1}{1+r}\sum_{j=1}^{\infty} \Psi_{t+1,t+j} \hat{E}_t[\Delta_{t+j}].$$

Matching coefficients, $\Lambda_{t,t} = 1$ and $\Psi_{t,t} = 0$. Advancing the analysis by one timestep, we find $\Lambda_{t+1,t+1} = 1$ and $\Psi_{t+1,t+1} = 0$. Therefore $\Lambda_{t,t+1} = 1/(1+r)$ and $\Psi_{t,t+1} = 0$. The lemma follows from repeating these steps for subsequent periods, deriving $\Lambda_{t+j,t+j}$ and $\Psi_{t+j,t+j}$, eventually $\Lambda_{t+1,t+j}$ and $\Psi_{t+1,t+j}$, and finally $\Lambda_{t,t+j}$ and $\Psi_{t,t+j}$.

I Proof of Proposition 4

The cost of emitting in period t is $D-(\hat{q}_t-\hat{E}[d_t])$. From (8),

$$D-(\hat{q}_t-\hat{E}[d_t]) = D-\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\hat{E}_t[d_{t+j}],$$

from which (7) implies

$$\begin{aligned}D-(\hat{q}_t-\hat{E}[d_t]) =& D-\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\hat{E}_t[r\,D-\Delta_{t+j}]\\ =&\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\hat{E}_t[\Delta_{t+j}].\end{aligned}$$

Using (6) and the properties of truncated normal distributions,

$$D-(\hat{q}_t-\hat{E}[d_t]) =C_0\,\alpha\left[\frac{1}{r}\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k+\frac{1}{r}\hat{\mu}_t-\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\chi_{t,t+j}\right], \tag{A-22}$$

where

$$\chi_{t,t+j}\triangleq\frac{\phi\left(\frac{\bar{\mu}-\hat{\mu}_t}{\Sigma_{t,t+j}}\right)}{\Phi\left(\frac{\bar{\mu}-\hat{\mu}_t}{\Sigma_{t,t+j}}\right)}\Sigma_{t,t+j}\geq 0$$

is the adjustment to the mean of a normal distribution (for time j random variables, using the time t information set) for the upper truncation point (from the deposit's definition in (5)), $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and cumulative density function for the standard normal distribution, and $\Sigma_{t,t+j}\triangleq\left(\widehat{Var}_t(\hat{\mu}_{t+j})\right)^{1/2}$ is independent of $\bar{\mu}$. Setting $\nu_t=D-\hat{q}_t$ in (A-15), time t emissions are as in (1) as $\chi_{t,t+j}\to 0$ for all $j>0$.

Assumption 1 ensures that some shares are outstanding. Applying the foregoing analysis to equation (A-21), we find that, if some shares are exercised,

$$p_t^R=\frac{1}{1+r}\sum_{j=1}^{\infty}\frac{1}{(1+r)^{j-1}}\hat{E}_t[\Delta_{t+j}].$$

Using in equation (A-10), we find that

$$\begin{aligned}\frac{c_t'(R_t)}{1-c_t(R_t)} =&\frac{1}{1+r}\sum_{j=1}^{\infty}\frac{1}{(1+r)^{j-1}}\hat{E}_t[\Delta_{t+j}]\\ =&C_0\,\alpha\left[\frac{1}{r}\sum_{k=1}^{N}\kappa_k\bar{\zeta}_k+\frac{1}{r}\hat{\mu}_t-\frac{1}{1+r}\sum_{j=1}^{\infty}\frac{1}{(1+r)^{j-1}}\chi_{t,t+j}\right],\end{aligned} \tag{A-23}$$

with $\chi_{t,t+j}$ as above. Time t removal of carbon emitted in all previous periods is as in (2) as $\chi_{t,t+j} \to 0$ for all $s \in \{1, ..., t\}$ and all $j > 0$.

$\chi_{t,t+j}$ decreases in $\bar{\mu}$ and goes to 0 as $\bar{\mu}$ goes to ∞ (i.e., the mean of a truncated-normal distribution increases in the upper bound and approaches the mean of the untruncated normal distribution as the truncation point goes to infinity). By the foregoing analysis, time t emissions and removal are as in (1) and (2) as $\bar{\mu} \to \infty$. So $\breve{L}_t \to 0$ as $\bar{\mu} \to \infty$. We have proved the proposition.

J Preliminaries for Proofs of Propositions 5 and 6

Let $\breve{\mu}_t$ and $\breve{\Omega}_t$ indicate the indicate the mean and variance for $\sum_{k=1}^{N} \kappa_k \zeta_k$ formed after observing $\tilde{\zeta}_s + \tilde{\lambda}_s$ and q_s for all $s < t$, with the corresponding variance of each ζ_i labeled $\breve{\tau}_t^2$ (note that this variance will be independent of i and that $\breve{\tau}_0^2 = \tau_0^2$). $\breve{\tau}_t^2$ will follow naturally from the analysis for $\breve{\mu}_t$. And to avoid repetition, use $\breve{q}_t$ to indicate the observed carbon share price, even in the case without noise traders (in which $\breve{q}_t = \breve{q}_t^*$).

Demand for carbon shares and market-clearing price

Conjecture that $\breve{q}_t$ is a linear function of the $\zeta_{it} + \lambda_{it}$ and that q_{t+1} is a linear function of $\tilde{\zeta}_t + \tilde{\lambda}_t$ and $\breve{q}_t$. In this case, $\breve{q}_t$ and q_{t+1} are normally distributed, and by standard properties of normal random variables and $D \to \infty$, the time t maximization problem for traders of type i is equivalent to:

$$\max_{X_{it}} - \exp\Bigg\{ - A_i(1+r)(w_{it} - X_{it}\breve{q}_t) - A_i(y_{it} + X_{it})\breve{E}_t[q_{t+1} + (1+r)d_t|\, \zeta_{it} + \lambda_{it}, \breve{q}_t] + \frac{1}{2}A_i^2(y_{it} + X_{it})^2 \breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]\Big|\, \zeta_{it} + \lambda_{it}, \breve{q}_t\right]\Bigg\},$$

where $\breve{E}_t$ and $\breve{Var}_t$ indicate the expectation and variance at the common time t beginning-of-period information set. Substituting for d_t from (7), their objective becomes

$$\max_{X_{it}} - \exp\Bigg\{ - A_i(1+r)(w_{it} - X_{it}\breve{q}_t) + \frac{1}{2}A_i^2(y_{it} + X_{it})^2 \breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]\Big|\, \zeta_{it} + \lambda_{it}, \breve{q}_t\right] - A_i(y_{it} + X_{it})\left(\breve{E}_t[q_{t+1}|\, \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left[rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t\Big|\, \zeta_{it} + \lambda_{it}, \breve{q}_t]\right]\right)\Bigg\}.$$

The first-order condition for a maximum is

$$X_{it} = \frac{1}{n_i}h_{it}\left(\breve{E}_t[q_{t+1}|\zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \breve{q}_t]\right) - (1+r)\breve{q}_t\right) - y_{it},$$

where

$$h_{it} \triangleq n_i \left(A_i \breve{Var}_t \left[q_{t+1} - (1+r) C_0 \alpha [\tilde{\zeta}_t + \tilde{\lambda}_t] \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right] \right)^{-1}. \tag{A-24}$$

The h_{it} are deterministic by standard properties of normal-normal updating. Aggregate net demand for carbon shares is

$$X_t = \sum_{i=1}^{N} n_i X_{it}.$$

Market-clearing requires $X_t = 0$. Substituting and rearranging, the market-clearing price is:

$$\begin{aligned}\breve{q}_t^* =& \frac{1}{(1+r)\sum_{i=1}^N h_{it}} \left[\sum_{i=1}^N h_{it} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, q_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) \right) \right. \\ & \left. - \sum_{i=1}^N y_{it} \right].\end{aligned}$$

Define

$$\breve{h}_{it} \triangleq \frac{h_{it}}{\sum_{i=1}^N h_{it}}. \tag{A-25}$$

We have:

$$\begin{aligned}\breve{q}_t^* =& \frac{1}{1+r} \sum_{i=1}^N \breve{h}_{it} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) \right) \\ & - \frac{1}{(1+r)\sum_{k=1}^N h_{kt}} \sum_{i=1}^N y_{it}.\end{aligned}$$

Observe that $\sum_{i=1}^N y_{it} = M_t - M_0$. Therefore

$$\begin{aligned}\breve{q}_t^* =& \frac{1}{1+r} \sum_{i=1}^N \breve{h}_{it} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) \right) \\ & - \frac{M_t - M_0}{(1+r)\sum_{k=1}^N h_{kt}}.\end{aligned} \tag{A-26}$$

Analyzing q_{t+1}

q_{t+1} is the equilibrium price determined by time $t+1$ agents who have common beliefs, so that Lemma 1 applies, modulo the information set. As $D \to \infty$,

$$\begin{aligned} q_{t+1} =& \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left[rD - C_0\alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_{t+1} \right] \\ =& \frac{1+r}{r} \left[rD - C_0\alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_{t+1} \right]. \end{aligned} \tag{A-27}$$

Define $\tilde{q}_t$ as the signal of $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k$ extracted from $\breve{q}_t$, which implies

$$\breve{E}_t[\tilde{q}_t] = \breve{\mu}_t. \tag{A-28}$$

Conjecture that

$$\breve{\mu}_{t+1} = a'_t \breve{\mu}_t + b'_t \left(\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) + B'_t \tilde{q}_t, \tag{A-29}$$

where a'_t, b'_t, and B'_t are constants to be determined. Taking the expectation of each side under the information set at the beginning of time t and using (A-28), we find that $1 = a'_t + b'_t + B'_t$. Under the conjecture,

$$\breve{E}_t[\breve{\mu}_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] = a'_t \breve{\mu}_t + b'_t \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right] + B'_t \tilde{q}_t.$$

Using this with (A-27),

$$\begin{aligned} \breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] =& \frac{1+r}{r} \Bigg[rD - C_0\alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \\ & - C_0 \alpha \left(a'_t \breve{\mu}_t + b'_t \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right] + B'_t \tilde{q}_t \right) \Bigg]. \end{aligned}$$

From (A-26),

$$\begin{aligned} \breve{q}^*_t =& \frac{1+r}{r} \left[rD - C_0\alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right] \\ & - \frac{1}{r} C_0 \alpha \left(a'_t \breve{\mu}_t + B'_t \tilde{q}_t \right) - \frac{b'_t + r}{r} C_0 \alpha \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right] \\ & - \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}}. \end{aligned} \tag{A-30}$$

Deriving $\tilde{q}_t$

From (A-30),

$$\sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right]$$
$$= \frac{1}{C_0 \alpha (b'_t + r)} \left[(1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha (a'_t \breve{\mu}_t + B'_t \tilde{q}_t) - r \breve{q}^*_t - r \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right]. \tag{A-31}$$

Consider setting

$$\tilde{q}_t = \frac{1}{C_0 \alpha (B'_t + b'_t + r)} \left[(1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 a'_t \breve{\mu}_t - r \breve{q}_t - r \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right]. \tag{A-32}$$

In the case without noise traders, (A-31) and $\breve{q}_t = \breve{q}^*_t$ imply

$$\tilde{q}_t = \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}^*_t \right]. \tag{A-33}$$

In the case with noise traders, (A-31) and the definition $\breve{q}_t \triangleq \breve{q}^*_t + \theta_t$ imply

$$\tilde{q}_t = \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right] - \frac{r}{C_0 \alpha (B'_t + b'_t + r)} \theta_t. \tag{A-34}$$

Recalling that $\breve{h}_{it} \in (0,1)$ and $\sum_{i=1}^{N} \breve{h}_{it} = 1$, in either case this $\tilde{q}_t$ satisfies the earlier definition of $\tilde{q}_t$ as the signal of $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k$ extracted from $\breve{q}_t$.

K Proof of Proposition 5

Section J contains preliminaries. An equilibrium is fully revealing if and only if

$$\breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t \middle| \breve{q}^*_t, \{\zeta_{it} + \lambda_{it}\}_{i=1}^{N} \right] = \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t \middle| \breve{q}^*_t \right].$$

And because the price cannot incorporate information not present in the economy,

$$\breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t \middle| \breve{q}^*_t, \{\zeta_{it} + \lambda_{it}\}_{i=1}^{N} \right] = \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t \middle| \{\zeta_{it} + \lambda_{it}\}_{i=1}^{N} \right].$$

Therefore an equilibrium is fully revealing if and only if

$$\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t\middle|\,\breve{q}_t^*\right]=\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t\middle|\,\{\zeta_{it}+\lambda_{it}\}_{i=1}^N\right]. \tag{A-35}$$

Assume that $\breve{q}_t^*$ is fully revealing. In that case,

$$\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t-\sum_{k=1}^N\kappa_k\bar{\zeta}_k\middle|\,\zeta_{it}+\lambda_{it},\breve{q}_t^*\right]=\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t-\sum_{k=1}^N\kappa_k\bar{\zeta}_k\middle|\,\breve{q}_t^*\right]$$

and, from (A-33), (A-35), and $\sum_{i=1}^N\breve{h}_{it}=1$,

$$\tilde{q}_t=\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t-\sum_{k=1}^N\kappa_k\bar{\zeta}_k\middle|\,\{\zeta_{it}+\lambda_{it}\}_{i=1}^N\right].$$

From (A-32) and $\breve{q}_t=\breve{q}_t^*$, the market clears with

$$\begin{aligned}r\breve{q}_t^*=&(1+r)\left(rD-C_0\alpha\sum_{k=1}^N\kappa_k\bar{\zeta}_k\right)-C_0a_t'\breve{\mu}_t-r\frac{M_t-M_0}{(1+r)\sum_{k=1}^N h_{kt}}\\&-C_0\alpha(B_t'+b_t'+r)\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t-\sum_{k=1}^N\kappa_k\bar{\zeta}_k\middle|\,\{\zeta_{it}+\lambda_{it}\}_{i=1}^N\right].\end{aligned} \tag{A-36}$$

The following lemma describes beliefs at the beginning of each period when prices are fully revealing:

Lemma 2. *If each $\breve{q}_t^*$ is fully revealing, then* $\lim_{\sigma^2\to0}\breve{\mu}_t=\lim_{\sigma^2\to0}\hat{\mu}_t$.

Proof. Proceed by induction. Begin by considering updating in period 0, as the basis step. Assume that $\breve{q}_0^*$ is fully revealing. And consider the conditional expectation

$$\breve{E}_0\left[\tilde{\zeta}_0+\tilde{\lambda}_0-\sum_{k=1}^N\kappa_k\bar{\zeta}_k\middle|\,\{\zeta_{i0}+\lambda_{i0}\}_{i=1}^N\right].$$

It is similar to that analyzed for $\hat{\mu}_1$ in Proposition 1 except not having observed $\tilde{\zeta}_0$ and predicting $\tilde{\zeta}_0$ instead of $\tilde{\zeta}_1$. If we defined $\breve{\tau}_0^{2*}\triangleq\tau_0^2+\sigma^2$, then the conditional expectation we seek follows from the proof of Proposition 1 with prior variance $\breve{\tau}_0^{2*}$ in place of τ_0^2 but covariance still $\Gamma\tau_0^2$ (because stochasticity and prior uncertainty here play symmetric roles, except with stochasticity not affecting the correlation), with $\sigma^2=0$ (because stochasticity was subsumed into prior uncertainty), and with $\tilde{\omega}^2 = \infty$ (which is equivalent to not observing $\tilde{\zeta}_0 + \tilde{\lambda}_0$). Therefore, following Proposition 1,

$$\begin{aligned}
&\breve{E}_0 \left[\tilde{\zeta}_0 + \tilde{\lambda}_0 - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \middle| \{\zeta_{i0} + \lambda_{i0}\}_{i=1}^N \right] \\
=& \frac{(1-\Gamma)\tau_0^2 + \sigma^2}{(1-\Gamma)\tau_0^2 + \sigma^2 + \omega^2} \sum_{k=1}^N \kappa_k \left[\zeta_{k0} + \lambda_{k0} - \bar{\zeta}_k \right] \\
&+ \frac{\omega^2}{(1-\Gamma)\tau_0^2 + \sigma^2 + \omega^2} \frac{N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2 + \omega^2 + N\Gamma\tau_0^2} \frac{1}{N} \sum_{k=1}^N \left[\zeta_{k0} + \lambda_{k0} - \bar{\zeta}_k \right]. \qquad \text{(A-37)}
\end{aligned}$$

Substituting this into (A-36) and following the logic of Bayesian updating, we find that

$$\lim_{\sigma^2 \to 0} \breve{E}_0 \left[\tilde{\zeta}_1 + \tilde{\lambda}_1 - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \middle| \breve{q}_0^*, \tilde{\zeta}_0 + \tilde{\lambda}_0 \right] = \lim_{\sigma^2 \to 0} \hat{\mu}_1.$$

We have established the basis step.

Now assume that $\lim_{\sigma^2 \to 0} \breve{\mu}_t = \lim_{\sigma^2 \to 0} \hat{\mu}_t$ for some $t > 0$. By following the same steps as the basis step, we find that $\lim_{\sigma^2 \to 0} \breve{\mu}_{t+1} = \lim_{\sigma^2 \to 0} \hat{\mu}_{t+1}$. We have established the induction step.

□

From this result, (A-33), and (A-37), the conjectured form (A-29) holds as $\sigma^2 \to 0$. Therefore from (A-36), the fully revealing price does clear the market when agents update as Bayesians.

Adapting (A-22) and (A-23) to the current informational environment and applying the conditions of the proposition, time t firms equate both the marginal cost of emission reductions and the marginal cost of carbon removal to $D - (q_t - \breve{E}_t[d_t])$. Using (A-27),

$$\begin{aligned}
D - (q_t - \breve{E}_t[d_t]) =& D - \left(\frac{1+r}{r} - 1 \right) \left[rD - C_0 \alpha \sum_{k=1}^N \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_t \right] \\
=& \frac{1}{r} C_0 \alpha \left[\sum_{k=1}^N \kappa_k \bar{\zeta}_k + \breve{\mu}_t \right].
\end{aligned}$$

The proposition follows from the proof of Proposition 4.

L Proof of Proposition 6

Section J contains preliminaries.

Time t signal $\tilde{q}_t$ as a function of time t information

The combination of normal random variables and an affine information structure implies that the posterior mean is a linear function of the prior and the signals:

$$\breve{E}_t\left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t\right] = a_{it}\breve{\mu}_t + b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) + B_{it}\tilde{q}_t \tag{A-38}$$

for yet-to-be-determined coefficients a_{it}, b_{it}, and B_{it}. Substituting into (A-34), we find:

$$\tilde{q}_t = \sum_{i=1}^{N}\breve{h}_{it}\left[a_{it}\breve{\mu}_t + b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) + B_{it}\tilde{q}_t\right] - \frac{r}{C_0\alpha(B'_t + b'_t + r)}\theta_t.$$

Solving for $\tilde{q}_t$ yields:

$$\tilde{q}_t = \frac{1}{1 - \sum_{i=1}^{N}\breve{h}_{it}B_{it}}\left(\sum_{i=1}^{N}\breve{h}_{it}a_{it}\breve{\mu}_t + \sum_{i=1}^{N}\breve{h}_{it}b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(B'_t + b'_t + r)}\theta_t\right).$$

Taking the expectation under the information set at the beginning of time t and setting it equal to (A-28), we find:

$$\breve{\mu}_t = \frac{\breve{\mu}_t}{1 - \sum_{i=1}^{N}\breve{h}_{it}B_{it}}\sum_{i=1}^{N}\breve{h}_{it}(a_{it} + b_{it}).$$

This holds if and only if $\sum_{i=1}^{N}\breve{h}_{it}B_{it} = 1 - \sum_{i=1}^{N}\breve{h}_{it}(a_{it} + b_{it})$. Define

$$\chi_t \triangleq 1 - \sum_{i=1}^{N}\breve{h}_{it}B_{it}. \tag{A-39}$$

Then:

$$\tilde{q}_t = \frac{1}{\chi_t}\left(\sum_{i=1}^{N}\breve{h}_{it}a_{it}\breve{\mu}_t + \sum_{i=1}^{N}\breve{h}_{it}b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(B'_t + b'_t + r)}\theta_t\right). \tag{A-40}$$

Deriving $\breve{E}_t\left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t\right]$

In time t but prior to observing $\zeta_{it} + \lambda_{it}$ and $\breve{q}_t$, the 3×1 random vector

$$\breve{S}_{it} = \begin{bmatrix} \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N}\kappa_k\bar{\zeta}_k \\ \zeta_{it} + \lambda_{it} - \bar{\zeta}_i \\ \tilde{q}_t \end{bmatrix} \tag{A-41}$$

is jointly normal with unconditional mean

$$\breve{E}_t[\breve{S}_{it}] = \begin{bmatrix} \breve{\mu}_t \\ \breve{\mu}_t \\ \breve{\mu}_t \end{bmatrix}$$

and covariance matrix

$$C\breve{o}v_t(\breve{S}_{it}) =$$

$$\begin{bmatrix} \sum_{j=1}^N \kappa_j^2[\breve{\tau}_t^2+\sigma^2]+2\Gamma\breve{\tau}_t^2\sum_{j=1}^N\sum_{k=j+1}^N \kappa_j\kappa_k+\tilde{\omega}^2 & \kappa_i\breve{\tau}_t^2+(1-\kappa_i)\Gamma\breve{\tau}_t^2+\kappa_i\sigma^2 & \sum_{k=1}^N(\kappa_k\breve{\tau}_t^2+(1-\kappa_k)\Gamma\breve{\tau}_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ \kappa_i\breve{\tau}_t^2+(1-\kappa_i)\Gamma\breve{\tau}_t^2+\kappa_i\sigma^2 & \breve{\tau}_t^2+\sigma^2+\omega^2 & (\breve{\tau}_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\breve{\tau}_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ \sum_{k=1}^N(\kappa_k\breve{\tau}_t^2+(1-\kappa_k)\Gamma\breve{\tau}_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & (\breve{\tau}_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\breve{\tau}_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{array}{c}\sum_{k=1}^N\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2\left(\breve{\tau}_t^2+\sigma^2+\omega^2\right) \\ +2\Gamma\breve{\tau}_t^2\sum_{j=1}^N\sum_{k=j+1}^N\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ +\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2\end{array} \end{bmatrix}. \tag{A-42}$$

From the projection theorem,

$$\breve{E}_t\left[\tilde{\zeta}_t+\tilde{\lambda}_t-\sum_{k=1}^N\kappa_k\bar{\zeta}_k \middle| \zeta_{it}+\lambda_{it},\breve{q}_t\right]$$

$$=\breve{\mu}_t+\begin{bmatrix}\kappa_i\breve{\tau}_t^2+(1-\kappa_i)\Gamma\breve{\tau}_t^2+\kappa_i\sigma^2 \\ \sum_{k=1}^N(\kappa_k\breve{\tau}_t^2+(1-\kappa_k)\Gamma\breve{\tau}_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\breve{\tau}_t^2+\sigma^2+\omega^2 & (\breve{\tau}_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\breve{\tau}_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ (\breve{\tau}_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\breve{\tau}_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{array}{c}\sum_{k=1}^N\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2\left(\breve{\tau}_t^2+\sigma^2+\omega^2\right) \\ +2\Gamma\breve{\tau}_t^2\sum_{j=1}^N\sum_{k=j+1}^N\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ +\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2\end{array}\end{bmatrix}^{-1}$$

$$\begin{bmatrix}\zeta_{it}+\lambda_{it}-\bar{\zeta}_i-\breve{\mu}_t \\ \tilde{q}_t-\breve{\mu}_t\end{bmatrix}.$$

Working through the matrix algebra and matching coefficients to (A-38), we find:

$$\begin{aligned}
b_{it} =&\frac{1}{det_{it}}\Bigg\{ \left(\kappa_i \frac{\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}+(1-\kappa_i)\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\right) \\
&\left[\sum_{k=1}^{N}\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2+2\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\sum_{j=1}^{N}\sum_{k=j+1}^{N}\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right. \\
&\left.+\frac{1}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2\right] \\
&-\left(\sum_{k=1}^{N}\left(\kappa_k\frac{\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}+(1-\kappa_k)\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\right)\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right) \\
&\left(\frac{\breve{h}_{it}b_{it}}{\chi_t}+\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)\Bigg\},
\end{aligned} \tag{A-43}$$

$$\begin{aligned}
B_{it} =&\frac{1}{det_{it}}\Bigg\{\sum_{k\neq i}\Bigg(\left[\kappa_k-\kappa_i\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\right]\frac{\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \\
&+\left[(1-\kappa_k)-(1-\kappa_i)\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\right]\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\Bigg)\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\Bigg\},
\end{aligned} \tag{A-44}$$

and

$$a_{it} = 1 - b_{it} - B_{it}, \tag{A-45}$$

where

$$\begin{aligned}
det_{it} \triangleq& \sum_{k\neq i}\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2+2\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\
&+\frac{1}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2-\left(\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\right)^2\left(\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2.
\end{aligned}$$

Simplifying, (A-43) becomes:

$$b_{it} = \Bigg\{ \sum_{k \neq i} \breve{h}_{kt}^2 b_{kt}^2 + 2\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \sum_{j=1, \neq i}^{N} \sum_{k=j+1, \neq i}^{N} \breve{h}_{jt} b_{jt} \breve{h}_{kt} b_{kt} + \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)} \right)^2 \frac{\Theta^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}$$
$$- \left(\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right)^2 \left(\sum_{k \neq i} \breve{h}_{kt} b_{kt} \right)^2 \Bigg\}^{-1}$$
$$\Bigg\{ \left(\kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right) \sum_{k \neq i} \breve{h}_{kt}^2 b_{kt}^2$$
$$+ \left(\kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right) \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)} \right)^2 \frac{\Theta^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}$$
$$+ \left(\kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right) 2\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \sum_{j=1, \neq i}^{N} \sum_{k=j+1, \neq i}^{N} \breve{h}_{jt} b_{jt} \breve{h}_{kt} b_{kt}$$
$$+ \left(\kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \breve{h}_{it} b_{it} \sum_{k \neq i} \breve{h}_{kt} b_{kt}$$
$$- \breve{h}_{it} b_{it} \sum_{k \neq i} \left(\kappa_k \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_k) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right) \breve{h}_{kt} b_{kt}$$
$$- \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \left(\sum_{k \neq i} \left(\kappa_k \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_k) \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \right) \breve{h}_{kt} b_{kt} \right) \sum_{k \neq i} \breve{h}_{kt} b_{kt} \Bigg\}.$$

Solve for b_{it}:

$$
\begin{aligned}
b_{it} = &\left(\kappa_i \frac{\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\right) \\
&\Bigg\{ \left(\breve{\tau}_t^2+\sigma^2+\omega^2-\Gamma\breve{\tau}_t^2\right)\left(\breve{\tau}_t^2+\sigma^2+\omega^2+\Gamma\breve{\tau}_t^2\right)\sum_{k\neq i}\breve{h}_{kt}^2 b_{kt}^2 \\
&\quad + 2\Gamma\breve{\tau}_t^2\left[\breve{\tau}_t^2+\sigma^2+\omega^2-\Gamma\breve{\tau}_t^2\right]\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\breve{h}_{jt}b_{jt}\breve{h}_{kt}b_{kt} \\
&\quad + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2(\breve{\tau}_t^2+\sigma^2+\omega^2) \\
&\quad - \Bigg[\sum_{k\neq i}\Bigg(\left[\kappa_k(\breve{\tau}_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\breve{\tau}_t^2\right](\breve{\tau}_t^2+\sigma^2) \\
&\qquad\qquad + \left[(1-\kappa_k)(\breve{\tau}_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\breve{\tau}_t^2\right]\Gamma\breve{\tau}_t^2\Bigg)\breve{h}_{kt}b_{kt}\Bigg] \\
&\qquad \sum_{k\neq i}\frac{\Gamma\breve{\tau}_t^2}{\kappa_i(\breve{\tau}_t^2+\sigma^2)+(1-\kappa_i)\Gamma\breve{\tau}_t^2}\breve{h}_{kt}b_{kt}\Bigg\} \\
&\Bigg\{ \left(\breve{\tau}_t^2+\sigma^2+\omega^2-\Gamma\breve{\tau}_t^2\right)\left(\breve{\tau}_t^2+\sigma^2+\omega^2+\Gamma\breve{\tau}_t^2\right)\sum_{k\neq i}\breve{h}_{kt}^2 b_{kt}^2 \\
&\quad + 2\Gamma\breve{\tau}_t^2\left[\breve{\tau}_t^2+\sigma^2+\omega^2-\Gamma\breve{\tau}_t^2\right]\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\breve{h}_{jt}b_{jt}\breve{h}_{kt}b_{kt} \\
&\quad + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2(\breve{\tau}_t^2+\sigma^2+\omega^2) \\
&\quad + \breve{h}_{it}\sum_{k\neq i}\Bigg(\left[\kappa_k(\breve{\tau}_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\breve{\tau}_t^2\right](\breve{\tau}_t^2+\sigma^2) \\
&\qquad\qquad + \left[(1-\kappa_k)(\breve{\tau}_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\breve{\tau}_t^2\right]\Gamma\breve{\tau}_t^2\Bigg)\breve{h}_{kt}b_{kt}\Bigg\}^{-1}. \qquad \text{(A-46)}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\left[\kappa_k(\breve{\tau}_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\breve{\tau}_t^2\right](\breve{\tau}_t^2+\sigma^2) + \left[(1-\kappa_k)(\breve{\tau}_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\breve{\tau}_t^2\right]\Gamma\breve{\tau}_t^2 \\
=&\left[(1-\Gamma)\breve{\tau}_t^2+\sigma^2+\omega^2\right]\left[\kappa_k(\breve{\tau}_t^2+\sigma^2)+(1-\kappa_i)\Gamma\breve{\tau}_t^2\right],
\end{aligned}
$$

which decreases in κ_i and is strictly positive as $\kappa_i \to 1$, and thus is strictly positive for all relevant κ_i. Because that expression is positive,

$$b_{it} < \kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1-\kappa_i)\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2},$$

and because this last inequality holds for arbitrary i, inspection of (A-46) shows that $b_{it} > 0$. Therefore the set of functions defined by (A-46) for each $i \in \{1, ..., N\}$ maps a vector from

$$\bigtimes_{i=1}^{N} \left[0, \kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1-\kappa_i)\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}\right]$$

into itself. By Brouwer's fixed-point theorem, there exists a fixed point in that space. By inspection, the fixed point does not have any b_{kt} on the boundary. Therefore, for each $i \in \{1, ..., N\}$,

$$b_{it} = \left(\kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1-\kappa_i)\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}\right) \breve{Z}_{it} \tag{A-47}$$

for some $\breve{Z}_{it} \in (0, 1)$. Observe that $\breve{Z}_{it}$ and b_{it} are deterministic because each $\breve{h}_{kt}$ is deterministic.

Deriving $\breve{\mu}_{t+1}$

Substituting (A-47) into (A-40) and using $\sum_{i=1}^{N} \breve{h}_{it} a_{it} = \chi_t - \sum_{i=1}^{N} \breve{h}_{it} b_{it}$ from (A-39) and (A-45),

$$\begin{aligned}\tilde{q}_t =& \frac{\chi_t - \sum_{i=1}^{N} \breve{h}_{it} b_{it}}{\chi_t} \breve{\mu}_t + \sum_{i=1}^{N} \left(\kappa_i \frac{\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + (1-\kappa_i)\frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}\right) \frac{\breve{h}_{it}\breve{Z}_{it}}{\chi_t}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) \\ &- \frac{r}{C_0 \alpha (B'_t + b'_t + r)} \frac{\theta_t}{\chi_t}. \end{aligned} \tag{A-48}$$

Define:

$$\breve{\kappa}_{it} \triangleq \breve{h}_{it} \breve{Z}_{it}, \tag{A-49}$$

$$\breve{w}_{it} \triangleq \kappa_i \breve{\kappa}_{it} \frac{(1-\Gamma)\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + \breve{\kappa}_{it} \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}, \tag{A-50}$$

$$\breve{\chi}_t \triangleq B'_t + b'_t.$$

$\breve{\kappa}_{it} \in (0, 1)$, $\breve{w}_{it} \in (0, 1)$, and $\breve{\chi}_t \in (0, 1)$. Using these,

$$\tilde{q}_t - \frac{\chi_t - \sum_{i=1}^{N} \breve{h}_{it} b_{it}}{\chi_t} \breve{\mu}_t = \sum_{i=1}^{N} \frac{\breve{w}_{it}}{\chi_t}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0 \alpha (\breve{\chi}_t + r)} \frac{\theta_t}{\chi_t}. \tag{A-51}$$

The left-hand side is measurable by agents at the beginning of time t based on their knowledge of earlier carbon share prices and earlier aggregate signals.

Consider the zero-mean $(t+2)\times 1$ random vector

$$\breve{s}_t \triangleq \begin{bmatrix} \sum_{k=1}^N \kappa_k \zeta_k \\ \tilde{q}_0 - \frac{\chi_0 - \sum_{i=1}^N \breve{h}_{i0} b_{i0}}{\chi_0} \breve{\mu}_0 \\ \vdots \\ \tilde{q}_{t-1} - \frac{\chi_{t-1} - \sum_{i=1}^N \breve{h}_{i(t-1)} b_{i(t-1)}}{\chi_{t-1}} \breve{\mu}_{t-1} \\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \end{bmatrix}.$$

Observe that

$$\breve{\mu}_t \triangleq \breve{E}_0\left[\sum_{k=1}^N \kappa_k \zeta_k \middle| \left\{\tilde{q}_j - \frac{\chi_j - \sum_{i=1}^N \breve{h}_{ij} b_{ij}}{\chi_j}\breve{\mu}_j\right\}_{j=0}^{t-1}, \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k, \right].$$

Let $\breve{\Psi}_t$ indicate the $(t+1)\times(t+1)$ covariance matrix of the final $t+1$ elements of $\breve{s}_t$ and $\breve{\Sigma}_t$ indicate the $1\times(t+1)$ vector of covariances between $\sum_{k=1}^N \kappa_k \zeta_k$ and the other $t+1$ elements of $\breve{s}_t$, so that

$$[\breve{\Sigma}_t]^\intercal \triangleq \begin{bmatrix} (1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k \breve{w}_{k0}\frac{1}{\chi_0} + \Gamma\tau_0^2 \sum_{k=1}^N \breve{w}_{k0}\frac{1}{\chi_0} \\ \vdots \\ (1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k \breve{w}_{k(t-1)}\frac{1}{\chi_{t-1}} + \Gamma\tau_0^2 \sum_{k=1}^N \breve{w}_{k(t-1)}\frac{1}{\chi_{t-1}} \\ (1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + \Gamma\tau_0^2 \end{bmatrix}.$$

From the projection theorem,

$$\breve{\mu}_t = \breve{\Sigma}_t \breve{\Psi}_t^{-1} \begin{bmatrix} \tilde{q}_0 - \frac{\chi_0 - \sum_{i=1}^N \breve{h}_{i0} b_{i0}}{\chi_0} \breve{\mu}_0 \\ \vdots \\ \tilde{q}_{t-1} - \frac{\chi_{t-1} - \sum_{i=1}^N \breve{h}_{i(t-1)} b_{i(t-1)}}{\chi_{t-1}} \breve{\mu}_{t-1} \\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \end{bmatrix} \tag{A-52}$$

$$= \tilde{\pi}_t \left(\frac{1}{t}\sum_{k=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k\right) + \sum_{j=0}^{t-1} \breve{\pi}'_{jt}\left(\tilde{q}_j - \frac{\chi_j - \sum_{i=1}^N \breve{h}_{ij} b_{ij}}{\chi_j}\breve{\mu}_j\right), \tag{A-53}$$

where $\tilde{\pi}_t$ and $\breve{\pi}'_{jt}$ are defined via the matrix multiplication in (A-52). Using (A-51), this becomes

$$\breve{\mu}_t = \tilde{\pi}_t \left(\frac{1}{t}\sum_{k=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^N \kappa_k \bar{\zeta}_k\right) + \sum_{j=0}^{t-1} \breve{\pi}'_{jt}\left(\sum_{i=1}^N \frac{\breve{w}_{ij}}{\chi_j}(\zeta_{ij} + \lambda_{ij} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(\breve{\chi}_j + r)}\frac{\theta_j}{\chi_j}\right).$$

Define

$$\breve{\pi}_{ij} \triangleq \breve{\pi}'_{jt}/\chi_j. \tag{A-54}$$

The following lemma establishes that division by χ_j cancels a χ_j in $\breve{\pi}'_{jt}$:

Lemma 3. *Holding all parameters fixed, $\tilde{\pi}_j$ and $\breve{\pi}_{jt}$ are independent of χ_j.*

Proof. See Appendix P. □

Using this definition and the definition of $\breve{w}_{jt}$ from (A-50), we obtain:

$$\begin{aligned}\breve{\mu}_t =&\tilde{\pi}_t \left(\frac{1}{t}\sum_{k=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k\right)\\ &+ \sum_{j=0}^{t-1} \breve{\pi}_{jt} \sum_{i=1}^{N} \left(\kappa_i \breve{\kappa}_{it} \frac{(1-\Gamma)\breve{\tau}_t^2 + \sigma^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} + \breve{\kappa}_{it} \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}\right)(\zeta_{ij} + \lambda_{ij} - \bar{\zeta}_i)\\ &- \sum_{j=0}^{t-1} \breve{\pi}_{jt} \frac{r}{C_0 \alpha (\breve{\chi}_j + r)} \theta_j.\end{aligned} \tag{A-55}$$

Emissions and carbon removal

Adapting (A-22) and (A-23) to the current informational environment and applying the conditions of the proposition, time t firms equate both the marginal cost of emission reductions and the marginal cost of carbon removal to $D - (q_t - \breve{E}_t[d_t])$. Using (A-27),

$$\begin{aligned}D - (q_t - \breve{E}_t[d_t]) =&D - \left(\frac{1+r}{r} - 1\right)\left[rD - C_0\alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_t\right]\\ =&\frac{1}{r} C_0 \alpha \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \breve{\mu}_t\right].\end{aligned}$$

The proposition follows from using (A-55) to define $\breve{\mu}_t$.

M Proof of Corollary 3

Consider updating beliefs at the beginning of time $t+1$ based on beliefs from the beginning of time t, on the time t aggregate measurement, and on the time t carbon share price. Define the vector of time t signals as

$$\breve{S}_t \triangleq \begin{bmatrix} \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \\ \tilde{q}_t - \frac{\chi_t - \sum_{i=1}^{N} \breve{\kappa}_{it}}{\chi_t} \breve{\mu}_t \end{bmatrix}.$$

Consider the random vector

$$\begin{bmatrix} \sum_{i=1}^N \kappa_i \zeta_i \\ \breve{S}_t \end{bmatrix}.$$

Using (A-51), its mean going into time t is

$$\begin{bmatrix} \breve{\mu}_t \\ \breve{\mu}_t \\ \breve{\breve{\mu}}_t \end{bmatrix},$$

where

$$\breve{\breve{\mu}}_t \triangleq \breve{E}_t \left[\sum_{i=1}^N \frac{\breve{w}_{it}}{\chi_t} (\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) \right].$$

Observe that

$$\breve{\mu}_{t+1} \triangleq \breve{E}_t \left[\sum_{k=1}^N \kappa_k \zeta_k \middle| \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k, \tilde{q}_t - \frac{\chi_t - \sum_{i=1}^N \breve{\kappa}_{it}}{\chi_t} \breve{\mu}_t \right].$$

Let $\breve{\Psi}_{S,t}$ indicate the 2×2 covariance matrix of $\breve{S}_t$ and $\breve{\Sigma}_{S,t}$ indicate the 1×2 vector of covariances between $\sum_{k=1}^N \kappa_k \zeta_k$ and the elements of $\breve{S}_t$, so that

$$\breve{\Psi}_{S,t} = \begin{bmatrix} [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] \sum_{i=1}^N \kappa_i^2 + \Gamma \breve{\tau}_t^2 + \tilde{\omega}^2 & [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] \sum_{i=1}^N \kappa_i \breve{w}_{it} \frac{1}{\chi_t} \\ & + \Gamma \breve{\tau}_t^2 \sum_{i=1}^N \breve{w}_{it} \frac{1}{\chi_t} \\ [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] \sum_{i=1}^N \kappa_i \breve{w}_{it} \frac{1}{\chi_t} & [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2 + \omega^2] \sum_{i=1}^N \breve{w}_{it}^2 \frac{1}{\chi_t^2} \\ + \Gamma \breve{\tau}_t^2 \sum_{i=1}^N \breve{w}_{it} \frac{1}{\chi_t} & + \Gamma \breve{\tau}_t^2 \left(\sum_{i=1}^N \breve{w}_{it} \right)^2 \frac{1}{\chi_t^2} \\ & + \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)} \right)^2 \frac{\Theta^2}{\chi_t^2} \end{bmatrix}$$

and

$$[\breve{\Sigma}_{S,t}]^\intercal \triangleq \begin{bmatrix} (1-\Gamma)\breve{\tau}_t^2 \sum_{k=1}^N \kappa_k^2 + \Gamma \breve{\tau}_t^2 \\ (1-\Gamma)\breve{\tau}_t^2 \sum_{k=1}^N \kappa_k \breve{w}_{kt} \frac{1}{\chi_t} + \Gamma \breve{\tau}_t^2 \sum_{k=1}^N \breve{w}_{kt} \frac{1}{\chi_t} \end{bmatrix}.$$

From the projection theorem,

$$\breve{\mu}_{t+1} = \breve{\mu}_t + \breve{\Sigma}_{S,t} \breve{\Psi}_{S,t}^{-1} \left(\breve{S}_t - \begin{bmatrix} \breve{\mu}_t \\ \breve{\breve{\mu}}_t \end{bmatrix} \right).$$

The time t carbon share price enters through the second element of $\breve{S}_t$, and earlier carbon share prices potentially enter through $\breve{\breve{\mu}}_t$ and $\breve{\mu}_t$ (observing that $\breve{\tau}_t^2$ is independent of realizations by familiar properties of normal-normal updating). The coefficient on

$$\tilde{q}_t - \frac{\chi_t - \sum_{i=1}^N \breve{\kappa}_{it}}{\chi_t} \breve{\mu}_t - \breve{\breve{\mu}}_t$$

is $\breve{\pi}'_{(t-1)t}$ from (A-53). Working through the matrix algebra,

$$\begin{aligned}\breve{\pi}'_{(t-1)t} =& \frac{1}{det(\breve{\Psi}_{S,t})}\Bigg\{ - \left[(1-\Gamma)\breve{\tau}_t^2 \sum_{k=1}^N \kappa_k^2 + \Gamma\breve{\tau}_t^2\right]\left[[(1-\Gamma)\breve{\tau}_t^2+\sigma^2]\sum_{i=1}^N \kappa_i \breve{w}_{it}\frac{1}{\chi_t} + \Gamma\breve{\tau}_t^2 \sum_{i=1}^N \breve{w}_{it}\frac{1}{\chi_t}\right] \\ &+ \left[(1-\Gamma)\breve{\tau}_t^2 \sum_{k=1}^N \kappa_k \breve{w}_{kt}\frac{1}{\chi_t} + \Gamma\breve{\tau}_t^2\sum_{k=1}^N \breve{w}_{kt}\frac{1}{\chi_t}\right]\left[[(1-\Gamma)\breve{\tau}_t^2+\sigma^2]\sum_{i=1}^N \kappa_i^2 + \Gamma\breve{\tau}_t^2 + \tilde{\omega}^2\right]\Bigg\} \\ =& \frac{1}{det(\breve{\Psi}_{S,t})\chi_t}\Bigg\{\sigma^2\Gamma\breve{\tau}_t^2\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{k=1}^N \breve{w}_{kt} - \sum_{i=1}^N \kappa_i \breve{w}_{it}\right] + \tilde{\omega}^2\left[(1-\Gamma)\breve{\tau}_t^2\sum_{k=1}^N \kappa_k \breve{w}_{kt} + \Gamma\breve{\tau}_t^2\sum_{k=1}^N \breve{w}_{kt}\right]\Bigg\}.\end{aligned}$$

Using (A-50), this becomes:

$$\begin{aligned}\breve{\pi}'_{(t-1)t} =& \frac{1}{det(\breve{\Psi}_{S,t})\chi_t}\Bigg\{\sigma^2\Gamma\breve{\tau}_t^2\Bigg(\frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{i=1}^N \kappa_i\breve{\kappa}_{it} - \sum_{i=1}^N \kappa_i^2\breve{\kappa}_{it}\right] \\ &+ \frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{i=1}^N \breve{\kappa}_{it} - \sum_{i=1}^N \kappa_i\breve{\kappa}_{it}\right]\Bigg) \\ &+ \tilde{\omega}^2\left[(1-\Gamma)\breve{\tau}_t^2\sum_{k=1}^N \kappa_k\breve{w}_{kt} + \Gamma\breve{\tau}_t^2\sum_{k=1}^N \breve{w}_{kt}\right]\Bigg\}. \qquad \text{(A-56)}\end{aligned}$$

Because $\breve{\kappa}_{it}$ is independent of i when each $\kappa_i = 1/N$, the top two lines on the right-hand side vanish when each $\kappa_i = 1/N$. And they clearly vanish if either $\Gamma = 0$ or $\sigma^2 = 0$. The bottom line vanishes if $\tilde{\omega}^2 = 0$. Therefore $\breve{\pi}'_{(t-1)t} = 0$ (and thus, from (A-54), $\breve{\pi}_{(t-1)t} = 0$) if $\tilde{\omega}^2 = 0$ and either $\Gamma = 0$, $\sigma^2 = 0$, or each $\kappa_i = 1/N$.

We have established that $\breve{\mu}_{t+1}$ does not depend on either $\tilde{q}_t$ or $\breve{\breve{\mu}}_t$ if $\tilde{\omega}^2 = 0$ and either $\Gamma = 0$, $\sigma^2 = 0$, or each $\kappa_i = 1/N$, although it does depend on $\breve{\mu}_t$. And because we showed this for arbitrary t, argument by induction shows that $\breve{\mu}_{t+1}$ does not depend on any earlier carbon share prices (and thus each $\breve{\pi}_{jt} = 0$ for $j \in \{0, ..., t-1\}$) if $\tilde{\omega}^2 = 0$ and either $\Gamma = 0$, $\sigma^2 = 0$, or each $\kappa_i = 1/N$.

Next consider when $\breve{\pi}'_{(t-1)t}$ is nonzero. The bottom line on the right-hand side of (A-56) is strictly positive if $\tilde{\omega}^2 > 0$ and zero otherwise. So $\breve{\pi}'_{(t-1)t}$ (and thus, from (A-54), $\breve{\pi}_{(t-1)t}$) is strictly positive if $\tilde{\omega}^2 > 0$ and either $\sigma^2 = 0$, $\Gamma = 0$, or each $\kappa_i = 1/N$.

Now consider the first two lines on the right-hand side of (A-56).

Lemma 4. *When $\sigma^2 > 0$ and $\Gamma > 0$, the first two lines in curly braces in* (A-56) *are minimized when each $\kappa_i = 1/N$.*

Proof. The κ_i and $\breve{\kappa}_{it}$ that minimize the first two lines in curly braces in (A-56) solve

$$\min_{\{\kappa_i\}_{i=1}^N,\{\breve{\kappa}_{it}\}_{i=1}^N} \Bigg\{ \frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{i=1}^N \kappa_i\breve{\kappa}_{it} - \sum_{i=1}^N \kappa_i^2\breve{\kappa}_{it}\right] + \frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{i=1}^N \breve{\kappa}_{it} - \sum_{i=1}^N \kappa_i\breve{\kappa}_{it}\right]\Bigg\}$$

$$\text{s.t.} \quad \sum_{i=1}^N \kappa_i = 1.$$

Form the Lagrangean:

$$\min_{\{\kappa_i\}_{i=1}^N,\{\breve{\kappa}_{it}\}_{i=1}^N} \Bigg\{ \frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{i=1}^N \kappa_i\breve{\kappa}_{it} - \sum_{i=1}^N \kappa_i^2\breve{\kappa}_{it}\right] + \frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{i=1}^N \kappa_i^2\right)\sum_{i=1}^N \breve{\kappa}_{it} - \sum_{i=1}^N \kappa_i\breve{\kappa}_{it}\right] + \lambda_t\left(1-\sum_{i=1}^N \kappa_i\right)\Bigg\}.$$

The first-order necessary condition for κ_i ($i \in \{1,...,N\}$) is:

$$0 = \frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[2\kappa_i\sum_{j=1}^N \kappa_j\breve{\kappa}_{jt} + \left(\sum_{j=1}^N \kappa_j^2\right)\breve{\kappa}_{it} - 2\kappa_i\breve{\kappa}_{it}\right] + \frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[2\kappa_i\sum_{j=1}^N \breve{\kappa}_{jt} - \breve{\kappa}_{it}\right] - \lambda_t.$$

The first-order necessary condition for $\breve{\kappa}_{it}$ ($i \in \{1,...,N\}$) is:

$$0 = \frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\left(\sum_{j=1}^N \kappa_j^2\right)\kappa_i - \kappa_i^2\right] + \frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[\sum_{j=1}^N \kappa_j^2 - \kappa_i\right],$$

which requires that each $\kappa_i = \sum_{j=1}^N \kappa_j^2$ and thus that each $\kappa_i = 1/N$. This symmetry implies that the $\breve{\kappa}_{it}$ are independent of i at a solution to the first-order conditions, so write $\breve{\kappa}_{it}$ as $\breve{\kappa}_t$. The first-order conditions for the κ_i imply $\lambda_t = \frac{1}{N}\frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2+N\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\breve{\kappa}_t$, so a solution permits any strictly positive $\breve{\kappa}_t$. It is easy to see that the objective is zero when each $\kappa_i = 1/N$.

Now consider the κ_i that minimize the first two lines on the right-hand side of (A-56) for any given $\breve{\kappa}_t$:

$$\min_{\{\kappa_i\}_{i=1}^N} \left\{ \frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \left[\left(\sum_{i=1}^N \kappa_i^2 \right) \sum_{i=1}^N \kappa_i \breve{\kappa}_t - \sum_{i=1}^N \kappa_i^2 \breve{\kappa}_t \right] + \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \left[\left(\sum_{i=1}^N \kappa_i^2 \right) \sum_{i=1}^N \breve{\kappa}_t - \sum_{i=1}^N \kappa_i \breve{\kappa}_t \right] \right\}$$
$$\text{s.t.} \quad \sum_{i=1}^N \kappa_i = 1.$$

Factor $\breve{\kappa}_t$ and substitute in the constraint:

$$\min_{\{\kappa_i\}_{i=1}^{N-1}} \left\{ \breve{\kappa}_t \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \left[N \sum_{i=1}^{N-1} \kappa_i^2 + N \left(1 - \sum_{i=1}^{N-1} \kappa_i \right)^2 - 1 \right] \right\}.$$

The first-order condition for κ_i is

$$0 = 2N\breve{\kappa}_t \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \left[\kappa_i - \kappa_N \right].$$

For $\Gamma > 0$, the solution requires that $\kappa_i = \kappa_N$ for all i and thus requires that each $\kappa_i = 1/N$. The $N-1 \times N-1$ Hessian has element (i,i)

$$4N\breve{\kappa}_t \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}$$

and element (i,j)

$$2N\breve{\kappa}_t \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}.$$

The Hessian is double-constant at the solution to the first-order conditions. Via standard results (e.g., O'Neill, 2020, Theorem 1), the determinant of its kth leading principal minor is

$$\left(N\breve{\kappa}_t \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \right)^k (4-2)^{k-1} (4-2+2k)$$
$$= \left(N\breve{\kappa}_t \frac{\Gamma \breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2} \right)^k 2^k (1+k).$$

This is strictly positive for all k. Therefore the second-order condition for a minimum holds when each $\kappa_i = 1/N$ with $\Gamma > 0$. And because there is no other critical point within the set of feasible $\{\kappa_i\}_{i=1}^N$, that minimum is a global minimum over the relevant domain.

□

Since the first two lines on the right-hand side of (A-56) are zero when each $\kappa_i = 1/N$, and since zero is the minimum (for $\Gamma > 0$), they are strictly positive when $\Gamma > 0$, $\sigma^2 > 0$, and some $\kappa_i \neq 1/N$. We have established that $\breve{\pi}'_{(t-1)t} > 0$ (and thus, from (A-54), $\breve{\pi}_{(t-1)t} > 0$) if either (i) $\tilde{\omega}^2 > 0$ or (ii) $\Gamma > 0$ and $\sigma^2 > 0$ with some $\kappa_i \neq 1/N$.

N Proof of Corollary 4

In (A-49), define $\breve{\kappa}_{it}^{BS} \triangleq \breve{Z}_{it}$ and $\breve{\kappa}_{it}^{RA} \triangleq \breve{h}_{it}$.

From (A-46) and (A-47), $\lim_{\Theta^2\to\infty} \breve{Z}_{it} = 1$. We have established part i of the corollary.

By inspection, equation (A-46) holds if $b_{it} = 0$ for all $i \in \{1, ..., N\}$ with $\Theta^2 = 0$, and because the denominator contains a term that is linear in the b_{kt} whereas the numerator contains only products of the b_{kt}, equation (A-46) holds if each b_{it} is arbitrarily small with Θ^2 arbitrarily small. We have established part ii of the corollary.

Parts iii and iv of the corollary follows from using (A-24) in (A-25) and taking limits.

Now analyze h_{it}. From the definition (A-24),

$$1/h_{it} = n_i^{-1} A_i \breve{Var}_t \left[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] + n_i^{-1} A_i \breve{Var}_t \left[(1+r) C_0 \alpha [\tilde{\zeta}_t + \tilde{\lambda}_t] \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right]. \quad \text{(A-57)}$$

From (A-29),

$$\breve{Var}_t[\mu_{t+1} | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = (b'_t)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t].$$

Using this in (A-27),

$$\breve{Var}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = \left(\frac{1+r}{r} C_0 \alpha b'_t \right)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t],$$

$$\breve{Cov}_t \left[q_{t+1}, -(1+r) C_0 \alpha [\tilde{\zeta}_t + \tilde{\lambda}_t] \middle| \zeta_{it} + \lambda_{it}, \breve{q}_t \right] = [(1+r) C_0 \alpha]^2 \frac{b'_t}{r} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t].$$

Substituting into (A-57),

$$1/h_{it} = n_i^{-1} A_i \left[\frac{1+r}{r} C_0 \alpha \right]^2 (b'_t + r)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t]. \quad \text{(A-58)}$$

Apply the projection theorem to (A-41), via (A-42):

$$
\begin{aligned}
&\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] \\
=&[\breve{\tau}_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + 2\Gamma \breve{\tau}_t^2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \kappa_j \kappa_k + \tilde{\omega}^2 \\
&- \begin{bmatrix} \kappa_i \breve{\tau}_t^2 + (1-\kappa_i)\Gamma\breve{\tau}_t^2 + \kappa_i \sigma^2 \\ \sum_{k=1}^N (\kappa_k \breve{\tau}_t^2 + (1-\kappa_k)\Gamma\breve{\tau}_t^2 + \kappa_k \sigma^2) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \end{bmatrix}^\mathsf{T}
\begin{bmatrix} \breve{\tau}_t^2 + \sigma^2 + \omega^2 & (\breve{\tau}_t^2 + \sigma^2 + \omega^2) \frac{\breve{h}_{it} b_{it}}{\chi_t} + \Gamma \breve{\tau}_t^2 \sum_{k \neq i} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \\ (\breve{\tau}_t^2 + \sigma^2 + \omega^2) \frac{\breve{h}_{it} b_{it}}{\chi_t} + \Gamma \breve{\tau}_t^2 \sum_{k \neq i} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} & \begin{matrix} \sum_{k=1}^N \left(\frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right)^2 (\breve{\tau}_t^2 + \sigma^2 + \omega^2) \\ + 2\Gamma \breve{\tau}_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \frac{\breve{h}_{jt} b_{jt}}{\chi_t} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \\ + \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)} \frac{\Theta}{\chi_t} \right)^2 \end{matrix} \end{bmatrix}^{-1} \\
&\begin{bmatrix} \kappa_i \breve{\tau}_t^2 + (1-\kappa_i)\Gamma\breve{\tau}_t^2 + \kappa_i \sigma^2 \\ \sum_{k=1}^N (\kappa_k \breve{\tau}_t^2 + (1-\kappa_k)\Gamma\breve{\tau}_t^2 + \kappa_k \sigma^2) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \end{bmatrix}
\end{aligned}
$$

Using the b_{it} and B_{it} from (A-43) and (A-44), this becomes

$$
\begin{aligned}
\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] =& [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + \Gamma \breve{\tau}_t^2 + \tilde{\omega}^2 - b_{it} \left[\kappa_i [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] + \Gamma \breve{\tau}_t^2 \right] \\
&- B_{it} \left[\sum_{k=1}^{N} (\kappa_k [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] + \Gamma \breve{\tau}_t^2) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \right].
\end{aligned}
$$

Substituting from (A-47),

$$
\begin{aligned}
\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] =& [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + \Gamma \breve{\tau}_t^2 + \tilde{\omega}^2 - \frac{\left[\kappa_i [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] + \Gamma \breve{\tau}_t^2 \right]^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \breve{Z}_{it} \\
&- B_{it} \left[\sum_{k=1}^{N} \frac{(\kappa_k [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] + \Gamma \breve{\tau}_t^2)^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2} \frac{\breve{h}_{kt} \breve{Z}_{kt}}{\chi_t} \right].
\end{aligned}
$$

Now consider part v of the corollary. If $\kappa_i = 1/N$ for each $i \in \{1, .., N\}$, then from (A-44)and (A-46), $\breve{Z}_{it}$ and B_{it} are independent of i. And we then have that $\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t]$ is independent of i. From that result, (A-58), and the definition (A-25), $\breve{h}_{kt} = n_k A_k^{-1} / \sum_{j=1}^N n_j A_j^{-1}$.

Finally, consider part vi of the corollary. We saw above that $\lim_{\Theta^2 \to \infty} \breve{Z}_{it} = 1$. And from (A-44), $\lim_{\Theta^2 \to \infty} B_{it} = 0$. Therefore,

$$
\lim_{\Theta^2 \to \infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + \Gamma \breve{\tau}_t^2 + \tilde{\omega}^2 - \frac{\left[\kappa_i [(1-\Gamma)\breve{\tau}_t^2 + \sigma^2] + \Gamma \breve{\tau}_t^2 \right]^2}{\breve{\tau}_t^2 + \sigma^2 + \omega^2}.
$$

Simplifying, we find:

$$\lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{it}+\lambda_{it},\tilde{q}_t] =\tilde{\omega}^2+[(1-\Gamma)\breve{\tau}_t^2+\sigma^2]\left(\sum_{j\neq i}\kappa_j^2+\frac{\omega^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\kappa_i^2\right) + \frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left((1-\kappa_i)^2[(1-\Gamma)\breve{\tau}_t^2+\sigma^2]+\omega^2\right).$$

Without loss of generality, order the sectors by increasing κ_i. Then $\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{it}+\lambda_{it},\tilde{q}_t]\geq \lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{jt}+\lambda_{jt},\tilde{q}_t]$ for all $j>i$ if, for all $\{n,n+k\}$ such that $\kappa_n<\kappa_{n+k}$,

$$\sum_{j\neq n}\kappa_j^2+\frac{\omega^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\kappa_n^2+\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}(1-\kappa_n)^2 \geq \sum_{j\neq n+k}\kappa_j^2+\frac{\omega^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\kappa_{n+k}^2+\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}(1-\kappa_{n+k})^2.$$

This sufficient condition is equivalent to

$$(\kappa_{n+k}^2-\kappa_n^2)+\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\left[-2(\kappa_n-\kappa_{n+k})+\kappa_n^2-\kappa_{n+k}^2\right]\geq\frac{\omega^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}(\kappa_{n+k}^2-\kappa_n^2),$$

and thus is equivalent to

$$\frac{(1-\Gamma)\breve{\tau}_t^2+\sigma^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}\frac{\kappa_n+\kappa_{n+k}}{2}\geq-\frac{\Gamma\breve{\tau}_t^2}{\breve{\tau}_t^2+\sigma^2+\omega^2}.$$

This condition clearly holds and thus so does the sufficient condition for $\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{it}+\lambda_{it},\tilde{q}_t]\geq\lim_{\Theta^2\to\infty}\breve{Var}_t[\tilde{\zeta}_t+\tilde{\lambda}_t|\zeta_{jt}+\lambda_{jt},\tilde{q}_t]$ for all $j>i$. Therefore from (A-58), the sequence $\{h_{1t},...,h_{Nt}\}$ is monotone increasing as $\Theta^2\to 0$ when sectors are ordered by increasing κ_i and $\{n_1A_1^{-1},...,n_NA_N^{-1}\}$ is weak monotone increasing. And from the definition (A-25), in that case the sequence $\{\breve{h}_{1t},...,\breve{h}_{Nt}\}$ is monotone increasing as $\Theta^2\to 0$, with $\breve{h}_{1t}\leq n_1A_1^{-1}/\sum_{j=1}^N n_jA_j^{-1}$ and $\breve{h}_{Nt}\geq n_NA_N^{-1}/\sum_{j=1}^N n_jA_j^{-1}$. The latter inequalities are strict if any $\kappa_i\neq 1/N$, which in turn is equivalent to $\kappa_1<1/N$.

O Proof of Corollary 5

Let G_t be cumulative revenue collected, which is invested with rate of return r:

$$G_{t+1}=(1+r)\left\{G_t+\nu_t\left[\sum_{i=1}^N e_{it}-R_t\right]\right\},$$

with $G_0 \geq 0$. The constraint in time t is now

$$G_t + \nu_t \left[\sum_{i=1}^{N} e_{it} - R_t \right] \geq 0.$$

This constraint binds only if $\sum_{i=1}^{N} e_{it} - R_t \leq 0$. Denote the smallest ν_t at which the constraint binds as $\bar{\nu}_t$. By the implicit function theorem,

$$\frac{\mathrm{d}\bar{\nu}_t}{\mathrm{d}G_t} = -\frac{1}{\sum_{i=1}^{N} e_{it}^*(\bar{\nu}_t) - R_t^*(\bar{\nu}_t) + \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t}\Big|_{\bar{\nu}_t} - \frac{\partial R_t^*}{\partial \nu_t}\Big|_{\bar{\nu}_t}} > 0,$$

where a star indicates market equilibrium. The sign follows from (A-18), (A-19), and emissions being net negative when the constraint binds.

The regulator solves the following Bellman equation:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \max_{\nu_t} \tilde{E}_t \Bigg[u\bigg((1 - c_t(R_t^*)) \prod_{i=1}^{N} [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i} \bigg) + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \Bigg],$$

where $\tilde{E}_t$ denotes expectations over the regulator's information set at the beginning of period t. Market outcomes will be insensitive to ν_t for $\nu_t > \bar{\nu}_t$, so the regulator's objective will be constant in ν_t for $\nu_t > \bar{\nu}_t$. Maximized welfare is therefore equivalent for a regulator who solves the following problem in which ν_t is constrained to be less than or equal to $\bar{\nu}_t$:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \max_{\nu_t \leq \bar{\nu}_t} \tilde{E}_t \Bigg[u\bigg((1 - c_t(R_t^*)) \prod_{i=1}^{N} [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i} \bigg) + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \Bigg].$$

At an interior solution, the regulator's first-order condition is

$$\begin{aligned} 0 = &\sum_{i=1}^{N} \frac{\kappa_i Y_e^{i\prime}(e_{it}^*)}{Y^{it}(e_{it}^*)} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t \Big[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \Big] \alpha \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} \\ &- \frac{c_t'(R_t^*)}{1 - c_t(R_t^*)} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t \Big[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \Big] \alpha \frac{\partial R_t^*}{\partial \nu_t} \\ &+ \tilde{E}_t \Big[\tilde{W}_G(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1}) \Big] \underbrace{\left\{ \sum_{i=1}^{N} e_{it}^* - R_t^* + \nu_t \left[\sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{\partial R_t^*}{\partial \nu_t} \right] \right\}}_{\text{Marginal effect of emission tax on revenue}}. \end{aligned} \tag{A-59}$$

From the envelope theorem,

$$\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \tilde{E}_t\left[\tilde{W}_G(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\right]$$

if $\nu_t < \bar{\nu}_t$ and

$$\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \frac{\mathrm{d}\bar{\nu}_t}{\mathrm{d}G_t}\frac{\partial \tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t)}{\partial \bar{\nu}_t}$$

if $\nu_t = \bar{\nu}_t$. Advancing these to later timesteps, we find

$$\begin{aligned}\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = &\sum_{s=1}^{\infty} Pr(\nu_{t+j} < \bar{\nu}_{t+j}\, \forall j \in \{1, ..., s-1\}) Pr(\nu_{t+s} = \bar{\nu}_{t+s}) \\ &\frac{\mathrm{d}\bar{\nu}_{t+s}}{\mathrm{d}G_{t+s}}\frac{\partial \tilde{W}(T_{t+s}, \tilde{\mu}_{t+s}, \tilde{\Omega}_{t+s}, G_{t+s})}{\partial \bar{\nu}_{t+s}}\end{aligned}$$

if $\nu_t < \bar{\nu}_t$. Because increasing $\bar{\nu}_{t+s}$ loosens a constraint, the final derivative is strictly positive. We saw above that the other derivative on the final line is strictly positive. And the probabilities on the first line are strictly positive because they depend on $\tilde{\mu}_{t+j}$ and $\tilde{\mu}_{t+s}$, which in turn depend on draws from normally distributed variables that have infinite support. Therefore $\tilde{W}_G > 0$.

Use ν_t^{NoLB} to denote the optimal ν_t implied by (A-17) and ν_t^{LB} to denote the optimal ν_t implied by (A-59). ν_t^{NoLB} is the tax described in Proposition 2 when emissions are strictly positive. If we evaluate (A-59) around ν_t^{NoLB}, then it reduces to its final line. Since we have established that $\tilde{W}_G > 0$, the sign of that final line of (A-59) matches the sign of the term in curly braces, which is the change in revenue due to a marginal change in the tax. If that change is positive, then this term increases the first-order condition and makes $\nu_t^{LB} > \nu_t^{NoLB}$ by concavity around a maximum. And if that change is negative, then this term decreases the first-order condition and makes $\nu_t^{LB} < \nu_t^{NoLB}$ by concavity around a maximum. We have established the first part of the corollary.

The curly braces in the final line of (A-59) are weakly (strictly) negative when net emissions are weakly (strictly) negative. Because net emissions weakly (strictly) decrease in ν_t when net emissions are weakly (strictly) negative, we have established the second part of the corollary in the case that the dynamic revenue constraint does not bind. And the second part of the corollary holds trivially when the dynamic revenue constraint does bind.

P Proof of Lemma 3

Consider $\breve{\Psi}_t^{-1}$. Label the $t \times t$ upper-left block of $\breve{\Psi}_t$ as $\breve{\Psi}_A$, the $t \times 1$ upper right block as $\breve{\Psi}_B$, the $1 \times t$ lower left block as $\breve{\Psi}_C$, and the 1×1 lower right block as $\breve{\Psi}_D$. From familiar results for block matrix inversion,

$$\breve{\Psi}_t^{-1} = \begin{bmatrix} \breve{\Psi}_A^{-1} + \breve{\Psi}_A^{-1}\breve{\Psi}_B(\breve{\Psi}_D - \breve{\Psi}_C\breve{\Psi}_A^{-1}\breve{\Psi}_B)^{-1}\breve{\Psi}_C\breve{\Psi}_A^{-1} & -\breve{\Psi}_A^{-1}\breve{\Psi}_B(\breve{\Psi}_D - \breve{\Psi}_C\breve{\Psi}_A^{-1}\breve{\Psi}_B)^{-1} \\ -(\breve{\Psi}_D - \breve{\Psi}_C\breve{\Psi}_A^{-1}\breve{\Psi}_B)^{-1}\breve{\Psi}_C\breve{\Psi}_A^{-1} & (\breve{\Psi}_D - \breve{\Psi}_C\breve{\Psi}_A^{-1}\breve{\Psi}_B)^{-1} \end{bmatrix}.$$

Using (A-51), element (j,j) of $\breve{\Psi}_A$ is

$$[(1-\Gamma)\tau_0^2+\sigma^2+\omega^2]\sum_{i=1}^N \breve{w}_{i(j-1)}^2\frac{1}{\chi_{j-1}^2}+\Gamma\tau_0^2\left(\sum_{i=1}^N \breve{w}_{i(j-1)}\right)^2\frac{1}{\chi_{j-1}^2}+\left(\frac{r}{C_0\alpha(\breve{\chi}_{j-1}+r)}\right)^2\Theta^2\frac{1}{\chi_{j-1}^2},$$

element (j,k) of $\breve{\Psi}_A$ is

$$(1-\Gamma)\tau_0^2\sum_{i=1}^N \breve{w}_{i(j-1)}\breve{w}_{i(k-1)}\frac{1}{\chi_{j-1}}\frac{1}{\chi_{k-1}}+\Gamma\tau_0^2\left(\sum_{i=1}^N \breve{w}_{i(j-1)}\right)\left(\sum_{i=1}^N \breve{w}_{i(k-1)}\right)\frac{1}{\chi_{j-1}}\frac{1}{\chi_{k-1}}$$
$$+\frac{r^2}{C_0^2\alpha^2(\breve{\chi}_{j-1}+r)(\breve{\chi}_{k-1}+r)}\Theta^2\frac{1}{\chi_{j-1}\chi_{k-1}}$$

for $j \neq k$, the jth element of $\breve{\Psi}_B$ and $\breve{\Psi}_C$ is

$$[(1-\Gamma)\tau_0^2+\sigma^2/t]\sum_{i=1}^N \kappa_i\breve{w}_{i(j-1)}\frac{1}{\chi_{j-1}}+\Gamma\tau_0^2\sum_{i=1}^N \breve{w}_{i(j-1)}\frac{1}{\chi_{j-1}},$$

and $\breve{\Psi}_D$ is

$$[(1-\Gamma)\tau_0^2+\sigma^2/t]\sum_{i=1}^N \kappa_i^2+\Gamma\tau_0^2+\tilde{\omega}^2.$$

In $\breve{\Psi}_A^{-1}$, element (j,k) is proportional to $\chi_{j-1}\chi_{k-1}$. $\breve{\Psi}_D - \breve{\Psi}_C\breve{\Psi}_A^{-1}\breve{\Psi}_B$ does not contain any χ, and element (j,k) of $\breve{\Psi}_A^{-1}\breve{\Psi}_B(\breve{\Psi}_D - \breve{\Psi}_C\breve{\Psi}_A^{-1}\breve{\Psi}_B)^{-1}\breve{\Psi}_C\breve{\Psi}_A^{-1}$ is proportional to $\chi_{j-1}\chi_{k-1}$. So, for $j,k \leq t$, element (j,k) of $\breve{\Psi}_t^{-1}$ is proportional to $\chi_{j-1}\chi_{k-1}$. Now observe that element k of $\breve{\Psi}_C\breve{\Psi}_A^{-1}$ is proportional to χ_{k-1} and element j of $\breve{\Psi}_A^{-1}\breve{\Psi}_B$ is proportional to χ_{j-1}. Therefore element $(t+1,k)$ of $\breve{\Psi}_t^{-1}$ is proportional to χ_{k-1} for $k \leq t$ and element $(j,t+1)$ of $\breve{\Psi}_t^{-1}$ is proportional to χ_{j-1} for $j \leq t$.

Using those results and the observation that element j of $\breve{\Sigma}_t$ is proportional to $1/\chi_{j-1}$ if $j \leq t$ and does not contain any χ term otherwise, we find that element $(1,k)$ of $\breve{\Sigma}_t\breve{\Psi}_t^{-1}$ is proportional to χ_{k-1} if $k \leq t$ and element $(1,t+1)$ of $\breve{\Sigma}_t\breve{\Psi}_t^{-1}$ does not contain any χ term.

We therefore have that $\tilde{\pi}_t$ is independent of all χ and that $\breve{\pi}'_{jt}$ is proportional to χ_j but independent of all other χ. Therefore $\breve{\pi}_{jt}$ is independent of all χ.

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