Information Processing: Contracts versus Communication*

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Abstract

We consider the trade-off between imperfect control and communication. A principal anticipates receiving private information and hires an agent to take an action. The principal can contractually tie the agent's action to the state, but this control is incomplete. States not covered by a contract induce a communication game. We show that close alignment of interests favors communication and, thus, ceding authority to the agent, and *vice versa*. In the uniform-quadratic setting, optimal contracts use clauses to separate events that induce distinct communication actions. Hence, equilibria of the contract writing game are partitional and monotonic. The separation of distinct communication events relaxes incentive compatibility and, therefore, helps equalize the size of communication events. This highlights the dual role of contracting as both substituting for and facilitating communication – the principal uses contracts not only to impose her favorite actions, but also to structure communication.

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1 Introduction

Organizations and contractual relationships need to deal with the anticipated arrival of private information. This is the case in *private sector procurement*, in which the procuring firm expects better information about final product specifications; with *contractors*, who learn about projects before their employees do; and, with *management*, which plans to use market analysis to direct product development. Ideally, the use of the information is completely controlled in advance, with actions pre-specified for every possible contingency. Private information, non-verifiability, insufficient language specificity, etc., however, may make it difficult to institute the rules and procedures necessary to exercise this degree of control. This introduces a potential role for non-binding communication and motivates our investigation of when and how cheap talk is used as a substitute for contractual control of the flow of information.

We consider a simple dyadic organization with imperfect control and the option of strategic information transmission (Crawford and Sobel (1982), henceforth CS). A principal (the sender, she) hires an agent (the receiver, he) to take an action for her. At the contracting stage, the sender faces a competitive market for receivers and can, therefore, determine the conditions of the hire. Prior to receiving private information about the state of the world, the sender writes a contract that prescribes actions as a function of the state. Contracts are lists of clauses, with each clause identifying a set of states and the action to be taken for that set.

We assume that contracts are incomplete: there is a finite upper bound on the number of clauses. In addition to this assumed contractual incompleteness, the sender can choose contracts to be *obligationally incomplete*:[1] contracts need not cover all contingencies. An obligationally incomplete contract induces a communication game, in which the sender has the option to provide information about states not covered by the contract, and the receiver is free to respond optimally to that information. There is no commitment in the communication game and messages are costless. Hence, communication is cheap talk.

The timing is as follows. In stage 1, the sender writes the contract. The contract commits the receiver to follow instructions for specified subsets of the state space ("conditions") and commits the sender to provide those instructions.[2] The contract is incomplete, in part because of the effort involved in clarifying the language to the point where the descriptions of conditions and instructions are sufficiently clear to permit third parties to verify that the contract was faithfully executed, once states covered by the contract become verifiable.[3] In stage 2, the state is realized and privately observed by the sender. In stage 3, the sender sends a message to the receiver. The sender cannot disclose the state. Messages are either instructions that appear in the contract or "other messages" (cheap talk). Instructions are the only messages that can be observed and understood by third parties. For states covered

[1] Ayres and Gertner (1992)

[2] When the contract gets executed only instructions need to be communicated and not any information about the state beyond what is implied by the instruction.

[3] In addition, contracting incentivizes the adoption of *measurement systems*, without which the states of nature would remain unobservable (Allen and Gale (1992)).

by the contract, the contract obliges the sender to communicate the stipulated instruction to the receiver. In stage 4, after having observed the sender's message, the receiver takes an action. Any communicated instruction obliges the receiver to take the action that matches the instruction. Any other communicated message leaves discretion to the receiver to take his preferred action. Finally, in stage 5, payoffs are realized and the states covered by the contract become verifiable: The contract establishes and clarifies the language that enables third parties to identify instructions and the conditions under which they apply. This makes it possible for third parties to check whether the sender gave the proper instruction given the state, and whether the receiver's action matched that instruction. Other communication is too informal or too private to allow a third-party to relate the message to the realized state and the action taken.

When writing a contract, the sender weighs the benefits of controlling the agent's actions against the responsiveness of those actions to information. Each contractual clause allows the sender to enforce her preferred action, conditional on a set of states. This control over the agent's actions, however, is imperfect because of the assumed contractual incompleteness. Ceding authority to the agent for some states and then relying on non-binding communication gives the principal an additional degree of freedom. At the expense of having the agent choose his preferred action rather than the principal's, ceding authority to the agent can help making the organization more responsive to information overall. The analysis of this trade-off between control and information responsiveness is the focus of the present paper.

Under general distributional and payoff assumptions, the sender always uses the maximal number of clauses. Optimal obligational incompleteness depends on this bound and on the degree of incentive alignment (the sender's bias relative to the receiver). For any fixed bias, if the bound on the number of clauses increases without limit, optimal contracts approximate obligationally complete contracts. Conversely, fixing the maximal number of clauses, with near-perfect incentive alignment, nearly all states will induce communication. In that case, the optimal contract will be highly obligationally incomplete.

Using the leading example of CS, with a uniform type distribution, quadratic loss functions, and a constant bias, we can be more explicit. For any maximal number of contract clauses, there is a value of the bias such that for any higher bias, any optimal contract will be obligationally complete – there will be no communication. For any fixed bias, there is a contract that allows for more actions to be induced by communication than in the standard cheap talk game without contracting.

Our main characterization result in this uniform-quadratic-constant-bias environment establishes that whenever there is communication, contract clauses will be used to separate events that induce distinct communication actions. As a consequence, equilibria of the contract writing game are partitional and monotonic. The use of contract clause to separate events that induce distinct communication actions relaxes incentive constraints in the communication game. This makes it possible to equalize the size of communication intervals relative to pure cheap talk. This highlights the dual role of contracting as both substituting for and facilitating communication – the sender uses contracts not only to impose her favorite actions, but also to structure communication.

Finally, we explore various extensions of our setup in a setting with quadratic loss functions, a comparatively large bias (that is small enough to make some communication optimal with contracting), and a single contract clause. We show that among conditions that are finite unions of disjoint intervals, intervals are optimal. In addition we explore the impact of non-constant biases, non-uniform distributions, and the role of transfers. With small departures from out baseline, optimal contracts remain close to those from the uniform-quadratic environment with a constant bias. In the case of a non-constant bias, the optimal condition shifts toward covering states with more conflict; with a non-uniform distribution the optimal condition shifts toward covering more likely states; and, with transfers the optimal condition shrinks in size.

Simon (1951) is the first to draw attention to the importance of contractual incompleteness. He notes that many contracts take the form of an "employment contract." An employment contract, in exchange for a fixed wage, transfers authority to the principal rather than providing a detailed specification of the agent's action. In our setting, also, the principal forgoes a detailed specification of the agent's actions, but unlike in Simon (1951), for actions not controlled by the contract, authority resides with the agent, and the principal resorts to communication to influence the agent's action.

Writing costs are sometimes used to rationalize contractual incompleteness. Dye (1985) is the first to make writing and monitoring cost explicit. He notes that contracts with specifications so detailed that they are sensitive to every state are prohibitively expensive to write. The contracts he considers consist of finite lists of clauses, with conditions partitioning the state space. The cost of writing a contract is increasing in the number of clauses.

Battigalli and Maggi (2002) explore the foundations of writing costs by making the language in which contracts are written explicit. A contract specifies a list of clauses and a transfer. Clauses map contingencies into instructions. More elaborate clauses require more "primitive sentences" and are therefore more costly. This results in two types of contractual incompleteness: *rigidity* – insufficient dependence on the state of the world; and *discretion* – insufficient precision in the prescription of behavior.

Our environment also gives rise to rigidity and discretion: whenever the optimal contract does not cover all states, the state space splits into a contracting region (states covered by the contract) and a communication region (states not covered by the contract). We have rigidity in the contracting region and discretion in the communication region. Greater alignment of interests, which facilitates communication, favors discretion, and *vice versa*.

Shavell (2006) (see also Schwartz and Watson (2013)) studies the impact of contract interpretation by courts on the writing of contracts. Again, contracts are lists of clauses, each comprised of a condition (a set of states of the world) and an instruction.[4] Because of writing costs contracts may contain *gaps* – sets of states not covered by any condition. Contracts may be incomplete in two senses: they may not be *fully detailed complete*, which would require a specific clause for each contingency, and they may not be *obligationally complete* (see Ayres and Gertner (1992)), having the above-mentioned gaps. One role of

[4]Heller and Spiegler (2008) allow for contradictory clauses, in which conditions overlap, but the corresponding instructions differ.

interpretation is to fill gaps, another to replace stated with interpreted clauses. The prospect of interpretation, like the prospect of communication in our setting, shapes how contracts are written.

Since Simon (1951), the interplay of information and authority has played an important role in the study of organizations. Aghion and Tirole (1997) distinguish the right to make a decision (formal authority) from the power to influence a decision (real authority). Either the principal or the agent has formal authority. Real authority requires information that players can acquire at a cost. There is no explicit model of communication.

Dessein (2002) examines the conditions under which an uninformed principal cedes authority to a better-informed agent.[5] He adopts an incomplete contracting approach in which authority, but not actions, can be contracted upon. The principal has a choice between delegating decision rights to the agent and making decisions herself after communicating with the agent. In our setting, the principal has the informational advantage but may cede authority to the agent if sufficiently closely aligned incentives make communication attractive.

Deimen and Szalay (2019) compare delegation with communication when there is endogenous information. In their setup, the agent can choose how much and what kind of information to acquire. The principal can choose whether to delegate decision rights or to rely on communication from the agent. In contrast, in our setup there is communication when the decision rights are left with the agent, and it is from the principal to the agent.

Aumann and Hart (2003), Golosov, Skreta, Tsyvinski and Wilson (2014), and Krishna and Morgan (2004) examine different versions of models with repeated cheap talk. One feature that these models have in common with ours is that new communication opportunities may arise as the result of subsets of types having been removed: if at some stage the sender sends a message that is only used by a strict subset of types, at the following stage the receiver can concentrate beliefs on the remaining types. Removing types may facilitate communication for the remaining types since fewer incentive constraints have to be dealt with. In Aumann and Hart (2003) and Krishna and Morgan (2004) types exit because they prefer not to take their chances in a jointly controlled lottery. In Golosov et al. (2014) types are induced to exit by receiver actions that follow each communication round. In our setting, types are removed from the communication game by being covered by a condition in the contract.

We abstain from modeling transfers explicitly in the main analysis, consistent with Battigalli and Maggi (2002), Shavell (2006), Dessein (2002), and others. In our environment, transfers play no role in providing incentives to supply information or to induce actions. We briefly discuss an example in the extensions that suggests that our results can be expected to generalize if we allow for *ex ante* transfers.[6]

[5]In the literature on optimal delegation (See, for example, Holmström (1977), Holmström (1984), Melumad and Shibano (1991), Szalay (2005), Alonso and Matouschek (2008), Kováč and Mylovanov (2009), and Amador and Bagwell (2013)), the uninformed principal decides how to optimally constrain the decision rights of the informed agent.

[6]Formally, our model corresponds to the limit of cases in which the agent cares primarily about the wage and only secondarily about the decision that is made. The more the agent cares about his wage, the less reason there is for the principal to compromise on the decision. In the extreme, when the agent has a

The paper is structured as follows. After presenting the model in Section 2, we introduce the communication subgame in Section 3. Section 4 offers a simple example of optimal contracts. In Section 5, we study properties of sender-optimal equilibria, first in a general framework, Section 5.1, and then under the assumption of having a uniform type distribution and quadratic payoff functions with a constant bias, Section 5.2. We discuss extensions in Section 6. In the final section we summarize our findings and suggest possible directions for future work. All proofs are in the Appendix.

2 Model

We consider a game with two players, a sender, S, and a receiver, R. They interact in two phases. In the first phase, prior to the sender receiving information about the state of the world, the sender writes a contract. That contract determines how information is dealt with in the second phase. Contract clauses specify actions as a function of the state of the world.[7] We refer to these actions as instructions. By assumption, the clauses in the contract are coarse, and the sender may choose a contract that does not cover all states of the world. For states covered by the contract, the instruction specified by the respective clause is implemented. For the remaining states, a communication game is played.

The payoff and information structure closely follows CS. The players' payoffs, $U^S(y, \theta, b)$ for the sender and $U^R(y, \theta)$ for the receiver, depend on the receiver's action $y \in \mathbb{R}$, the state of the world $\theta \in [0, 1]$, and a parameter $b > 0$ that measures the divergence of preferences between the sender and the receiver.[8] The state is drawn from a common prior distribution F with continuous density f that is positive everywhere; $f(\theta) > 0$ for all $\theta \in [0, 1]$. The payoff functions U^i, for $i = R, S$, are assumed to be twice continuously differentiable. Denoting derivatives by subscripts, we assume that the payoff functions are strictly concave: $U^i_{11} < 0$; the sorting condition $U^i_{12} > 0$ holds; and, for all θ, there is an action $y^i(\theta)$ such that $U^i_1(y^i(\theta), \theta) = 0$. We assume that $y^S(\theta) > y^R(\theta)$ for all $\theta \in [0, 1]$.

At the beginning of the *contract-writing game* G, the sender writes a *contract* $\mathcal{C} = \{(C_k, x_k)\}_{k=1}^K$. The contract specifies K *clauses* (C_k, x_k), $k = 1, \ldots, K$. There is an exogenous maximal number of clauses $\widehat{K}$.[9] Each clause (C_k, x_k) consists of a *condition* $C_k \subseteq [0, 1]$ and an *instruction* $x_k \in \mathbb{R}$. The interpretation is that if condition C_k holds – i.e., $\theta \in C_k$ is realized – then the receiver is instructed to take the action $y = x_k$. Contracts must satisfy: $C_{k'} \cap C_{k''} = \emptyset$ for all $k' \neq k''$ (to avoid contradictions); C_k is an interval for each $k = 1, \ldots, K$

lexicographic preference that favors his wage, any action the principal prescribes in the contract will be her own favored action, conditional on the available information, as is the case in the model we analyze in the paper.

[7]Note that we assume that there are no transfers, and therefore there is no incentive provision through contingent transfers.

[8]For notational convenience, we will sometimes suppress the dependence of the sender's payoff on the bias b.

[9]This corresponds to a limiting case of writing costs that are increasing in the number of clauses (see, e.g., Dye (1985)). Writing costs are zero for the first $\widehat{K}$ clauses and prohibitive thereafter.

(motivated by keeping contracts simple);[10] and $\bigcup_{k=1}^{K} C_k$ is a closed set. Denote the lower (upper) endpoint of the interval C_k by $\underline{C}_k$ ($\overline{C}_k$). For any $\delta \in \mathbb{R}$, we refer to the clause $(C_k + \delta, x_k + \delta)$ as the *δ-translation of the clause* (C_k, x_k) and to the condition $C_k + \delta$ as the *δ-translation of the condition* C_k.[11] We allow for an empty contract without clauses, in which case we adopt the convention that $K = 0$. An *obligationally complete contract* covers the entire state space, in which case $\bigcup_{k=1}^{K} C_k = [0, 1]$. Denote the set of all contracts by $\mathfrak{C}$. Sometimes, it will be convenient to highlight the maximal number of clauses and the sender's bias, in which case we make the dependence of the game on these parameters explicit and write $G(\widehat{K}, b)$ for the contract writing game.

After the contract $\mathcal{C}$ is written and observed by the receiver, the state θ is realized and privately observed by the sender. For any state covered by the contract – e.g., $\theta \in C_{k'}$ – the instruction stipulated for that state, $x_{k'}$, is implemented. For any state not covered by the contract $\mathcal{C}$, the sender sends a message $m \in M$ to the receiver, where M is an infinite measurable space. After observing the sender's message, the receiver takes an action $y \in \mathbb{R}$.

Every contract $\mathcal{C}$ induces a *communication subgame*, $\Gamma^{\mathcal{C}}$, in the event that the state θ belongs to the *gap* $\mathcal{L}(\mathcal{C}) := [0, 1] \setminus \bigcup_{k=1}^{K} C_k$ in the contract – i.e., $\theta \in \mathcal{L}(\mathcal{C})$. In this communication subgame, the commonly known type distribution $F^{\mathcal{C}}$ is the prior F concentrated on the set $\mathcal{L}(\mathcal{C})$. If the contract $\mathcal{C}$ is empty, we denote the resulting communication subgame by Γ^0. The communication subgame Γ^0 is simply a CS game. If we want to make the dependence of the communication subgame on the bias parameter explicit, we write $\Gamma^{\mathcal{C}}(b)$. A (behavior) strategy $\sigma : \mathcal{L}(\mathcal{C}) \to \Delta(M)$ of the sender in the communication subgame $\Gamma^{\mathcal{C}}$ maps states to distributions over messages. A strategy $\rho : M \to \mathbb{R}$ for the receiver in $\Gamma^{\mathcal{C}}$ maps messages to actions. Given the strict concavity of the receiver's utility, the restriction to pure receiver strategies is without loss of generality. A sender strategy $\left(\mathcal{C}; \left(\sigma^{\mathcal{C}'}\right)_{\mathcal{C}' \in \mathfrak{C}}\right)$ in the contract-writing game G specifies a contract $\mathcal{C}$ and for every possible communication subgame $\Gamma^{\mathcal{C}'}$ a strategy $\sigma^{\mathcal{C}'}$. A receiver strategy $\left(\left(\rho^{\mathcal{C}'}\right)_{\mathcal{C}' \in \mathfrak{C}}\right)$ in the game G specifies a strategy $\rho^{\mathcal{C}'}$ for every possible communication subgame $\Gamma^{\mathcal{C}'}$. We are interested in sender-optimal subgame-perfect equilibria of the contract-writing game $G(\widehat{K}, b)$, denoted by $e(\widehat{K}, b)$. We refer to the contracts chosen in these equilibria as *optimal contracts*.

3 Communication

For a strategy profile $(\sigma^{\mathcal{C}}, \rho^{\mathcal{C}})$ in communication subgame $\Gamma^{\mathcal{C}}$, we say that a *communication action y* is *induced* by that profile if there is a type θ and a message m in the support of

[10]Sets other than intervals require more detailed and, therefore, more costly descriptions. In Section 6.1 we explore a departure from the assumption that conditions are intervals. We show that in the leading example of CS, for a sufficiently large bias (that is small enough to make some communication optimal with contracting), contracts with single conditions that are intervals are optimal among contracts with single conditions that are non-trivial unions of disjoint intervals.

[11]Here, for any set $C \subset \mathbb{R}$ and any $\delta \in \mathbb{R}$, $C + \delta$ denotes the Minkowski sum of the sets C and $\{\delta\}$ – i.e., $C + \delta := \{c' \in \mathbb{R} | \exists c \in C \text{ s.t. } c' = c + \delta\}$.

$\sigma^{\mathcal{C}}(\theta)$ such that $\rho^{\mathcal{C}}(m) = y$. If, in addition, $(\sigma^{\mathcal{C}}, \rho^{\mathcal{C}})$ is an equilibrium profile, we say that action y is *induced in equilibrium*. As in CS, if the actions that are induced in equilibrium are $0 < y_1 < y_2 < \ldots < y_{n-1} < y_n < 1$, there are $n+1$ *critical types* $0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n = 1$ such that type θ_j is indifferent between actions y_j and y_{j+1} for $j = 1, \ldots, n-1$. We follow the convention of referring to the indifference requirement for critical type θ_j, $j = 1, \ldots, n-1$, as that type's *arbitrage condition*. Since a critical type may belong to a condition C_k, unlike in CS, critical types do not necessarily bound the sets of types who induce a common action. In an equilibrium that induces n actions, we refer to the interval (θ_{j-1}, θ_j) as *step* j, for $j = 1, \ldots, n$. We call an equilibrium that induces n actions an n*-step equilibrium*.

Given an equilibrium $e^{\mathcal{C}}$ of the communication subgame $\Gamma^{\mathcal{C}}$, we refer to an interval $\left(\underline{\theta}, \overline{\theta}\right)$ of types as a *communication interval* if there is an action y that is induced with positive probability in $e^{\mathcal{C}}$, $\underline{\theta} = \inf\{\theta \in [0,1] | \theta \text{ induces } y\}$, and $\overline{\theta} = \sup\{\theta \in [0,1] | \theta \text{ induces } y\}$. Observe that for each action y_j that is induced in equilibrium, the corresponding communication interval is a (possibly strict) subset of the step (θ_{j-1}, θ_j). For an illustration, see Figure 1.

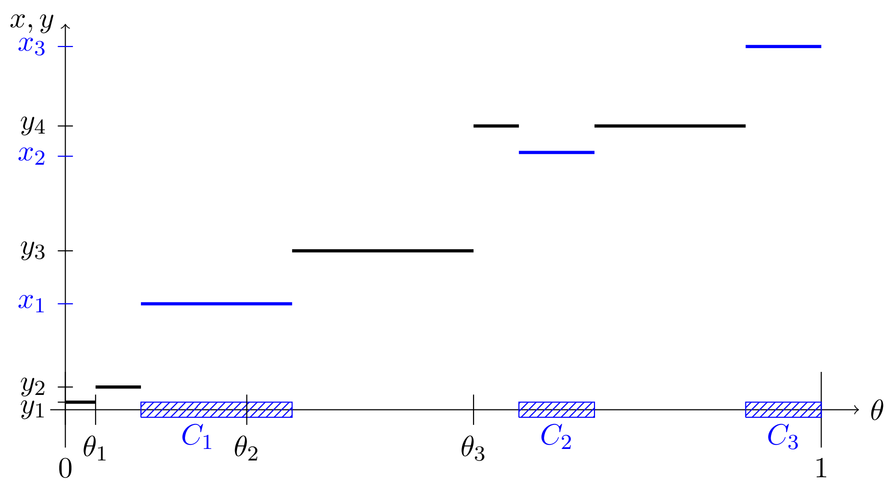

Figure 1: Contract with conditions C_i and instructions x_i for $i = 1, 2, 3$ and induced 4-step equilibrium with critical types $\theta_1, \theta_2, \theta_3$ and actions y_j for $j = 1, 2, 3, 4$.

The standard communication game introduced by CS is included in our setup as subgame Γ^0, in which no contract is written. It is straightforward to see that CS's Lemma 1 holds for all communication subgames, including those induced by contracts with a positive number of clauses.

Lemma 1 *(CS Lemma 1) There exists an* $\varepsilon > 0$*, uniform over all communication subgames* $\Gamma^{\mathcal{C}}$*, such that for every equilibrium in* $\Gamma^{\mathcal{C}}$ *and all actions* y *and* y' *induced in that equilibrium,* $|y - y'| \geq \varepsilon$*. There is an upper bound on the number of actions that are induced in equilibrium that is uniform across all communication subgames*

Proof. See CS Lemma 1. That ε is uniform over all communication subgames follows from the fact that the type distribution plays no role in the proof. $\square$

The maximal number of actions $N(\mathcal{C})$ that can be induced in communication subgame $\Gamma^{\mathcal{C}}$ can vary with the distribution $F^{\mathcal{C}}$ that is induced by contract $\mathcal{C}$. At the same time, since ε is uniform across communication subgames, there is an upper bound $\widehat{N} \in \mathbb{N}$ on the number of equilibrium actions that is uniform across all communication subgames – i.e., $N(\mathcal{C}) \leq \widehat{N} \in \mathbb{N}$ for all $\mathcal{C} \in \mathfrak{C}$.

Another fact familiar from CS remains true in our setting: For every communication subgame $\Gamma^{\mathcal{C}}$, all equilibria are interval partitional. That is, for every equilibrium action, the set of types who induce that action is of the form $I \cap \mathcal{L}(\mathcal{C})$, where $I \subset [0,1]$ is an interval.

4 Example

Suppose that payoff functions are quadratic, $U^S(y,\theta,b) = -(\theta+b-y)^2$, $U^R(y,\theta) = -(\theta-y)^2$, and the type distribution is uniform on $[0,1]$. Consider $G\left(\widehat{K}, b\right) = G\left(1, \frac{1}{3}\right)$, the contracting game with maximally one clause and a bias $b = \frac{1}{3}$. Recall that this bias is too large for more than one action to be induced in a CS game – i.e., with only cheap talk and no contract. In contrast, we will see that in $G\left(1, \frac{1}{3}\right)$ the optimal contract with a single clause gives rise to a communication game with an equilibrium that induces two communication actions.[12]

We call a contract an n-step contract if n is the maximal number of communication actions possible in equilibrium with that contract. Let $\mathcal{C}_n^*$ be an optimal contract among n-step contracts and $EU^S\left(\mathcal{C}_n^*\right)$ the corresponding sender-optimal equilibrium payoff. When no contract is written, denote the sender's payoff from a sender-optimal equilibrium by $EU^S(0)$.

Clearly, for any bias $b > 0$ and irrespective of the contract $\mathcal{C}$, it is the case that any two equilibrium actions in the induced cheap-talk game $\Gamma^{\mathcal{C}}$ must be at least a distance $2b$ apart. Therefore, for biases $b \in \left(\frac{1}{4}, \frac{1}{2}\right)$ in the game $G(1,b)$ no contract exists that induces an equilibrium with more than two communication actions. Hence, there are four candidates for optimality: no contract, and 0-step, 1-step, or 2-step contracts. An optimal contract in this example maximizes the sender's expected payoff among the optima of these four options.

The first option – where no contract is written – results in the standard cheap talk game, Γ^0, being played. For $b = \frac{1}{3}$, in this game, there is no information conveyed in any equilibrium. The receiver's action after every equilibrium message is $y^* = \frac{1}{2}$, and the sender's expected payoff is $EU^S(0) = -\frac{1}{12} - \frac{1}{9} = -0.194$ (for an illustration, see Figure 2 first left panel).

The second option is for the sender to write an optimal obligationally complete contract with one condition $\mathcal{C}_0^* = \left\{\left([0,1], \frac{1}{2} + \frac{1}{3}\right)\right\}$ that covers the entire type set $[0,1]$ and imposes her optimal action (for an illustration, see Figure 2 second left panel). This contract increases her expected payoff to $EU^S\left(\mathcal{C}_0^*\right) = -\frac{1}{12} = -0.083$. It leaves no room for communication,

[12]The details of our analysis can be found in the Appendix.

raising the question of whether the sender can gain from reducing the size of the condition and allowing for some communication.

The third option has the sender write a contract that allows for a 1-step equilibrium. It is not optimal for the sender to place the single condition $C = \left[\underline{C}, \overline{C}\right]$ in the interior of the state space, since moving it to either extreme of $[0,1]$ reduces variance and there are no incentive issues. It is without loss of generality to consider the case with $\overline{C} = 1$. Given the condition $[\underline{C}, 1]$, the optimal instruction is given by $x = \frac{1+\underline{C}}{2} + \frac{1}{3}$. The sender's problem of determining the optimal $\underline{C}$ is

$$\max_{\underline{C}} \int_0^{\underline{C}} -\left(s + \frac{1}{3} - \frac{\underline{C}}{2}\right)^2 ds + \int_{\underline{C}}^1 -\left(s + \frac{1}{3} - \frac{\underline{C}+1}{2} - \frac{1}{3}\right)^2 ds,$$

where the first integral is the expected sender payoff from communication, while the second integral is the expected sender payoff for states covered by the contract. The solution to this maximization problem is reached at $\underline{C} = \frac{1}{2}(1 - \frac{4}{9}) = 0.278$. Hence, a 1-step optimal contract is given by $\mathcal{C}_1^* = \left\{\left(\left[\frac{1}{2}(1-\frac{4}{9}), 1\right], \frac{1}{2} + \frac{1}{4}(1-\frac{4}{9}) + \frac{1}{3}\right)\right\}$. For an illustration, see the third left panel of Figure 2. For this contract, the sender's expected payoff is higher than for the previous ones: $EU^S\left(\mathcal{C}_1^*\right) = -0.064$.

The sender's fourth option is to write a contract that makes a 2-step equilibrium possible. We can limit our attention to contracts and equilibria in which the single condition C contains the sender's critical type, θ_1, that is indifferent between the two communication actions. This follows because (a) having the condition C belonging to the interior of the lower communication interval would be inconsistent with $b = \frac{1}{3}$ and (b) having the condition C belonging to the interior of the upper communication interval can be ruled out by the following improvement: First, shift C down to the original critical type (thus improving payoffs by variance reduction) and then, if necessary, shift C again to restore incentive compatibility, while, at the same time, taking advantage of the benefit from equalizing the length of communication intervals. That is, it suffices to consider $\theta_1 \in [\underline{C}, \overline{C}]$. Therefore, we can find an optimal contract by solving

$$\max_{\underline{C},\overline{C}} \; -\int_0^{\underline{C}} \left(s + \frac{1}{3} - \frac{\underline{C}}{2}\right)^2 ds - \int_{\underline{C}}^{\overline{C}} \left(s + \frac{1}{3} - \left(\frac{\overline{C}+\underline{C}}{2} + \frac{1}{3}\right)\right)^2 ds - \int_{\overline{C}}^1 \left(s + \frac{1}{3} - \frac{(\overline{C}+1)}{2}\right)^2 ds$$

$$\text{s.t.} \;\; \underline{C} + \frac{1}{3} - \frac{\underline{C}}{2} \leq \frac{(\overline{C}+1)}{2} - \underline{C} - \frac{1}{3}.$$

The first integral in the objective function is the sender's expected payoff conditional on the lower communication action being taken; the third integral is the sender's expected payoff conditional on the higher communication action being taken; and, the middle integral is the sender's expected payoff conditional on the contract action being taken. The constraint is analogous to the usual arbitrage condition in sender-receiver games. It ensures that types in the interval $[0, \underline{C}]$ prefer the lower communication action to the higher one. We can ignore the constraint that requires types in the interval $[\overline{C}, 1]$ to prefer the higher communication action to the the lower communication action because it is automatically satisfied in the solution to the above problem; if it were not satisfied we could shift the condition C, thereby make the length of the two communication intervals more equal, and hence increase the expected sender payoff. The solution is given by $\mathcal{C}_2^* = \left\{ \left(\left[\frac{1}{3}\left(2-\sqrt{1+\frac{12}{9}}\right), \frac{1}{3}\left(1+\sqrt{1+\frac{12}{9}}\right)\right], \frac{1}{2}+\frac{1}{3}\right)\right\}$. See the fourth left panel of Figure 2, for an illustration. The sender's expected utility in this case equals $EU^S(\mathcal{C}_2^*) = -0.062$.

Thus, with $\widehat{K} = 1$ and $b = \frac{1}{3}$, the sender-optimal contract is unique and given by $\mathcal{C}_2^*$.[13] The optimal contract induces two communication actions, y_1 and y_2, while without a contract, the maximal feasible number of communication actions would be one. In this sense, contracting facilitates communication. Notice also that the two communication intervals are of equal length. This contrasts with non-trivial communication in CS equilibria of the game with a uniform type distribution and a constant bias, where higher communication actions are associated with longer communication intervals. In CS, this difference in the length of communication intervals, which is dictated by incentive constraints, is costly to the sender. The sender would prefer having all communication intervals to be of equal length. In our setting, this can be achieved with a contract: The contract relaxes incentive constraints for types adjacent to the contract condition. As long as those constraints are slack, it pays to shift the condition in the contract in the direction of equalizing the lengths of communication intervals.

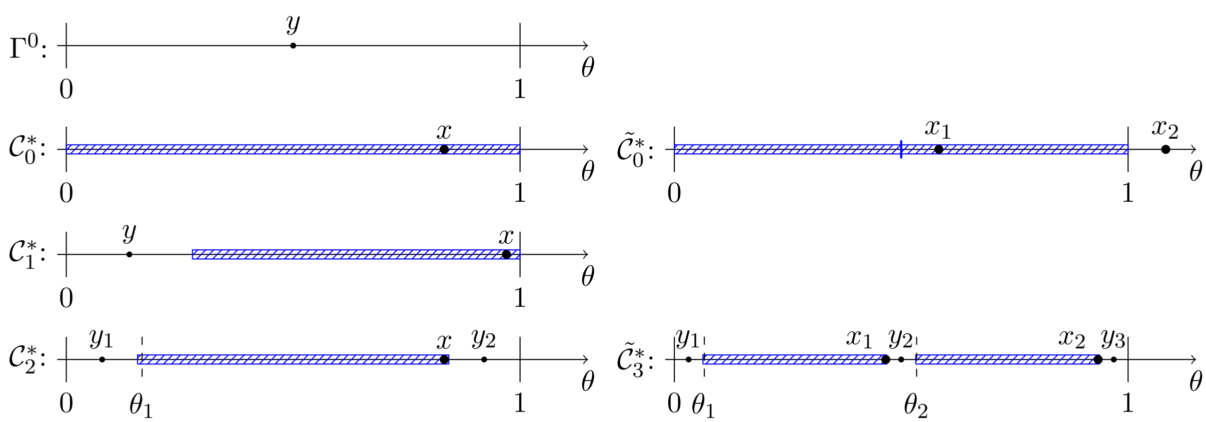

Figure 2: Left: the four candidates for optimal contracts, with $\widehat{K} = 1$ and $b = \frac{1}{3}$: $\Gamma^0, \mathcal{C}_0^*, \mathcal{C}_1^*, \mathcal{C}_2^*$. Right: two candidates for optimal contracts with $\widehat{K} = 2$: obligationally complete $\tilde{\mathcal{C}}_0^*$ for $b = \frac{1}{3}$, and 3-step $\tilde{\mathcal{C}}_3^*$ for $b = \frac{1}{5}$.

Next, we want to see how the optimal contract changes with the parameters b and $\widehat{K}$. Consider, first, the case in which we keep $b = \frac{1}{3}$ and relax the constraint on the number of conditions by letting $\widehat{K} = 2$. In that case, the optimal contract will be obligationally com-

[13] Our analysis in Section 6.1 below implies that this remains the optimal contract if we permit conditions that are finite unions of disjoint intervals.

plete with the two conditions dividing the state space into two equal-length intervals, $\tilde{\mathcal{C}}_0^* = \{([0, 0.5], 0.583), \ ([0.5, 1], 1.083)\}$. For the second case, we now lower the bias to $b = \frac{1}{5}$ and keep $\widehat{K} = 2$. There is a unique optimal contract $\tilde{\mathcal{C}}_3^* = \{([0.063, 0.468], 0.466), ([0.532, 0.937], 0.934)\}$, which induces three communication actions. This is, again, more than the maximal number of two actions that can be induced in an equilibrium of the communication game without contracting. See the right panel of Figure 2, for an illustration.

Hence, if we keep the bias fixed while increasing the bound on the number of clauses, contracting drives out communication. If, instead, we lower the bias while fixing the upper bound on the number of contract clauses, communication replaces contracting. We will see that both of these observations generalize.

In both cases, $\left(\widehat{K} = 1, b = \frac{1}{3}\right)$ and $\left(\widehat{K} = 2, b = \frac{1}{5}\right)$, the conditions in the optimal contracts contain critical types, θ_1 and θ_1, θ_2. We will find that this fact – that at an optimum, every maximal connected set of conditions contains a critical type – also generalizes.

We discuss variations of this example with a condition composed of multiple disjoint intervals, with non-constant bias, with a nonuniform distribution, and with the possibility of transfers, in Section 6.

5 Sender-Optimal Equilibria

In this section, we characterize sender-optimal equilibria of the contract writing game $G(\widehat{K}, b)$ and the contracts that the sender writes in those equilibria.

For general preferences, we begin by showing that an optimal contract always exhausts the bound on the number of clauses. We then provide two limit results in terms of the bound on the number of clauses, $\widehat{K}$, and the size of the bias, b. Increasing the bound, in the limit, contracts drive our communication. Conversely, fixing the bound, if we let the bias converge to zero, communication takes over.

If we restrict attention to quadratic payoff functions, a constant bias, and a uniform type distribution, we can be specific about how much communication is possible with a (not necessarily optimal) contract. We use that to show how more communication is possible with, rather than without, a contract. We give a sufficient condition in terms of the bias and the bound on the number of clauses for optimal contracts to be obligationally complete. Finally, we examine how contracts are used to structure communication. Here, we find that in any sender-optimal equilibrium, any maximal connected union of conditions, which we call a *condition cluster*, contains a critical type. If communication induces more than one action, there is at least one cluster that separates two communication intervals – i.e., this cluster contains a critical type that is not 0 or 1. Having such an "interior" critical type belong to a cluster implies that the incentive constraints cannot be tight: either the highest type in the communication interval below or the lowest type in the communication interval above that cluster does not have to satisfy the usual arbitrage condition. Such a contract relaxes the incentive constraints and this facilitates communication.

5.1 General Preferences

When writing a contract, the sender trades off the benefit from directly controlling the action against the resulting rigidity. By writing a condition, the sender benefits from being able to prescribe her preferred action for that condition. At the same time, when increasing the size of a condition, the sender incurs both a *rigidity cost* and a potential *communication loss*: since there is a bound on the number of clauses, any increase in the set of states covered by the contract requires increasing the size of a condition. A downside from increasing the size of a condition is that, on average, the mandated action matches the sender's preferred action less closely. This entails a rigidity cost. Increasing the set of states covered by the contract also impacts communication. Here, the impact is ambiguous and depends on the bias. If the conflict of interest is large, there is little role for communication, and the sender may prefer mandating an action over ceding authority to the receiver. If, however, incentives are closely aligned, communication can be used to make the action highly sensitive to the state. Trying to substitute contracting for communication can result in breaking that close link and, thus, in a communication loss.

While there is a rigidity cost and a potential communication loss from increasing the coverage of the contract with a fixed number of clauses, it is always beneficial to use all available clauses. There is clearly a strict gain from using at least one clause: take any sender-optimal equilibrium of the communication subgame Γ^0 that is induced by an empty contract. Then, for any equilibrium action y, introduce a clause with the following properties: the condition is equal to the interval of types who induce that action y; and the instruction is equal to the sender's favorite action given that set of types. Since the condition corresponds to a step in the communication equilibrium, we still have an equilibrium in the new communication game. Furthermore, since we are substituting the sender-optimal action for the receiver optimal action after turning the step into a condition, the sender's payoff strictly increases. The following result then follows because any clause that is used can be improved upon by subdividing it into two clauses, with additional gains for the sender.

Proposition 1 *If $\mathcal{C} = \{(C_k, x_k)\}_{k=1}^{K}$ is an optimal contract in $G(\widehat{K}, b)$, then $K = \widehat{K}$.*

It is perhaps worth noting that the literature has found that, sometimes, *prima facie* useful and readily available clauses will not be included in a contract. Allen and Gale (1992) and Spier (1992) have pointed out that in the presence of asymmetric information agents may prefer non-contingent contracts. This is the case when proposing a contingent contract would send an unfavorable signal. Bernheim and Whinston (1998) observe that if some aspects of performance are non-verifiable, it may be advantageous not to include other, verifiable, aspects in a contract. In essence, once a contract needs to be incomplete in some dimensions, the contract will give rise to some form of strategic interaction. In that case, there can be instances in which the quality of that strategic interaction can be improved by not specifying some obligations, even when they are verifiable. In our case, the signaling aspect is absent and while there is strategic interaction for states not covered by the contract, any given contract that does not use all available clauses can be improved upon without impacting the strategic interaction.

As it becomes easier to write detailed contracts (with increasing $\widehat{K}$), we might expect that contracts replace communication. The following result makes this intuition precise. For any event $\Phi \subseteq [0,1]$ with $\text{Prob}(\Phi) > 0$, define player i's optimal action conditional on the event Φ by

$$y^{*i}(\Phi) := \arg\max_y \int_\Phi U^i(y,\theta)dF(\theta),\ i = S, R.$$

Before stating and proving the result, we note the following helpful observation.

Lemma 2 *For all $b \geq 0$ and all $\eta > 0$, there exists a $\gamma > 0$ such that for all $\Phi \subseteq [0,1]$ with* $\text{Prob}(\Phi) \geq \eta$,

$$\int_\Phi U^S(y^S(\theta),\theta,b)dF(\theta) - \int_\Phi U^S(y^{*S}(\Phi),\theta,b)dF(\theta) > \gamma.$$

This observation establishes that for every sufficiently likely set of types Φ, there is a strictly positive lower bound γ for the sender's utility gain from receiving her ideal action $y^S(\theta)$ for every type in that set, rather than the action $y^{*S}(\Phi)$ that maximizes her expected payoff across types in that set. For every strictly positive probability η, this bound, γ, is uniform across all events that have at least that probability η. With this in hand, we can show that as the bound on the number of clauses grows without limit, the probability measure of the gap in the optimal contract converges to zero.

Proposition 2 *For any sequence $\{\mathcal{L}_{\widehat{K}}\}_{\widehat{K}=1}^\infty$ of gaps arising in sender-optimal equilibria $e(\widehat{K},b)$ of contract-writing games $G(\widehat{K},b)$, $\widehat{K} = 1,2,\ldots,$*

$$\lim_{\widehat{K}\to\infty} \text{Prob}(\mathcal{L}_{\widehat{K}}) = 0.$$

The proof shows that, not leaving any gaps and using all available clauses, with an increasing number of clauses, it is possible to approximate the sender's ideal payoff arbitrarily closely. For any gap, on the other hand, we know from Lemma 1, that there is a limit to how many actions can be induced. Thus, with a nonvanishing gap, there will be a nonnegligible set of types who receive a common action. Lemma 2, however, implies that, on this set of types, there will be a significant loss relative to the sender's ideal payoff.

If, instead, we fix the bound on the number of clauses $\widehat{K}$, with sufficiently small biases communication dominates nearly all the information processing. To obtain this result, we impose the following continuity property: for any sequence of biases $\{b_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} b_i = 0$ and any sequence $\{e(b_i)\}_{i=1}^\infty$ of sender-optimal equilibria in the games $\{\Gamma^0(b_i)\}_{i=1}^\infty$, the sender's payoffs in those equilibria converge to $\int_{[0,1]} U^S(y^S(\theta),\theta,0)dF(\theta)$.[14]

Proposition 3 *For any sequence $\{\mathcal{L}_i\}_{i=1}^\infty$ of gaps in sender-optimal equilibria $e(b_i)$ of games $G(\widehat{K},b_i)$ with $\lim_{i\to\infty} b_i = 0$,*

$$\lim_{i\to\infty} \text{Prob}(\mathcal{L}_i) = 1.$$

[14]Spector (2000), Agastya, Bag and Chakraborty (2015), and Dilmé (2018) provide conditions on primitives that ensure that this continuity property holds.

Here, the proof establishes that without any clauses, as the bias approaches zero, there is a sequence of communication equilibria that approximate the sender's ideal payoff. If, on the other hand, there is a nonvanishing set of types for which actions are controlled by the contract, then there must be one nonnegligible set of types who receive a common action. Once again, Lemma 2 implies that on this set of types, there will be a significant loss relative to the sender's ideal payoff.

5.2 Quadratic Losses and constant bias

We now assume that the players' payoff functions are quadratic and that the bias is constant.[15] The ability to write clauses changes the communication environment and the number of actions that can be induced through communication. The following result holds without imposing additional assumptions on the type distribution. It characterizes the maximal number of communication actions that can be induced (without regard for optimality).

Proposition 4 *For any b, there exist a $\widehat{K}$, a contract $\mathcal{C}$, and an equilibrium of the communication subgame $\Gamma^{\mathcal{C}}$ with n induced actions if and only if $n < 1 + \frac{1}{2b}$.*

For the proof of sufficiency, we construct equilibria in which communication intervals are very short, and clauses are used to generate just sufficient separation to satisfy the sender's incentive compatibility constraints.

For the remainder of the paper, we assume that the type distribution is uniform on $[0, 1]$. The following well-known fact will be useful to indicate how to improve the sender's expected payoff.

Observation 1 *Suppose that, given any distribution over $[\underline{\theta}, \overline{\theta}] \subseteq [0, 1]$, the sender/receiver takes an optimal action. Then, the sender's expected payoff is decreasing in the variance of that distribution.*

As an illustration of how we use this observation, see Figure 3. Suppose that we have a contract $\mathcal{C}_1$ such that in a sender-optimal equilibrium $e^{\mathcal{C}_1}$ of $\Gamma^{\mathcal{C}_1}$ for some action y, the set $[\underline{\theta}, \overline{\theta}] \cap \mathcal{L}(\mathcal{C}_1)$ is the set of types inducing that action. Suppose, further, that there is an alternative contract $\mathcal{C}_2$ that differs from $\mathcal{C}_1$ only in that conditions $C_j, j \in J$, in the interior of $[\underline{\theta}, \overline{\theta}]$ are replaced by conditions $C'_j \subset [\underline{\theta}, \overline{\theta}], j \in J$, such that for each $j \in J$, C'_j is a translation of C_j and $[\underline{\theta}, \overline{\theta}] \cap \mathcal{L}(\mathcal{C}_2)$ forms an interval. Then, if $\Gamma^{\mathcal{C}_2}$ has an equilibrium $e^{\mathcal{C}_2}$ in which types in $[0, 1] \setminus [\underline{\theta}, \overline{\theta}]$ behave as before and types in $[\underline{\theta}, \overline{\theta}] \cap \mathcal{L}(\mathcal{C}_2)$ send a common distinct message y', the sender's payoff from $e^{\mathcal{C}_2}$ exceeds that from $e^{\mathcal{C}_1}$.

A second way to improve the sender's payoff also proves useful.

Observation 2 *Let $\underline{\theta}_i < \overline{\theta}_i \leq \underline{\theta}_j < \overline{\theta}_j$ and $\overline{\theta}_j - \underline{\theta}_j - \delta > \overline{\theta}_i - \underline{\theta}_i + \delta$. Suppose that the receiver takes action $y_i^\delta = \frac{\underline{\theta}_i + \overline{\theta}_i + \delta}{2}$ for types in $\left(\underline{\theta}_i, \overline{\theta}_i + \delta\right)$ and action $y_j^\delta = \frac{\underline{\theta}_j + \overline{\theta}_j + \delta}{2}$ for types in $\left(\underline{\theta}_j + \delta, \overline{\theta}_j\right)$. Then, the expected sender-payoff conditional on $\left(\underline{\theta}_i, \overline{\theta}_i + \delta\right) \cup \left(\underline{\theta}_j + \delta, \overline{\theta}_j\right)$ is increasing in δ.*

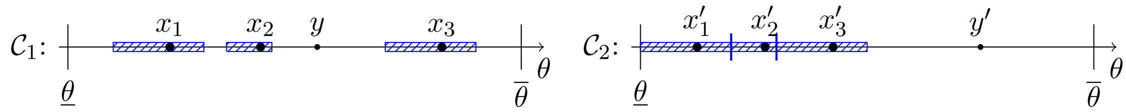

Figure 3: A payoff improvement by translations of C_j for $J = 3$ and a bias $b = 0.025$.

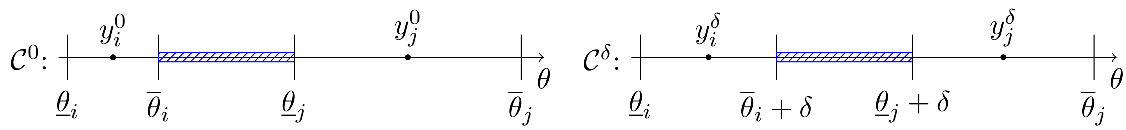

Figure 4: A payoff improvement by a δ-translation of C.

For an illustration, see Figure 4, in which the left panel depicts $\delta = 0$. Enlarging (shrinking) a communication interval increases (decreases) the variance in this interval. Considering two intervals of different lengths, if the smaller interval is enlarged by the same amount δ as a larger interval is decreased, having a symmetric strictly concave utility function in which payoffs only depend on the distance from the ideal point implies that the gain in the larger outweighs the loss in the smaller. Equalizing the lengths of the intervals reduces the expected conditional variance.

We are now equipped to study the sender's problem. We first compare the maximal number of receiver actions that can be induced in equilibrium in our setup with the maximal number in the standard CS game without a contract. Proposition 4 states that the maximal number of actions that can be induced in an equilibrium of an appropriately chosen communication subgame equals

$$\hat{n} := \left\lceil \frac{1}{2b} \right\rceil .$$

By contrast, the maximal number of actions that can be induced in a CS equilibrium with a uniform type distribution equals

$$n^* := \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{2b}} \right\rfloor .$$

For $b < \frac{1}{2}$, it is the case that

$$n^* < \hat{n}.$$

Thus, for sufficiently small b, the maximal number of actions that can be induced in an equilibrium of a suitably chosen communication subgame strictly exceeds the maximal number of actions than can be induced in an equilibrium of the corresponding CS game. With $b = \frac{1}{10}$, for example, we have $\hat{n} = 5$, whereas $n^* = 2$.

[15]We discuss an example with a non-constant bias in the extensions in Section 6.2.

In the general setup, Proposition 2 shows that if we increase the number of clauses, contracts drive out communication. For the uniform-quadratic setting, the next proposition gives an explicit number of clauses such that an optimal contract covers the entire state space.

Proposition 5 *If $\widehat{K} > \frac{1}{2b}$, then any optimal contract is obligationally complete.*

The intuition for the proof is similar to that for Proposition 2. An upper bound on what can be achieved from communication is given by full revelation. Using this fact, we get an upper bound on the sender's payoff from a contract that leaves a communication region of size λ. Differentiating this upper bound with respect to λ, we find that the derivative is negative for sufficiently large $\widehat{K}$. Hence, for any sufficiently large $\widehat{K}$, we want to reduce the size λ of the communication region to zero.

The conditions in a contract can be separated or contiguous. For convenience, we introduce the following notation. Given a contract $\mathcal{C} = \{(C_k, x_k)\}_{k=1}^{K}$, a union of conditions C_k is *maximal connected* if it is connected and not contained in a larger connected union. We call each maximal connected union of conditions $\boldsymbol{C}$ a *condition cluster.*

The following proposition shows that no condition cluster can be strictly inside of a communication interval. Moreover, if there is influential communication, there is at least one condition cluster that contains an interior critical type.

Proposition 6 *Suppose that the contract $\mathcal{C} = \{(C_k, x_k)\}_{k=1}^{\widehat{K}}$ is optimal in the contract-writing game G, and the equilibrium $e^{\mathcal{C}}$ is sender-optimal in the communication subgame $\Gamma^{\mathcal{C}}$. Then, for every condition cluster $\boldsymbol{C}$, there is a critical type θ with $\boldsymbol{C} \cap \{\theta\} \neq \emptyset$. If, in addition, the equilibrium $e^{\mathcal{C}}$ induces at least two communication actions, then there is a condition cluster $\boldsymbol{C}$ and a critical type $\theta \neq 0, 1$ with $\boldsymbol{C} \cap \{\theta\} \neq \emptyset$.*

We know from Proposition 1 that an optimal contract uses all available clauses. The more clauses the contract has, the less rigid it is. Since the number of clauses is finite, however, there is always some residual rigidity. The sender can reduce this rigidity by writing an obligationally incomplete contract. Introducing a gap makes it possible to induce at least one additional action, which is a communication action. Thus, leaving a gap makes the actions more sensitive to the state of the world. If, in addition, the bias is small, Proposition 3 shows that it becomes feasible and attractive to use communication to induce a large number of actions. With all available clauses being used and communication inducing more than one action, Proposition 6 shows that there is an interesting interaction between contracts and communication. The sender uses contracts not only to impose her favorite actions, but also to structure communication. Contract clauses are used to separate events that induce distinct communication actions and, therefore, to relax incentive constraints in the communication game. The relaxation of incentive constraints makes it possible to equalize the size of communication intervals relative to pure cheap talk.[16] This highlights the dual role of contracting as both substituting for and facilitating communication.

[16]A similar effect arises in Kolotilin, Li and Li (2013). There the receiver has the power to commit a set of actions from which to chose.

Intuitively, the sender uses condition clusters to relax the sender's incentive constraints that pin down the bounds of the communication intervals. The first part of the proof of Proposition 6 shows that there is a critical type contained in every condition cluster. The second part then proves that, for influential communication, there is a condition cluster that contains an interior critical type.

To prove the first part of Proposition 6, we start with any contract that does not satisfy the properties indicated in the proposition. We proceed by modifying that contract in several steps. We ensure in each step that the sender's payoff increases: the typical argument is that properly translating a condition cluster increases shorter communication intervals while it decreases longer intervals. At the end of the first part, we check that, indeed, we obtain an equilibrium. In the first step, we use the fact that there can be no more than one condition in any communication interval (see Lemma A.3 in the Appendix). We consider a candidate-optimal contract $\mathcal{C}$ and a corresponding equilibrium $e^{\mathcal{C}}$ with a communication interval containing a single condition in its interior. We then translate that condition to the lower bound of the communication interval. The new contract is $\mathcal{C}_0$. In the second step, we adjust the strategies in the communication game such that, locally (in between condition clusters), incentive compatibility is restored. The resulting game is called $\Gamma^{\mathcal{C}_1}$, with contract $\mathcal{C}_1 = \mathcal{C}_0$. We sketch steps one and two in Figure 5.

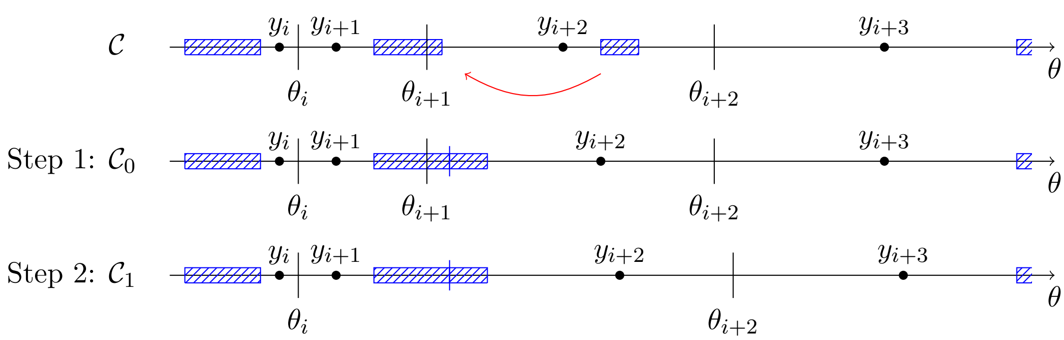

Figure 5: Sketch of the first two steps in the first part of the proof of Proposition 6.

In order to restore incentive compatibility locally, we have to raise the action y_{i+2}. This makes the action less attractive for the type $\tilde{\theta}$ that is at the top at the newly created condition cluster (see Figure 6 for an illustration). In fact, it may make action y_{i+1} more attractive than y_{i+2}. In the third step, we address incentive-compatibility problems of this kind – that is, for types that are separated by condition clusters. To do so, we identify the highest condition cluster such that a type $\tilde{\theta}$ at the upper boundary of that cluster prefers to deviate to a message inducing an action below the cluster. In multiple steps that maintain the local equilibrium conditions, we properly translate the respective condition cluster upwards to restore incentive compatibility for type $\tilde{\theta}$. The resulting contract is $\mathcal{C}_2$. We iterate the third step for all lower condition clusters to obtain a global equilibrium.

In the second part of the proof of Proposition 6, we show that the sender's payoff can

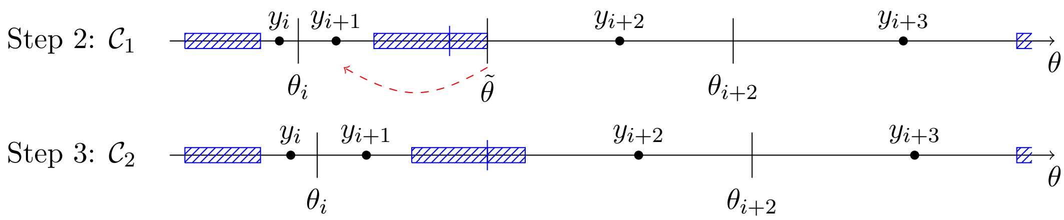

Figure 6: Sketch of the third step in the first part of the proof of Proposition 6.

be increased when more than one action is induced in equilibrium and all condition clusters are at the extremes. For an illustration of the steps in the argument, see Figure 7.

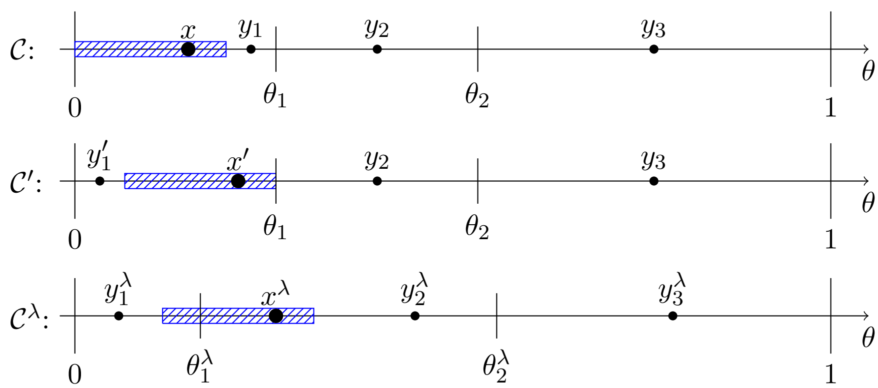

Figure 7: Second part of proof: payoff improvement by translations.

The first panel of Figure 7 shows a contract $\mathcal{C}$ with a single condition located at the left extreme of the type space and a corresponding three-step communication equilibrium. We replace contract $\mathcal{C}$ by a new contract $\mathcal{C}'$ that translates the condition upwards such that the first critical type θ_1 becomes its new upper boundary. Since we do not change the length of any communication interval, payoffs remain the same. However, type θ_1 now strictly prefers action y_2 over y_1'. Moreover, the length of the communication interval inducing action y_1' is smaller than the length of the communication interval inducing y_2. Together, this implies that we can translate the condition further upwards to $\mathcal{C}^\lambda$ while maintaining incentive compatibility and increasing payoffs. This shows that the contract $\mathcal{C}$ that we started with cannot be optimal.

As a consequence of Proposition 6, we obtain that sender-optimal equilibria of the contract-writing game G are partitional and monotonic:

Corollary 1 *Suppose that the contract $\mathcal{C} = \{(C_k, x_k)\}_{k=1}^{\widehat{K}}$ is part of a sender-optimal equilibrium e^G in the contract-writing game G and induces a sender-optimal n-step equilibrium $e^{\mathcal{C}}$ in the communication subgame $\Gamma^{\mathcal{C}}$. Then, the equilibrium e^G is*

1. *partitional – there is a partition* $\mathcal{P} = \left\{P_1, P_2, \ldots, P_{\widehat{K}+n}\right\}$ *of the type space* $[0,1]$ *into intervals such that each* $P \in \mathcal{P}$ *is either a condition of* $\mathcal{C}$ *or a communication interval in* $e^{\mathcal{C}}$*; and,*

2. *monotonic – for any two* $P, P' \in \mathcal{P}$, $P \neq P'$, *with* $\inf(P') \geq \sup(P)$, *the actions* $a(P')$ *and* $a(P)$ *taken for states in* P' *and* P *satisfy* $a(P') > a(P)$.

While the partitional structure given in Part 1. is an immediate consequence of Proposition 6, monotonicity requires slightly more thought. The relevant case to consider is the one in which a communication interval P' is directly above a condition P. For this case, we show in the Appendix, if we had $a(P') \leq a(P)$ (and keeping in mind that $a(P')$ is the receiver-optimal action on P' and $a(P)$ is the sender-optimal action on P), we could form a new contract clause $(\overline{P \cup P'}, a(P) + \epsilon)$ that, for sufficiently small ϵ, would result in a payoff improvement for the sender, resulting in a contradiction. This implies that any sender-optimal equilibrium of G is monotonic.

With only a single condition, the following result illustrates the impact of using contracting to relax incentive constraints in the communication subgame. Whenever the optimal contract induces non-trivial communication in the communication subgame, the lengths of the communication intervals flanking the condition differ by less than $4b$ (the amount by which the lengths of communication intervals increase in CS from one to the next). This shows how contracting helps to equalize the lengths of communication intervals.

Corollary 2 *Suppose* $\widehat{K} = 1$, *the contract* $\mathcal{C}$ *with condition* $[\underline{C}, \overline{C}]$ *is optimal, and* $\mathcal{C}$ *induces at least two communication actions in the sender-optimal equilibrium* $e^{\mathcal{C}}$ *of the communication subgame* $\Gamma^{\mathcal{C}}$. *If* θ_{i-1}, θ_i, *and* θ_{i+1} *are critical types in the equilibrium* $e^{\mathcal{C}}$ *with* $\theta_i \in \left[\underline{C}, \overline{C}\right]$, *then* $\left|\theta_{i+1} - \overline{C}\right| < |\underline{C} - \theta_{i-1}| + 4b$*; and, if* $\theta_i \in \left(\underline{C}, \overline{C}\right)$, *then* $\left|\theta_{i+1} - \overline{C}\right| \leq |\underline{C} - \theta_{i-1}|$.

In summary, in the uniform-quadratic environment with a constant bias, optimal contracts never place conditions in the interior of communication intervals; if there is nontrivial communication, conditions are used to relax incentive constraints; equilibria of the contract-writing game G are partitional and monotonic; and, the ability to relax incentive constraints is used to equalize the lengths of communication intervals.

6 Extensions

In this section, we explore various extensions in the context of our example from Section 4. We, first, consider allowing more general conditions than just intervals. Here, we find that among conditions that are finite unions of disjoint intervals, intervals are optimal. In the subsequent subsections, we explore the impact of non-constant biases, non-uniform distributions, and the role of transfers. We find that with these small departures, optimal contracts remain close to those from the uniform-quadratic environment with a constant bias. In the case of a non-constant bias, optimal conditions shift toward covering states with more conflict; with a non-uniform distribution optimal conditions shift toward covering more likely states; and, with transfers optimal conditions shrink in size.

6.1 Finite unions of disjoint closed intervals as conditions

We assume, in our model, that conditions take the form of intervals. In this section, we show that, in the framework of the example, it is indeed never optimal to split the condition into finitely many disjoint intervals. The sender can improve her payoff by translating all these disjoint intervals so that they merge into one contiguous interval.

Let C be a condition that is a finite union of components D_j, $j = 1, \ldots, J$, with $J > 1$; each component is a closed interval, $\left[\underline{D}_j, \overline{D}_j\right]$, and components are disjoint. Suppose that there is a two-step equilibrium e in the induced communication subgame that is characterized by a critical type θ_1. Let y_1 (y_2) be the action induced by types $\theta < \theta_1$ (types $\theta > \theta_1$) in the equilibrium e. We define j' as the index of the maximal component that is entirely below θ_1 – if there is any. Similarly, j'' is the index of the minimal component that is not entirely below θ_1 – if there is any.

We aim at increasing the sender's payoff by constructing a new contiguous condition $\tilde{C}$ and then, if necessary, shifting it to restore a two-step equilibrium in the induced communication subgame. Let

$$\tilde{C} = [\underline{\tilde{C}}, \overline{\tilde{C}}] = \left[\min\{\theta_1, \underline{D}_{j''}\} - \sum_{j=1}^{j'} \overline{D}_j - \underline{D}_j, \min\{\theta_1, \underline{D}_{j''}\} + \sum_{j=j''}^{J} \overline{D}_j - \underline{D}_j\right].$$[17]

Thus, $\underline{\tilde{C}}$ is obtained by translating all components D_j for $j \leq j'$ that are entirely below the critical type θ_1 upward so that they are contiguous and bounded on the right-hand side by θ_1 (if $\theta_1 < \underline{D}_{j''}$) or by $\underline{D}_{j''}$ (if $\theta_1 \in D_{j''}$). Similarly, $\overline{\tilde{C}}$ is obtained by translating all components D_j for $j \geq j''$ that are above $\min\{\theta_1, \underline{D}_{j''}\}$ downward so that they are contiguous and bounded on the left-hand side by $\min\{\theta_1, \underline{D}_{j''}\}$. Let $\tilde{y}_1$ ($\tilde{y}_2$) denote the receiver's best reply to prior beliefs concentrated on the set $[0, \underline{\tilde{C}}]$ ($[\overline{\tilde{C}}, 1]$). For an illustration, see Figure 8.

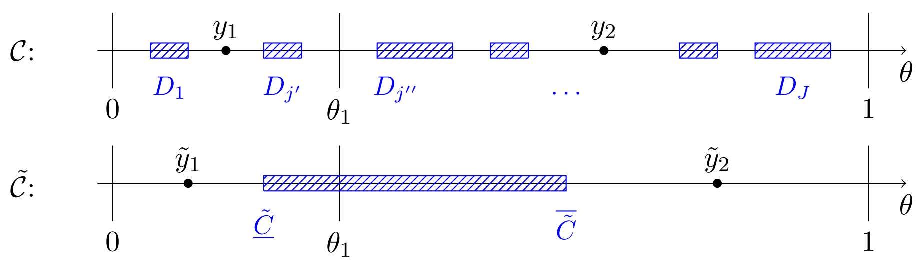

Figure 8: Improving payoffs by making the condition-components D_j contiguous.

We, then, have the following observation.

Lemma 3 *Suppose that* $\{j \leq J | \overline{D}_j < \theta_1\}$ *is nonempty. Then* $\min\{\theta_1, \underline{D}_{j''}\} - y_1 \geq \underline{\tilde{C}} - \tilde{y}_1$.

[17] If all components are strictly below θ_1 and therefore there is no $D_{j''}$, we adopt the convention that $\min\{\theta_1, \underline{D}_{j''}\} = \theta_1$.

To understand the significance of this observation, recall that in the equilibrium e type $\min\{\theta_1, \underline{D}_{j''}\}$ weakly prefers action y_1 to action y_2. Lemma 3 implies that types $[0, \underline{\tilde{C}}]$ prefer inducing action $\tilde{y}_1$ to action y_2. Thus, translating the components that are entirely below θ_1 upward, and having types below θ_1 – that are not covered by the new contract – send a common message in the resulting communication game preserves incentive compatibility for low types.

With this in hand, we can prove that dividing a condition into components cannot be optimal.

Proposition 7 *Suppose that we allow contracts with conditions C that are finite unions of disjoint closed intervals. Then, for $b > \frac{1}{4}$ and $\widehat{K} = 1$, any optimal contract is nonempty and the condition in that contract is a single interval.*

To prove the result, we observe that, by Proposition 4, there cannot be more than two communication actions in equilibrium. If there is only one communication action, by Observation 1, it cannot be optimal to split the condition into disjoint components. Thus, consider a two-step equilibrium e that is characterized by some critical type θ_1 and induced by a condition C that is a finite union of disjoint components D_j, $j \leq J$.

The sender strictly gains by replacing condition C by condition $\tilde{C}$ defined above. First, note that the sender's payoff conditional on the event $\tilde{C}$ exceeds the sender's payoff conditional on the event C in the original contract by Observation 1. Second, suppose the receiver responds to messages sent by types in $[0, \underline{\tilde{C}}]$ with action $\tilde{y}_1$ and to messages sent by types in $[\overline{\tilde{C}}, 1]$ with action $\tilde{y}_2$. Then, the sender's payoff conditional on the event $[0, 1] \setminus \tilde{C}$ exceeds the sender's payoff conditional on the event$[0, 1] \setminus C$ in the original contract, again, by Observation 1.

Since $\tilde{y}_2 \geq y_2$, Lemma 3, implies that types below $\underline{\tilde{C}}$ prefer action $\tilde{y}_1$ to action $\tilde{y}_2$. Thus, if we also had the reverse hold for types above $\overline{\tilde{C}}$, we would have an equilibrium in the communication game induced by condition $\tilde{C}$ and we would be done.

Consider then the possibility that incentive compatibility is violated for types above $\overline{\tilde{C}}$ – i.e., that type $\overline{\tilde{C}}$ strictly prefers action $\tilde{y}_1$ to action $\tilde{y}_2$. Then, since $b > 0$, it must be the case that the length of the interval $[\overline{\tilde{C}}, 1]$ exceeds that of the interval $[0, \underline{\tilde{C}}]$. Thus, there exists $\delta > 0$ such that type $\overline{\tilde{C}} + \delta$ is the critical type characterizing the two-step equilibrium induced by condition $\tilde{C} + \delta$. Note that in this new equilibrium the communication intervals are of more equal length than the intervals $[0, \underline{\tilde{C}}]$ and $[\overline{\tilde{C}}, 1]$. Hence, by Observation 2, the sender's payoff conditional on the event $[0, 1] \setminus (\tilde{C} + \delta)$ exceeds that from the event $[0, 1] \setminus \tilde{C}$.

6.2 Non-constant Bias

In the main text, we first assume an arbitrary strictly positive bias and then, in the uniform-quadratic setup, a strictly positive constant bias. If the bias is non-constant, it is intuitive that the sender rather covers states with higher disagreement in the contract than states with lower disagreement. In this section, we show in the context of Example 1 that the sender indeed optimally covers states with higher bias in the contract relative to the constant-bias case. However, if the maximal bias is sufficiently low, the sender still prefers an obligationally incomplete contract that allows for communication. While the structure of the optimal contract in the example was symmetric, here, the gaps differ in their length: the gap is larger for states in which the interests are more aligned.

As in Example 1, we assume that the players' loss functions are quadratic and the state is uniformly distributed on $[0, 1]$. Here, instead of a constant bias $b = \frac{1}{3}$, the sender's bias is state-dependent and of the form $b(\theta) = \frac{1}{3} + \frac{1}{30}\theta$.

We compare all cases analyzed in the example. The optimal obligationally complete contract has an instruction $x_b = 0.85$, compared to $x = 0.83$ with a constant bias. The sender's payoff equals -0.0890. Allowing for one-step communication, the optimal contract with state-dependent bias is $\mathcal{C}_{1b}^* = \{[0.28, 1], 0.99\}$; with constant bias we have $\mathcal{C}_1^* = \{[0.285, 1], 0.97\}$. Given that the states close to 1 – with high bias – are covered under both types of biases, it is intuitive that the optimal contracts are similar. The sender's payoff equals -0.0672. Finally, considering two-step communication, the optimal contract with state-dependent bias is $\mathcal{C}_{2b}^* = \{[0.25, 0.96], 0.96\}$, compared to the case with constant bias $\mathcal{C}_2^* = \{[0.16, 0.84], 0.83\}$. The sender's payoff equals -0.0640.[18]

For an illustration see Figure 9. The conditions on top of the axes refer to the optimal contracts with constant bias while the conditions below the axes indicate the optimal contracts with state-dependent bias. The figure illustrates the intuitive impact of an increasing bias. If the bias is increasing in the state instead of constant, the optimal instruction as well as optimal condition shift upwards. The sender prefers covering states with a higher bias to covering states with a smaller bias because under communication the receiver's action diverges more from the sender's preferred action.

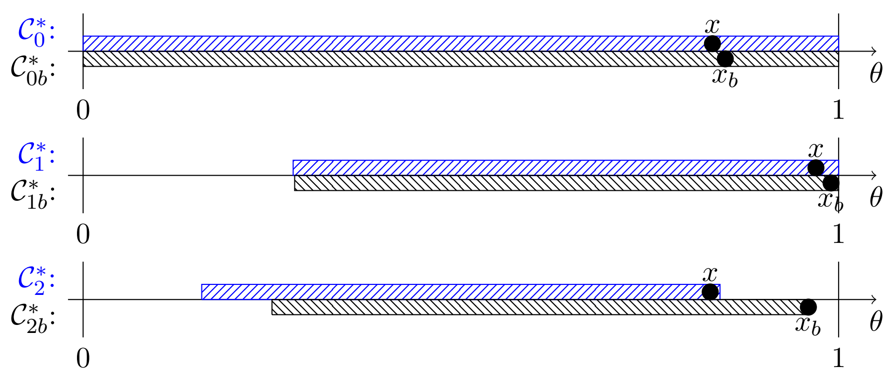

Figure 9: Example 1 with constant bias $b = \frac{1}{3}$ on top and with state-dependent bias, $b(\theta) = \frac{1}{3} + \frac{1}{30}\theta$ below: the obligationally complete contracts, 1-step optimal contracts, and the 2-step optimal contracts.

[18]We can show that for the state-dependent bias considered in this section, the optimal two-step equilibrium hast the same structure as in the constant bias case – i.e., the critical type optimally is in the contracting region.

While communication intervals are no longer of identical size, it remains the case that contracting relaxes incentive constraints. With the bias function of the example, no communication would be possible without contracting: for any candidate first communication interval, the length of the second communication interval that is implied by incentive compatibility would be too long. Here, as in the constant bias case, that constraint no longer binds. The second communication interval is shorter than the first.

6.3 Nonuniform Distribution

Here, we explore in the context of Example 1 how the optimal contract changes under a non-uniform distribution. We find that the sender prefers to cover more likely states in the contract. Compared to a uniform distribution where the solution in Example 1 is symmetric, when higher states are more likely the contracting regions are shifted upwards to the more likely states and the optimal solution is no longer symmetric.

As in Example 1, we assume that the players' loss functions are quadratic and the bias is a constant $b = \frac{1}{3}$. Instead of a uniform distribution $\mathcal{U}([0,1])$, we assume that the state is distributed on $[0,1]$ with density $f(\theta) = \frac{9}{10} + \frac{2}{10}\theta$.

We compare all four cases analyzed in the example. In the case of no contract, the receiver takes his preferred action $x_f = 0.52$. The sender's payoff, in this case, equals -0.1942. The optimal obligationally complete contract has an instruction $x_f = 0.85$, compared to $x = 0.83$ with a uniform distribution. The sender's payoff equals -0.0831. Allowing for one-step communication, the optimal contract under the distribution f is $\mathcal{C}^*_{1f} = \{[0.29, 1], 0.99\}$; while with a uniform distribution, we have $\mathcal{C}^*_1 = \{[0.28, 1], 0.97\}$. The sender's payoff equals -0.0625. Finally, considering two-step communication, the optimal contract under f is $\mathcal{C}^*_{2f} = \{[0.23, 0.91], 0.91\}$, compared to the case with constant bias $\mathcal{C}^*_2 = \{[0.16, 0.84], 0.83\}$. The sender's payoff equals -0.0620.

For an illustration see Figure 10. The conditions on top of the axes refer to the optimal contracts with uniform distribution while the conditions below the axes indicate the optimal contracts with distribution f.

For the distribution with an increasing density compared to a uniform distribution, the optimal instruction as well as the optimal condition shift upwards. The sender prefers covering states that occur more frequently in the contract rather than states that have a lower probability: for states covered by the condition the sender gets her preferred action rather than the receiver's preferred action. Once more, the contract is used to relax incentive constraints, and thereby makes communication feasible, when otherwise it would not be.

6.4 Transfers

For the main analysis, we abstain from modeling transfers from the principal to the agent. We believe that this does not entail a significant loss of generality. Two common uses of transfers in the literature do not apply to our setup. Under moral hazard, the agent needs to be incentivized to take particular actions; here, however, actions that are governed by the

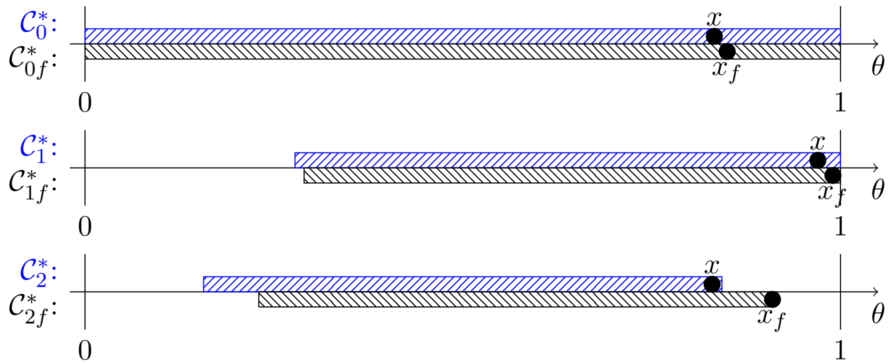

Figure 10: Example 1 with uniform distribution on top and with distribution, $f(\theta) = \frac{9}{10} + \frac{2}{10}\theta$ below: the obligationally complete contracts, 1-step optimal contracts, and the 2-step optimal contracts.

contract are fully under the control of the principal. Under screening, the principal tries to gather information about the agents private type, whereas in our setup, there is no private information on the agent's side. Whatever role remains for transfers is minimal as long as the agent cares primarily about his wage. At the extreme, the agent has lexicographic preferences for a higher wage. This case matches our model.

In the context of our example (Section 4), we here discuss a slightly less extreme case in which the agent assigns some, but small, weight to his payoff from the action. We find that adding transfers has little effect on the optimal contract as long as he agent cares primarily about the wage.

We denote by w the receiver's wage and by $\overline{u}^R$ the receiver's reservation utility. The players' payoffs can be rewritten in the following form $U^S(y, \theta, b, w) = -(\theta + b - y)^2 - w$ and $U^R(y, \theta, w) = -\alpha(\theta - y)^2 + (1-\alpha)w$, where $\alpha > 0$ denotes the (small) weight the receiver puts on the payoff that depends on his action relative to his wage. The sender's objective now is to maximize $\mathbb{E}U^S(y, \theta, b, w)$ subject to the individual rationality (IR) constraint $\mathbb{E}U^R(y, \theta, w) \geq \overline{u}^R$. Since the IR constraint is binding at the maximum, this amounts to maximizing weighted joint surplus, $U^S(y, \theta, b, w) = -(\theta + b - y)^2 - \frac{\alpha}{1-\alpha}(\theta - y)^2 - \frac{\overline{u}^R}{1-\alpha}$.

We compare all four cases analyzed in the example. The bias is $b = \frac{1}{3}$ and we vary $\alpha = 0.1$ $(= 0.5)$. For the case of no contract nothing changes. Considering optimal contracts we obtain the following. In case of an obligationally complete contract, the optimal instruction with transfers is $x_t = 0.80$ $(= 0.67)$ compared to $x = 0.83$ without transfers. Allowing for one-step communication, an optimal contract with transfers is $\mathcal{C}_{1t}^* = \{[0.32, 1], 0.96\}$ $(= \{[0.44, 1], 0.89\})$, while without transfers we have $\mathcal{C}_1^* = \{[0.28, 1], 0.97\}$. Finally, considering two-step communication, the optimal contract with transfers is $\mathcal{C}_{2t}^* = \{[0.19, 0.81], 0.8\}$ $(= \{[0.28, 0.72], 0.67\})$, compared to the case without transfers with $\mathcal{C}_2^* = \{[0.16, 0.84], 0.83\}$.

For an illustration see Figure 11. The condition on top of the axis refers to the optimal contract without transfers while the condition below the axis indicates the optimal contract with transfers, for $\alpha = 0.1$ ($= 0.5$) on the left-hand-side (right-hand-side).

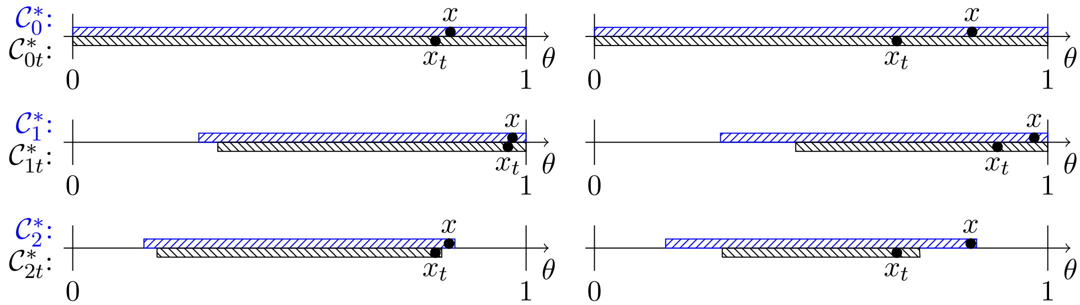

Figure 11: Example 1 with transfers, $b = \frac{1}{3}$, $\alpha = 0.1$ left panel and $\alpha = 0.5$ right panel: the obligationally complete contracts, 1-step optimal contracts, and the 2-step optimal contracts.

Recall that the introduction of transfers changes the sender's problem to one closer to joint-surplus maximization. In her optimization problem, the sender has to take the receiver's payoff into account. It is, therefore, intuitive that the optimal instruction decreases in the direction of the receiver-preferred action which is at the midpoint of the condition. Moreover, the trade-off between communication (receiver-optimal action) and contract (sender-optimal instruction) becomes less extreme. As a result, the length of the condition shrinks and becomes more equal to the length of the communication intervals. This reduces the overall variance and increases the sender's payoff.

7 Conclusion

In this paper, we study how a principal who anticipates receiving private decision-relevant information and who needs to rely on an agent to act on that information organizes the use of that information, when her ability to control that use is limited. We capture limited control by restricting the principal to contracts that allow for only a bounded number of clauses, and are, therefore, with an infinite state space, not fully detailed complete.

In such an environment, in addition to being constrained to writing contracts that are not fully detailed complete, the principal may opt for contracts that are not obligationally complete. That is, the principal may elect to have some states of the world not covered by the contract. The principal may be motivated to do so when the interests of the principal and the agent are sufficiently closely aligned. In this case, cheap-talk communication in conjunction with leaving discretion to the agent becomes an attractive substitute for mandating actions via coarse contract clauses.

We find that the principal always makes use of the maximal number of contract clauses, but sometime excludes states from being covered by the contract. With little conflict of interest, communication drives out contracting and *vice versa*. The sender uses contracts not only to impose her favorite actions, but also to structure communication. Contract clauses are used to separate events that induce distinct communication actions and, therefore, to relax incentive constraints in the communication game. The relaxation of incentive constraints makes it possible to equalize the size of communication intervals relative to pure cheap talk. This highlights the dual role of contracting as both substituting for and facilitating communication.

There are a number of natural variations on the theme of our paper. We already explored conditions that are unions of multiple disjoint intervals, non-constant biases, non-uniform type distributions, and transfers in our example. In the case of conditions that are unions of multiple disjoint intervals we find that single intervals are optimal; with a non-constant bias, we find that optimal conditions shift toward covering states with more conflict; with a non-uniform distribution optimal conditions shift toward covering more likely states; and, with transfers optimal conditions shrink in size. For these variations, we expect the main insight from our paper – that contracts are used, in part, to facilitate communication – survive.

A more significant departure from our basic model would be to allow for imperfect enforcement of contract clauses. It is, for example, clear that sometimes at the interim stage the principal would prefer to send one of the cheap talk messages, rather than insist on having the applicable contract clause enforced. Giving the principal that option with some probability would remove the clean separation of contract actions and communication actions that we maintain throughout. In this paper, we deliberately completely decouple the choice of contract actions and non-contract actions. As a result, once the contract is in place, there is a well-defined cheap-talk game that can be analyzed in isolation. Relaxing this decoupling would considerably complicate the analysis, and is, therefore, left to future work.

A Appendix

Example Section 4 In the CS game Γ^0, the only equilibrium for $b = \frac{1}{3}$ is the babbling equilibrium, where the receiver takes the action $y = \frac{1}{2}$. The resulting payoff for the sender is given by

$$\int_0^1 -\left(s+b-\frac{1}{2}\right)^2 ds = -\frac{1}{12} - b^2 = -0.194.$$

The sender's expected payoff from an obligationally complete contract prescribing the optimal instruction $x = \frac{1}{2} + b$ is given by

$$\int_0^1 -\left(s+b-\frac{1}{2}-b\right)^2 ds = -\frac{1}{12} = -0.083.$$

The sender's problem of writing an obligationally incomplete contract $\left(\left[\underline{C},1\right], \frac{1+\underline{C}}{2}+b\right)$ allowing for one-step communication is given by

$$\begin{aligned}&\max_{\underline{C}} \int_0^{\underline{C}} -\left(s+b-\frac{\underline{C}}{2}\right)^2 ds + \int_{\underline{C}}^1 -\left(s+b-\frac{\underline{C}+1}{2}-b\right)^2 ds\\ =&\max_{\underline{C}} -\frac{\left(1-\underline{C}\right)^3}{12} - \frac{\underline{C}}{9} - \frac{\underline{C}^3}{12}.\end{aligned}$$

For $b = \frac{1}{3}$, solving the first order condition yields $\underline{C} = \frac{1-4b^2}{2} = 0.278$ and a resulting payoff for the sender of $-\frac{1}{48} - \frac{b^2}{2} + b^4 = -0.064$.

The sender's optimization problem for writing an obligationally incomplete contract $\left(\left[\underline{C},\overline{C}\right], \frac{\underline{C}+\overline{C}}{2}+b\right)$ allowing for 2-step communication is given by

$$\max_{\underline{C},\overline{C}} \int_0^{\underline{C}} -\left(s+b-\frac{\underline{C}}{2}\right)^2 ds + \int_{\underline{C}}^{\overline{C}} -\left(s+b-\left(\frac{\overline{C}+\underline{C}}{2}+b\right)\right)^2 ds + \int_{\overline{C}}^1 -\left(s+b-\frac{\left(\overline{C}+1\right)}{2}\right)^2 ds.$$

$$\text{s.t. } \underline{C}+b-\frac{\underline{C}}{2} \leq \frac{\left(\overline{C}+1\right)}{2} - \underline{C} - b,$$

where the (sender's IC) constraint ensures that types below $\underline{C}$ prefer to induce the lower of the two communication actions. Simplifying the objective yields

$$-\frac{\underline{C}^3}{12} - b^2\underline{C} - \frac{\left(\overline{C}-\underline{C}\right)^3}{12} - \frac{\left(1-\overline{C}\right)}{12}\left(12b^2 + \left(1-\overline{C}\right)^2\right).$$

Solving the first order conditions results in

$$C = \left[\underline{C},\overline{C}\right] = [\frac{1}{3}\left(2-\sqrt{1+12b^2}\right), \frac{1}{3}\left(1+\sqrt{1+12b^2}\right)] = \left[0.157, 0.843\right].$$

Note that the solution satisfies the sender's IC constraint for all $b \in \left(\frac{1}{4}, \frac{1}{2}\right)$. This follows from the equivalence of the following three inequalities and the fact that the third inequality is satisfied for all b.

$$\underline{C} + b - \frac{\underline{C}}{2} \leq \frac{\left(\overline{C}+1\right)}{2} - \underline{C} - b$$
$$\frac{1}{3}\left(2-\sqrt{1+12b^2}\right) + b - \frac{1}{6}\left(2-\sqrt{1+12b^2}\right) \leq \frac{1}{6}\left(1+\sqrt{1+12b^2}\right) + \frac{1}{2} - \frac{1}{3}\left(2-\sqrt{1+12b^2}\right) - b$$
$$\frac{1}{3} + 2b \leq \frac{2}{3}\sqrt{1+12b^2}.$$

The sender's resulting payoff is -0.062. Comparing the resulting payoffs for $b = \frac{1}{3}$, it is straightforward to see that

$$EU^S(\mathcal{C}_0, 1) < EU^S\left(\left([0,1], \frac{1}{2}+b\right), 1\right) < EU^S\left(\mathcal{C}_1^*, 1\right) < EU^S\left(\mathcal{C}_2^*, 2\right). \tag{1}$$

For biases $b \in \left(\frac{1}{4}, \frac{1}{2}\right)$, the first two inequalities follow since we have

$$\begin{aligned} EU^S(\mathcal{C}_0, 1) &= -\frac{1}{12} - b^2 \\ < EU^S\left(\left([0,1], \frac{1}{2}+b\right), 1\right) &= -\frac{1}{12} \\ < EU^S\left(\mathcal{C}_1^*, 1\right) &= -\frac{1}{12} + \left(b^2 - \frac{1}{4}\right)^2. \end{aligned}$$

To show the last inequality in (1), $EU^S\left(\mathcal{C}_1^*, 1\right) < EU^S\left(\mathcal{C}_2^*, 2\right)$, we have to show that $\left(b^2 - \frac{1}{4}\right)^2 - \frac{1}{12} = b^4 - \frac{b^2}{2} - \frac{1}{48} < \frac{1}{108}\left(-5 + 4\sqrt{1+12b^2} + 48b^2\left(-3+\sqrt{1+12b^2}\right)\right)$. Note that for $b = 0$ we have

$$\left(b^2 - \frac{1}{4}\right)^2 - \frac{1}{12} = -\frac{1}{12} < -\frac{1}{108} = \frac{1}{108}\left(-5 + 4\sqrt{1+12b^2} + 48b^2\left(-3+\sqrt{1+12b^2}\right)\right).$$

Moreover, for $b = \frac{1}{2}$, we obtain

$$\left(b^2 - \frac{1}{4}\right)^2 - \frac{1}{12} = -\frac{1}{12} = \frac{1}{108}\left(-5 + 4\sqrt{1+12b^2} + 48b^2\left(-3+\sqrt{1+12b^2}\right)\right).$$

Thus the difference between these utilities, $EU^S\left(\mathcal{C}_2^*, 2\right) - EU^S\left(\mathcal{C}_1^*, 1\right)$, is zero at $b = \frac{1}{2}$. The result follows, because the difference between these utilities is monotone decreasing in b for all $b \in \left(\frac{1}{4}, \frac{1}{2}\right)$:

$$\frac{d}{db}\left(EU^S\left(\mathcal{C}_2^*, 2\right) - EU^S\left(\mathcal{C}_1^*, 1\right)\right)$$

$$= b - 4b^3 + \frac{1}{108}\left(\frac{48b}{\sqrt{1+12b^2}} + \frac{576b^3}{\sqrt{1+12b^2}} + 96b\left(-3+\sqrt{1+12b^2}\right)\right)$$
$$= \frac{b}{3}\left(-5 + \frac{4}{\sqrt{1+12b^2}} + b^2\left(-12 + \frac{48}{\sqrt{1+12b^2}}\right)\right)$$
$$= -\frac{b}{3}\left(5 + 12b^2 - 4\sqrt{1+12b^2}\right).$$

The expression on the right-hand side is negative if and only if $5 + 12b^2 > 4\sqrt{1+12b^2}$. Since both sides of the inequality are positive, this is equivalent to $144b^4 - 72b^2 + 9 > 0$. The polynomial on the left-hand side has two zeros, $b = \pm\frac{1}{2}$, is strictly positive at $b = 0$ and is therefore strictly positive for $b \in \left(\frac{1}{4}, \frac{1}{2}\right)$. This implies that the derivative of the difference of the utilities is strictly negative for $b \in \left(\frac{1}{4}, \frac{1}{2}\right)$. Hence, we have that $EU^S\left(\mathcal{C}_2^*, 2\right) > EU^S\left(\mathcal{C}_1^*, 1\right)$.

Finally, consider $\widehat{K} = 2$ and $b = \frac{1}{5}$. The sender's problem for optimal contracts with 3-step equilibria is

$$\max_{C_1, C_2} - \int_0^{\underline{C}_1}\left(s + b - \frac{\underline{C}_1}{2}\right)^2 ds - \int_{\underline{C}_1}^{\overline{C}_1}\left(s + b - \left(\frac{\underline{C}_1 + \overline{C}_1}{2} + b\right)\right)^2 ds$$
$$- \int_{\overline{C}_1}^{\underline{C}_2}\left(s + b - \frac{\overline{C}_1 + \underline{C}_2}{2}\right)^2 ds - \int_{\underline{C}_2}^{\overline{C}_2}\left(s + b - \left(\frac{\underline{C}_2 + \overline{C}_2}{2} + b\right)\right)^2 ds - \int_{\overline{C}_2}^{1}\left(s + b - \frac{\overline{C}_2 + 1}{2}\right)^2 ds$$
$$\text{subject to } \underline{C}_1 + b - \frac{\underline{C}_1}{2} \le \frac{\overline{C}_1 + \underline{C}_2}{2} - \underline{C}_1 - b \text{ and } \underline{C}_2 + b - \frac{\overline{C}_1 + \underline{C}_2}{2} \le \frac{\overline{C}_2 + 1}{2} - \underline{C}_2 - b.$$

For $b = \frac{1}{5}$ the contract that solves this problem is

$$\mathcal{C}_3^* = \{([0.063, 0.468], 0.466), ([0.532, 0.937], 0.934)\}.$$

This contract yields an expected sender payoff of -0.01874. □

Proof of Proposition 1. For any measurable set $\Phi \subseteq [0,1]$ with $\text{Prob}(\Phi) > 0$ define

$$y^{*i}(\Phi) := \arg\max_y \int_\Phi U^i(y, \theta) dF(\theta), \ i = S, R.$$

Suppose $\mathcal{C}$ is an optimal contract in $G(\widehat{K}, b)$. If the contract is empty, $K = 0$, or the union of conditions has measure zero, $\mu\left(\bigcup_{k=1}^K C_k\right) = 0$, then $\Gamma^{\mathcal{C}}$ is a CS game. Hence, each equilibrium action in an equilibrium of $\Gamma^{\mathcal{C}}$ is induced by an interval of types. Consider a sender optimal equilibrium $e^{\mathcal{C}}$ of $\Gamma^{\mathcal{C}}$. Since there are only finitely many equilibrium actions, there is an action $\hat{y}$ that is induced with positive probability. Let $[\underline{\theta}, \overline{\theta}]$ be the closure of the set of types who induce action $\hat{y}$ in $e^{\mathcal{C}}$. For every $\varepsilon > 0$ such that $\tau + \varepsilon < \overline{\theta}$, there is a set $[\tau, \tau + \varepsilon] \subset [\underline{\theta}, \overline{\theta}]$ with $y^{*R}([\tau, \tau+\varepsilon]) = \hat{y}$. Evidently, also $y^{*R}\left([\underline{\theta}, \overline{\theta}] \setminus [\tau, \tau+\varepsilon]\right) = \hat{y}$. Since $y^S(\theta) \neq y^R(\theta)$ and both y^S and y^R are continuous and $[0,1]$ is compact, there exists $\varepsilon_0 > 0$ such that $|y^S(\theta) - y^R(\theta)| > \varepsilon_0$ for all $\theta \in [0,1]$. Continuity of y^S and y^R and compactness of

$[0,1]$ further imply that there exists $\delta > 0$ such that $|y^S(\theta) - y^R(\theta+\delta)| > \varepsilon_0$ for all $\theta \in [0,1]$. Hence, if we choose $\varepsilon < \delta$ then $y^{*S}([\tau, \tau+\varepsilon]) > y^{*R}([\tau,\tau+\varepsilon)] = \hat{y}$. Hence, the alternative contract $\mathcal{C}' = \{(C_1, x_1)\}$, where $C_1 = [\tau, \tau+\varepsilon)]$ and $x_1 = y^{*S}([\tau,\tau+\varepsilon])$ allows an equilibrium $e^{\mathcal{C}'}$ in $\Gamma^{\mathcal{C}'}$ in which types outside of $[\tau, \tau+\varepsilon]$ induce the same actions and receive the same payoffs as in the equilibrium $e^{\mathcal{C}}$ in $\Gamma^{\mathcal{C}}$, while the sender is strictly better off if condition C_1 is realized. It follows that $K \geq 1$, and therefore an optimal contract is never empty.

Consider any contract $\mathcal{C}$ with $K < \widehat{K}$ and a sender optimal equilibrium in the communication game $\Gamma^{\mathcal{C}}$. Consider replacing the contract $\mathcal{C}$ by a contract $\mathcal{C}'$ that splits the condition $C_K = [\underline{C}_K, \overline{C}_K]$ (taking the condition C_K to be closed is without loss of generality) into two conditions $\tilde{C}_K = [\underline{C}_K, \tilde{C})$ and $\tilde{\tilde{C}}_K = [\tilde{C}, \overline{C}_K)$ with $\underline{C}_K < \tilde{C} < \overline{C}_K$ and leaves all other clauses unchanged. Then $U^S_{11} < 0$ and $U^S_{12} > 0$ imply that $y^{*S}(\tilde{C}_K) < y^{*S}(C_K) < y^{*S}(\tilde{\tilde{C}}_K)$, which implies that the sender is strictly better off under the new contract, conditional on the event C_K being realized, while incentives in the communication games $\Gamma^{\mathcal{C}'}$ and $\Gamma^{\mathcal{C}}$ are identical. This implies that optimal contracts must have $K = \widehat{K}$. □

Proof of Lemma 2. By continuity of f and compactness of $[0,1]$, f is bounded. Therefore, for all $\delta > 0$ there is an $\epsilon_0 > 0$ such that for all $\Phi \subseteq [0,1]$ with $\text{Prob}(\Phi) > \delta$, $\ell(\Phi) > \epsilon_0$ (where ℓ denotes Lebesgue measure). Hence, for all $\delta > 0$ there is an $\epsilon_1 > 0$ such that for all $\Phi \subseteq [0,1]$ with $\text{Prob}(\Phi) > \delta$, for all $\theta \in [0,1]$ there exists $\Psi \subseteq \Phi$ such that $|\theta - \theta'| > \epsilon_1$ for all $\theta' \in \Psi$ and $\ell(\Psi) > \epsilon_1$. This and the fact that $y^{*S}(\Phi)$ is the ideal point of some type $\theta(\Phi) \in [0,1]$ imply that for all $\delta > 0$ there is an $\epsilon_1 > 0$ such that for all $\Phi \subseteq [0,1]$ with $\text{Prob}(\Phi) > \delta$, there exists $\Psi \subseteq \Phi$ such that $|\theta(\Phi) - \theta'| > \epsilon_1$ for all $\theta' \in \Psi$ and $\ell(\Psi) > \epsilon_1$.

Since the derivative of y^S is strictly positive and continuous it has a strictly positive lower bound. Therefore, for all $\epsilon_1 > 0$ we can find $\epsilon_2 > 0$ such that for all $\theta, \theta' \in [0,1]$ with $|\theta - \theta'| > \epsilon_1$, we have $|y^S(\theta) - y^S(\theta')| > \epsilon_2$. This and the continuity of U^S imply that for all $\epsilon_1 > 0$ we can find $\epsilon_3 > 0$ such that for all $\theta, \theta' \in [0,1]$ with $|\theta - \theta'| > \epsilon_1$, we have $U^S(y^S(\theta), \theta)$ - $U^S(y^S(\theta'), \theta) > \epsilon_3$. This, the fact that f is everywhere positive, and the observation at the end of the previous paragraph imply the statement. □

Proof of Proposition 2. Suppose not. Then there is a sequence of gaps $\{\mathcal{L}_{\widehat{K}}\}_{\widehat{K}=1}^{\infty}$ with a subsequence $\{\mathcal{L}_{\widehat{K}_i}\}_{i=1}^{\infty}$ and $\kappa > 0$ such that $\text{Prob}(\mathcal{L}_{\widehat{K}_i}) > \kappa$ for all i. From Lemma 1, there is an upper bound $\widehat{N}$ on the number of actions induced in any equilibrium of any communication subgame. Hence for every $\widehat{K}_i$, $i = 1, \ldots$, there is an action that is induced by a subset $\Phi_{\widehat{K}_i}$ of $\mathcal{L}_{\widehat{K}_i}$ that has at least probability $\frac{\kappa}{\widehat{N}}$. Hence, by Lemma 2 there exists $\epsilon > 0$ such that

$$\int_{\Phi_{\widehat{K}_i}} U^S(y^S(\theta), \theta) dF(\theta) - \int_{\Phi_{\widehat{K}_i}} U^S(y^{*S}(\Phi_{\widehat{K}_i}), \theta) dF(\theta) > \epsilon$$

for all $i = 1, \ldots$. This implies that for every $i = 1, \ldots$ the sender's payoffs in $e(\widehat{K}_i, b)$ are bounded from above by

$$\int_{[0,1]} U^S(y^S(\theta), \theta) dF(\theta) - \epsilon.$$

Continuity of y^S follows from the maximum theorem and uniform continuity from the fact that $[0,1]$ is compact. By assumption U^S is continuous. Uniform continuity of U^S follows from compactness of $\left[\min_{\theta\in[0,1]} y^S(\theta), \max_{\theta\in[0,1]} y^S(\theta)\right] \times [0,1]$. For any $\widehat{K}$, partition the interval $[0,1]$ into $\widehat{K}$ equal length intervals $I_1 := [\theta_0, \theta_1]$ and $I_k := (\theta_{k-1}, \theta_k]$, $k = 2, \dots, \widehat{K}$. For each $\widehat{K} = 1, 2, \dots$, define the function $U^S_{\widehat{K}} : [0,1] \to \mathbb{R}$ by the property that $U^S_{\widehat{K}}(\theta) = U^S(y^S(\theta_k), \theta)$ for all $\theta \in I_k$ and all $k = 1, \dots, \widehat{K}$. Then $\int_{[0,1]} U^S_{\widehat{K}}(\theta) dF(\theta)$ is the sender's payoff from writing the contract $\mathcal{C}_{\widehat{K}} = \{(C_k, x_k)\}_{k=1}^{\widehat{K}}$ where $C_k = I_k$ and $x_k = y^S(\theta_k)$. Uniform continuity of y^S and U^S imply that for any $\tilde{\epsilon} > 0$ we can choose $\widehat{K}$ sufficiently large (and therefore $\delta := \theta_k - \theta_{k-1}$ appropriately small) such that $0 \leq U^S(y^S(\theta), \theta) - U^S_{\widehat{K}}(\theta) < \tilde{\epsilon}$ for all $\theta \in [0,1]$. Therefore we have

$$\lim_{\widehat{K}\to\infty} \int_{[0,1]} U^S_{\widehat{K}}(\theta) dF(\theta) = \int_{[0,1]} U^S(y^S(\theta), \theta) dF(\theta),$$

and therefore a contradiction with the supposition that $e(\widehat{K}_i, b)$ is sender optimal in $G(\widehat{K}_i, b)$ for all $i = 1, 2, \dots$. □

Proof of Proposition 3. Suppose not. Then there is an $\epsilon_0 > 0$ and a subsequence $\{\mathcal{L}_j\}_{j=1}^{\infty}$ (reindexed for convenience) with $\text{Prob}(\mathcal{L}_j) < 1 - \epsilon_0$ for all j. Hence, for every j there is a condition C^j in the contract $\mathcal{C}^j$ that is part of the equilibrium $e(b_j)$ with $\text{Prob}(C^j) \geq \frac{\epsilon_0}{\widehat{K}}$. By Lemma 2 there is an $\epsilon_1 > 0$ such that

$$\int_{C^j} U^S(y^S(\theta), \theta, 0) dF(\theta) - \int_{C^j} U^S(y^{*S}(C^j), \theta, 0) dF(\theta) > \epsilon_1$$

for all j. The space of intervals of length ℓ, $\frac{\epsilon_0}{\widehat{K}} \leq \ell \leq 1$ is compact. Hence, the sequence $\{C^j\}_{j=1}^{\infty}$ has a convergent subsequence. After reindexing, use $\{C^j\}_{j=1}^{\infty}$ to denote that subsequence in the sequel, and denote the limit by C. By continuity,

$$\int_{C} U^S(y^S(\theta), \theta, 0) dF(\theta) - \int_{C} U^S(y^{*S}(C), \theta, 0) dF(\theta) \geq \epsilon_1.$$

Hence, appealing to continuity again, for sufficiently large j,

$$\int_{C^j} U^S(y^S(\theta), \theta, b_j) dF(\theta) - \int_{C^j} U^S(y^{*S}(C^j), \theta, b_j) dF(\theta) \geq \frac{\epsilon_1}{2}.$$

This implies that for sufficiently large j in this subsequence the sender's payoffs in the equilibria $e(b_j)$ are bounded away from $\int_{[0,1]} U^S(y^S(\theta), \theta, 0) dF(\theta)$. This contradicts optimality of the equilibria in the sequence $\{e(b_j)\}$, since by the continuity property the communication games $\Gamma^0(b_j)$ have equilibria whose payoffs converge to $\int_{[0,1]} U^S(y^S(\theta), \theta, 0) dF(\theta)$ with $j \to \infty$. □

Proof of Proposition 4. Consider necessity first. For each action y_j with $j < n$ that is induced in equilibrium define $t_j := \sup\{\theta \in [0,1] | \theta \text{ induces } y_j\}$. The receiver's ideal action if he knew the type to be t_j would be $y = t_j$. Therefore, by single crossing, $y_j \le t_j$. Incentive compatibility requires that $(t_j + b - y_j)^2 \le (t_j + b - y_{j+1})^2$. Since $y_{j+1} > y_j$, we have $t_j + b - y_j > t_j + b - y_{j+1}$. Thus incentive compatibility and the fact that $t_j + b - y_j > t_j - y_j \ge 0$ imply that

$$t_j + b - y_j \le y_{j+1} - t_j - b,$$

and since $t_j \ge y_j$, we need $y_{j+1} - y_j \ge 2b$. Since every action is induced by a positive measure of types and by the single crossing property, we have $y_1 > 0$. Hence, if n actions are induced, then $(n-1)2b < 1$.

For the converse, consider a contract $\mathcal{C}$ with $n-1$ conditions C_k, $k = 1, 2, \ldots, n-1$, where each condition is of length $2b + \epsilon$, any two adjacent conditions C_k and C_{k+1} are separated by an interval $\left(\overline{C}_k, \underline{C}_{k+1}\right)$ of length ϵ, and $C_1 = [\epsilon, 2b + 2\epsilon]$. Since we assume that $n < 1 + \frac{1}{2b}$, we can choose $\epsilon > 0$ such that $n\epsilon + (n-1)(2b + \epsilon) = 1$, and hence such a contract exists. In the communication subgame $\Gamma^{\mathcal{C}}$ let the sender use a strategy $\sigma^{\mathcal{C}}$ that prescribes that types in any interval $\left(\overline{C}_k, \underline{C}_{k+1}\right)$ separating two adjacent conditions C_k and C_{k+1} send a common message different from the message sent by any other such interval. Then the receiver has a best reply $\rho^{\mathcal{C}}$ to the sender's strategy $\sigma^{\mathcal{C}}$ that prescribes an action $y_k \in \left(\overline{C}_k, \underline{C}_{k+1}\right)$ for the message send by types in the interval $\left(\overline{C}_k, \underline{C}_{k+1}\right)$. Notice that

$$\underline{C}_{k+1} + b - \overline{C}_k = b + \epsilon = 2b + \epsilon - b = \overline{C}_{k+1} - \underline{C}_{k+1} - b.$$

Therefore, (for any distribution) given the receiver's strategy, types in $\left(\overline{C}_k, \underline{C}_{k+1}\right)$ have no incentive to mimic types in $\left(\overline{C}_{k+1}, \underline{C}_{k+2}\right)$ and *a fortiori* any higher types. Similarly, since

$$\underline{C}_{k+1} - \overline{C}_k - b = \epsilon - b < 3b + \epsilon = \overline{C}_k + b - \underline{C}_k$$

types in $\left(\overline{C}_k, \underline{C}_{k+1}\right)$ have no incentive to mimic types in $\left(\overline{C}_{k-1}, \underline{C}_k\right)$ and *a fortiori* any lower types. This implies global incentive compatibility for the sender strategy $\sigma^{\mathcal{C}}$ against $\rho^{\mathcal{C}}$. Hence $\left(\sigma^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ is an equilibrium strategy pair for the communication subgame $\Gamma^{\mathcal{C}}$. □

Proof of Proposition 5. By Observation 1, the expected payoff from partitional communication is always bounded from above by the expected payoff from fully revealing communication. Therefore, the expected payoff from a contract with $\widehat{K}$ conditions that specifies a communication region of size λ is bounded from above by

$$\begin{aligned} & -\lambda b^2 - \widehat{K} \int_0^{\frac{1-\lambda}{\widehat{K}}} \left(x - \frac{1-\lambda}{2\widehat{K}}\right)^2 dx \\ = \quad & -\lambda b^2 - \frac{1}{12}\frac{1}{\widehat{K}^2}(1-\lambda)^3. \end{aligned}$$

The derivative of this expression with respect to λ, $-b^2 + \frac{(1-\lambda)^2}{4\widehat{K}^2}$, is negative for $b^2 > \frac{1}{4\widehat{K}^2}$. Therefore, for $\widehat{K} > \frac{1}{2b}$ it is optimal to reduce the size λ of the communication region to zero. □

Lemma A.1 *Suppose that for an equilibrium $e^{\mathcal{C}}$ there is a communication interval $\left(\underline{\theta},\overline{\theta}\right)$ for which the conditions C_ℓ, $\ell=1,\ldots,k$, are the ones satisfying $C_\ell \subset \left(\underline{\theta},\overline{\theta}\right)$. Then the action induced by types in $\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C})$ is*

$$y^{*R}\left(\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C})\right)=\frac{1}{2}\frac{\overline{\theta}^2-\sum_{\ell=1}^k\overline{C}_\ell^2+\sum_{\ell=1}^k\underline{C}_\ell^2-\underline{\theta}^2}{\overline{\theta}-\sum_{\ell=1}^k\overline{C}_\ell+\sum_{\ell=1}^k\underline{C}_\ell-\underline{\theta}}.$$

Proof of Lemma A.1. The action induced by the types in $\left(\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C})\right)$ solves

$$\max_y -\int_{\underline{\theta}}^{\underline{C}_1}(s-y)^2\,ds-\sum_{\ell=1}^{k-1}\int_{\overline{C}_\ell}^{\underline{C}_{\ell+1}}(s-y)^2\,ds-\int_{\overline{C}_k}^{\overline{\theta}}(s-y)^2\,ds.$$

The FOC is given by

$$\left(\underline{C}_1-y\right)^2-\left(\underline{\theta}-y\right)^2+\sum_{\ell=1}^{k-1}\left(\underline{C}_{\ell+1}-y\right)^2-\sum_{\ell=1}^{k-1}\left(\overline{C}_\ell-y\right)^2+\left(\overline{\theta}-y\right)^2-\left(\overline{C}_k-y\right)^2=0.$$

Rearranging, we get

$$\left(\underline{C}_1^2-\underline{\theta}^2+\sum_{\ell=1}^{k-1}\underline{C}_{\ell+1}^2-\sum_{\ell=1}^{k-1}\overline{C}_\ell^2+\overline{\theta}^2-\overline{C}_k^2\right)-2a\left(\underline{C}_1-\underline{\theta}+\sum_{\ell=1}^{k-1}\underline{C}_{\ell+1}-\sum_{\ell=1}^{k-1}\overline{C}_\ell+\overline{\theta}-\overline{C}_k\right)=0,$$

equivalent to $\left(-\underline{\theta}^2+\sum_{\ell=1}^k\underline{C}_\ell^2-\sum_{\ell=1}^k\overline{C}_\ell^2+\overline{\theta}^2\right)-2a\left(-\underline{\theta}+\sum_{\ell=1}^k\underline{C}_\ell-\sum_{\ell=1}^k\overline{C}_\ell+\overline{\theta}\right)=0$. We conclude by observing that the SOC is negative. □

Lemma A.2 *Suppose that for an equilibrium $e^{\mathcal{C}}$ there is a communication interval $\left(\underline{\theta},\overline{\theta}\right)$ such that the following holds: the conditions C_ℓ, $\ell=1,\ldots,k$, are the ones satisfying $C_\ell\subset\left(\underline{\theta},\overline{\theta}\right)$, and the boundaries of the conditions satisfy $\underline{C}_1>\underline{\theta}$, $\overline{C}_k<\overline{\theta}$, and for $i<j$, $\underline{C}_i>\overline{C}_{i-1}$ if $i>1$ and $\overline{C}_j<\overline{C}_{j+1}$ if $j<k$. Then, for any sufficiently small $\varepsilon>0$, there exist $\delta>0$ and a contract $\mathcal{C}'$ that differs from contract $\mathcal{C}$ only in the following way: condition C_i is replaced by its $(-\varepsilon)$-translation, condition C_j is replaced by its δ-translation, these translations continue to satisfy $C_i,C_j\subset\left(\underline{\theta},\overline{\theta}\right)$, they do not change the ordering of the conditions, and*

$$y^{*R}\left(\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C}')\right)=y^{*R}\left(\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C})\right).$$

Any $\varepsilon>0$ and $\delta>0$ for which this is the case satisfy $\delta=\varepsilon\frac{\overline{C}_i-\underline{C}_i}{\overline{C}_j-\underline{C}_j}$.

Proof of Lemma A.2. Replacing C_i by its $(-\varepsilon)$-translation raises the action $y^{*R}\left(\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C}')\right)$ and replacing C_j by its δ-translation lowers it. Furthermore, $y^{*R}\left(\left(\underline{\theta},\overline{\theta}\right)\cap\mathcal{L}(\mathcal{C}')\right)$ varies continuously with ε and δ. Thus, existence follows from continuity.

By Lemma A.1,

$$y^{*R}\left((\underline{\theta},\overline{\theta})\cap\mathcal{L}(\mathcal{C})\right)=\frac{1}{2}\frac{\overline{\theta}^2-\sum_{l=1}^{k}\overline{C}_l^2+\sum_{l=1}^{k}\underline{C}_l^2-\underline{\theta}^2}{\overline{\theta}-\sum_{l=1}^{k}\overline{C}_l+\sum_{l=1}^{k}\underline{C}_l-\underline{\theta}}.$$

Similarly, $y^{*R}\left((\underline{\theta},\overline{\theta})\cap\mathcal{L}(\mathcal{C}')\right)=$

$$\frac{1}{2}\frac{\overline{\theta}^2-\sum_{l\neq i,j}\overline{C}_l^2-\left(\overline{C}_i-\varepsilon\right)^2-\left(\overline{C}_j+\delta\right)^2+\sum_{l\neq i,j}\underline{C}_l^2+\left(\underline{C}_i-\varepsilon\right)^2+\left(\underline{C}_j+\delta\right)^2-\underline{\theta}^2}{\overline{\theta}-\sum_{l\neq i,j}\overline{C}_l-\left(\overline{C}_i-\varepsilon\right)-\left(\overline{C}_j+\delta\right)+\sum_{l\neq i,j}\underline{C}_l+\left(\underline{C}_i-\varepsilon\right)+\left(\underline{C}_j+\delta\right)-\underline{\theta}}.$$

Setting both expressions equal to each other and noting that the denominators are identical, we can simplify to get

$$-\overline{C}_i^2-\overline{C}_j^2+\underline{C}_i^2+\underline{C}_j^2=-\left(\overline{C}_i-\varepsilon\right)^2-\left(\overline{C}_j+\delta\right)^2+\left(\underline{C}_i-\varepsilon\right)^2+\left(\underline{C}_j+\delta\right)^2.$$

Hence, $\varepsilon\left(\overline{C}_i-\underline{C}_i\right)=\delta\left(\overline{C}_j-\underline{C}_j\right)$. □

Lemma A.3 *For any optimal contract $\mathcal{C}$ and any communication interval $(\underline{\theta},\overline{\theta})$ of a sender-optimal equilibrium $e^{\mathcal{C}}$ of the communication subgame $\Gamma^{\mathcal{C}}$ there is no more than one condition C with $C\subset(\underline{\theta},\overline{\theta})$.*

Proof of Lemma A.3. Suppose that the contract $\mathcal{C}$ contains more than one clause (C,x) with $C\subset(\underline{\theta},\overline{\theta})$. Since there is a finite number of clauses and the corresponding conditions are ordered, there is a minimal condition, $C_{\min}$, and a maximal condition, $C_{\max}$, satisfying $C_{\min}\subset(\underline{\theta},\overline{\theta})$ and $C_{\max}\subset(\underline{\theta},\overline{\theta})$. We now show that we can improve upon the assumed contract$\mathcal{C}$. For any $\varepsilon>0$ let $\delta=\varepsilon\frac{\overline{C}_{\min}-\underline{C}_{\min}}{\overline{C}_{\max}-\underline{C}_{\max}}$. Consider $\varepsilon>0$ such that $\varepsilon<\underline{C}_{\min}-\underline{\theta}$ and $\delta<\overline{\theta}-\overline{C}_{\max}$. Consider the contract $\mathcal{C}'$ which differs from contract $\mathcal{C}$ only in that the clauses $(C_{\min},x_{\min})$ and $(C_{\max},x_{\max})$ have been replaced by the $(-\varepsilon)$-translation of $(C_{\min},x_{\min})$ and the δ-translation of $(C_{\max},x_{\max})$. Let $\hat{y}$ be the action induced by types in $(\underline{\theta},\overline{\theta})\cap\mathcal{L}(\mathcal{C})$ in the postulated sender optimal equilibrium $e^{\mathcal{C}}$. By Lemma A.2, if all types in $(\underline{\theta},\overline{\theta})\cap\mathcal{L}(\mathcal{C}')$ send a common message $m(\hat{y})$ that is among the messages to which the receiver responds with action $\hat{y}$ in $e^{\mathcal{C}}$, then $m(\hat{y})$ also induces action $\hat{y}$. This implies that the game $\Gamma^{\mathcal{C}'}$ has an equilibrium $e^{\mathcal{C}'}$ in which the receiver strategy is the same as in $e^{\mathcal{C}}$, the sender strategy is the same for all types in $\mathcal{L}(\mathcal{C}')\setminus(\underline{\theta},\overline{\theta})$, and types in $(\underline{\theta},\overline{\theta})\cap\mathcal{L}(\mathcal{C}')$ send $m(\hat{y})$. The change in payoffs from replacing the contract-equilibrium pair $(\mathcal{C},e^{\mathcal{C}})$ by the pair $(\mathcal{C}',e^{\mathcal{C}'})$ is given by:

$$
\begin{aligned}
& \int_{\overline{C}_{\min}-\varepsilon}^{\overline{C}_{\min}} -(s+b-\hat{y})^2\,ds - \int_{\underline{C}_{\min}-\varepsilon}^{\underline{C}_{\min}} -(s+b-\hat{y})^2\,ds \\
& + \int_{\underline{C}_{\max}}^{\underline{C}_{\max}+\delta} -(s+b-\hat{y})^2\,ds - \int_{\overline{C}_{\max}}^{\overline{C}_{\max}+\delta} -(s+b-\hat{y})^2\,ds \\
= \; & \varepsilon \frac{\overline{C}_{\min}-\underline{C}_{\min}}{\overline{C}_{\max}-\underline{C}_{\max}} \Big(\left(\overline{C}_{\max}+\underline{C}_{\max}-\overline{C}_{\min}-\underline{C}_{\min}\right)\left(\overline{C}_{\max}-\underline{C}_{\max}\right) \\
& + \varepsilon\left(\overline{C}_{\max}-\underline{C}_{\max}+\overline{C}_{\min}-\underline{C}_{\min}\right)\Big).
\end{aligned}
$$

This expression is strictly positive since $\varepsilon > 0$, $\overline{C}_{\min} - \underline{C}_{\min} > 0$, $\overline{C}_{\max} - \underline{C}_{\max} > 0$, and $\overline{C}_{\max} + \underline{C}_{\max} > \overline{C}_{\min} + \underline{C}_{\min}$. □

Proof of Proposition 6. **Part I.** Under the assumptions of the proposition, for every condition cluster $\boldsymbol{C}$ there is a critical type $\theta_{\boldsymbol{C}}$ with $\boldsymbol{C} \cap \{\theta_{\boldsymbol{C}}\} \neq \emptyset$:

Since for every equilibrium in which the sender mixes there is an outcome equivalent equilibrium in which her strategy is pure, it is without loss of generality to have the sender strategy be pure in the equilibrium $e^{\mathcal{C}}$. Denote the strategy profile corresponding to the equilibrium $e^{\mathcal{C}}$ by $f^{\mathcal{C}} = \left(\sigma^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$. It follows from Lemma A.3 that it suffices to look at the case where the interior of each communication interval of the equilibrium $e^{\mathcal{C}}$ contains at most one condition. Hence, it suffices to show that for any $k = 1, \ldots, \widehat{K}$, the condition C_k does not belong to the interior of a communication interval for the equilibrium $e^{\mathcal{C}}$.

Suppose otherwise, i.e., for the contract $\mathcal{C}$ and the equilibrium $e^{\mathcal{C}}$ there is at least one communication interval with a condition in its interior. We will gradually replace the contract $\mathcal{C}$ by other contracts and the strategy profile $f^{\mathcal{C}}$ by other strategy profiles. At each iteration we will ensure that sender payoffs strictly increase. At the end we will verify that the strategy profile we obtain is an equilibrium profile.

Let the equilibrium $e^{\mathcal{C}}$ have n steps, and therefore n communication intervals I_j, $j = 1, \ldots, n$. For each communication interval I_j let the sender send message m_j and denote the action induced by types in I_j by y_j. Denote the critical types from equilibrium $e^{\mathcal{C}}$ by $\theta_j^{\mathcal{C}}$, $j = 0, 1, \ldots, n$. At each replacement of the prevailing contract and strategy profile, the number of steps as well as the number communication intervals remains constant at n. Types in communication interval I_j continue to send message m_j after each replacement and the receiver best responds to the sender's replacement strategy. After all unsent messages, have the receiver use the same response as after message m_1. As the response to m_1 changes with each replacement, change the response to unsent messages in the same way.

Step 1. Replace the contract $\mathcal{C}$ and the strategy profile $f^{\mathcal{C}}$ by a new contract $\mathcal{C}_0$ and a new strategy profile $f^{\mathcal{C}_0}$:

(a) Change the contract as follows: Consider any condition C_k such that there is a communication interval I_j with $C_k \subset (\underline{\theta}_j, \overline{\theta}_j)$. If $\underline{\theta}_j$ does not belong to a condition, replace

C_k by its $-(\underline{C}_k - \underline{\theta}_j)$-translation. If $\underline{\theta}_j$ does belong to a condition, replace C_k by the $-(\underline{C}_k - \underline{\theta}_j)$-translation of the left-open interval $C_k \setminus \{\underline{C}_k\}$.

(b) Change the sender strategy as follows: For any communication interval I_j that was affected by a translation (i.e., there was a condition $C_k \subset (\underline{\theta}_j, \overline{\theta}_j)$), after the translation have the sender send message m_j for types θ with $\underline{\theta}_j + (\overline{C}_k - \underline{C}_k) < \theta < \overline{\theta}_j$. For any communication interval I_j that was not affected by a translation have the sender continue to send message m_j.

(c) Change the receiver strategy as follows: Let the receiver best respond to the new sender strategy and respond to all unsent messages the same way he responds to message m_1.

We make no claim that the new strategy profile $f^{\mathcal{C}_0}$ is an equilibrium profile of the communication game $\Gamma^{\mathcal{C}_0}$. The question of equilibrium is addressed after the final iteration. By Observation 1, we have a strict payoff improvement for the sender over the payoff from $e^{\mathcal{C}}$ in $\Gamma^{\mathcal{C}}$ if players adopt the strategy profile $f^{\mathcal{C}_0}$ in the communication game $\Gamma^{\mathcal{C}_0}$.

After the replacement of the contract $\mathcal{C}$ by the contract $\mathcal{C}_0$ there is some number $L \leq \widehat{K}$ of condition clusters $\boldsymbol{C}_\ell$, $\ell = 1, \ldots, L$. Denote the minimal (maximal) type in each condition cluster $\boldsymbol{C}_\ell$ by $\underline{\boldsymbol{C}}_\ell$ ($\overline{\boldsymbol{C}}_\ell$). Refer to the communication interval with lower bound $\overline{\boldsymbol{C}}_\ell$ by $I^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ and let $y^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ be the receiver's best reply to beliefs concentrated on $I^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$. Similarly, let $I^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ stand for the communication interval with upper bound $\underline{\boldsymbol{C}}_\ell$ and let $y^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ be the receiver's best reply to beliefs concentrated on $I^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$.

Observe that type $\underline{\boldsymbol{C}}_\ell$ (weakly) prefers action $y^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ to action $y^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$: $y^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ is no further from $\underline{\boldsymbol{C}}_\ell$ than that type's preferred equilibrium action under the original equilibrium $e^{\mathcal{C}}$ and $y^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ is no closer to $\underline{\boldsymbol{C}}_\ell$ than that type's preferred equilibrium action under $e^{\mathcal{C}}$.

Step 2. As noted before, the strategy profile $f^{\mathcal{C}_0}$ will generally violate incentive compatibility for the sender given the contract $\mathcal{C}_0$ and the receiver's strategy. With the ultimate goal of reestablishing equilibrium, we begin by restoring incentive compatibility locally by replacing the strategy profile $f^{\mathcal{C}_0}$ by a new strategy profile $f^{\mathcal{C}_1}$ while leaving the prevailing contract unchanged, i.e., $\mathcal{C}_1 = \mathcal{C}_0$.

Between any two condition clusters $\boldsymbol{C}_\ell$ and $\boldsymbol{C}_{\ell+1}$ with $\ell < L$, and similarly between $\boldsymbol{C}_L$ and 1, restore equilibrium locally. In order to obtain a *local equilibrium* between $\boldsymbol{C}_\ell$ and $\boldsymbol{C}_{\ell+1}$, alter the sender strategy in that range and the receiver's responses to messages sent by types in that range so that the receiver best responds to those messages and sender types in that range have no incentive to mimic other types in that range. For now, ignore incentives to mimic types between other condition clusters. We address those incentives later. Modify strategies as follows:

(a) If none of the critical types $\theta^{\mathcal{C}}$ from the equilibrium $e^{\mathcal{C}}$ satisfy $\overline{\boldsymbol{C}}_\ell < \theta^{\mathcal{C}} < \underline{\boldsymbol{C}}_{\ell+1}$, leave sender and receiver strategies unchanged – they already satisfy the local-equilibrium condition. Otherwise, suppose that the critical types $\theta^{\mathcal{C}}$ satisfying $\overline{\boldsymbol{C}}_\ell < \theta^{\mathcal{C}} < \underline{\boldsymbol{C}}_{\ell+1}$

are $\theta_i^{\mathcal{C}}, \ldots, \theta_{i'}^{\mathcal{C}}$. Note that given the postulated receiver behavior in $f^{\mathcal{C}_0}$, type $\theta_i^{\mathcal{C}}$ is the only critical type in the range $\left(\overline{\boldsymbol{C}}_\ell, \underline{\boldsymbol{C}}_{\ell+1}\right)$ for which incentive compatibility is violated. Define $\lambda^{\mathcal{C}} := \theta_i^{\mathcal{C}} - \overline{\boldsymbol{C}}_\ell$.

(b) In order to restore equilibrium locally between $\boldsymbol{C}_\ell$ and $\boldsymbol{C}_{\ell+1}$, consider replacing $\theta_i^{\mathcal{C}}, \ldots, \theta_{i'}^{\mathcal{C}}$ in the specification of the sender's and receiver's strategies by $\theta_i, \ldots, \theta_{i'}$, where $\theta_i = \overline{\boldsymbol{C}}_\ell + \lambda$ and $\theta_{j+1} - \theta_j = \theta_{j+1}^{\mathcal{C}} - \theta_j^{\mathcal{C}} - \frac{\lambda - \lambda^{\mathcal{C}}}{i'+1-i}$, $j = i, \ldots, i'-1$, and $\lambda^{\mathcal{C}} \leq \lambda \leq \left(\theta_{i+1}^{\mathcal{C}} - \theta_i^{\mathcal{C}}\right)(i' + 1 - i) + \lambda^{\mathcal{C}}$. The last condition ensures that the length of the second step $\theta_{i+1} - \theta_i$ (and thus all subsequent steps) remains positive. For types in the range $\left(\overline{\boldsymbol{C}}_\ell, \underline{\boldsymbol{C}}_{\ell+1}\right)$, have the new sender strategy prescribe that the sender send message m_i in the interval $(\overline{\boldsymbol{C}}_\ell, \theta_i)$, message m_j in (θ_{j-1}, θ_j), $j = i+1, \ldots, i'$, and message $m_{i'+1}$ for types in $\left(\theta_{i'}, \underline{\boldsymbol{C}}_{\ell+1}\right)$. Otherwise, leave the sender strategy unchanged. Adjust the receiver's strategy so that the receiver best responds to messages m_j, $j = i, \ldots, i'+1$, given the new sender strategy, leaving all other responses unchanged.

(c) For $\lambda = \lambda^{\mathcal{C}}$, type θ_i (weakly) prefers the action that is induced by types in the interval $(\overline{\boldsymbol{C}}_\ell, \theta_i)$ to the action that is induced by types in the interval (θ_i, θ_{i+1}). If θ_i is indifferent, we are done. Otherwise, it must be the case that the length of the interval (θ_i, θ_{i+1}) exceeds that of $(\overline{\boldsymbol{C}}_\ell, \theta_i)$. Consider increasing λ from $\lambda = \lambda^{\mathcal{C}}$ to the value λ'' at which the lengths of these two intervals become the same. At that point type θ_i strictly prefers the action that is induced by types in the interval (θ_i, θ_{i+1}) to the action that is induced by types in the interval $(\overline{\boldsymbol{C}}_\ell, \theta_i)$. Therefore, existence of a λ' with $\lambda'' \geq \lambda' \geq \theta_i - \overline{\boldsymbol{C}}_\ell$ that restores equilibrium locally between $\boldsymbol{C}_\ell$ and $\boldsymbol{C}_{\ell+1}$ follows from continuity the payoff function, the intermediate value theorem, and the fact that as we vary λ in the manner described, the arbitrage conditions for types θ_j, $j = i+1, \ldots, i'$ continue to be satisfied, since the lengths of adjacent intervals (θ_{j-1}, θ_j), $j = i+1, \ldots, i'$, and $\left(\theta_{i'}, \underline{\boldsymbol{C}}_{\ell+1}\right)$, continue to differ by $4b$.

The total change of behavior required to restore equilibrium locally between $\boldsymbol{C}_\ell$ and $\boldsymbol{C}_{\ell+1}$, as just described, can be decomposed into $i' + 1 - i$ steps. In the kth step λ is increased by $\frac{\lambda' - \lambda^{\mathcal{C}}}{i'+1-i}$, the intervals $(\theta_{i+(k'-1)}, \theta_{i+k'})$ with $1 \leq k' < k$ are all shifted up by that amount, and the interval $(\theta_{i+k-1}, \theta_{i+k})$ is reduced in size by the same amount by keeping θ_{i+k} fixed while θ_{i+k-1} increases. In the final step the interval whose size is reduced is $\left(\theta_{i'}, \underline{\boldsymbol{C}}_{\ell+1}\right)$. By Observation 2 we have a payoff improvement at every step. Denote the strategy profile that results from restoring local equilibria in the game $\Gamma^{\mathcal{C}_1}$ between all pairs of adjacent condition clusters by $f^{\mathcal{C}_1}$.

Step 3. We next turn to addressing incentive constraints that involve types that are separated by condition clusters.

Observe that when we replace $f^{\mathcal{C}_0}$ by $f^{\mathcal{C}_1}$ in $\Gamma^{\mathcal{C}_1}$, for any condition cluster $\boldsymbol{C}_\ell$, we have $\left|I^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_1})\right| \geq \left|I^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})\right|$ and $\left|I^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_1})\right| \leq \left|I^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})\right|$. In combination with type $\underline{\boldsymbol{C}}_\ell$ having preferred action $y^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ to action $y^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_0})$ prior to the strategy-profile replacement, this implies that none of the types equal to or less than $\underline{\boldsymbol{C}}_\ell$, have an incentive to induce any action greater than $y^-(\boldsymbol{C}_\ell, f^{\mathcal{C}_1})$ available to them given the profile $f^{\mathcal{C}_1}$. Therefore, if none of the types $\overline{\boldsymbol{C}}_\ell$, $\ell = 1, \ldots, L$ have an incentive to induce an action less than $y^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_1})$ available to them given the profile $f^{\mathcal{C}_1}$, the combination of local equilibria forms an equilibrium overall.

If instead there is a type $\overline{\boldsymbol{C}}_\ell$ who prefers inducing an action less than $y^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_1})$ that is available given the profile $f^{\mathcal{C}_1}$, let $\hat{\ell}$ be the maximal ℓ such that this is the case. Consider the set of actions that are induced by types $\theta > \overline{\boldsymbol{C}}_{\hat{\ell}}$. Refer to the types who are indifferent among adjacent actions in this set of actions as $\hat{\ell}$-critical types. Use $\tilde{\ell}$ to denote the minimal $\ell > \hat{\ell}$ such that there is an $\hat{\ell}$-critical type $\tilde{\theta} \in \left[\underline{\boldsymbol{C}}_\ell, \overline{\boldsymbol{C}}_\ell\right)$, if there is such a type. If there is no $\hat{\ell}$-critical type $\tilde{\theta} \in \left[\underline{\boldsymbol{C}}_\ell, \overline{\boldsymbol{C}}_\ell\right)$ for all $\ell > \hat{\ell}$, proceed without introducing $\tilde{\ell}$. Note that if this case we have that either $\overline{\boldsymbol{C}}_\ell$ is an $\hat{\ell}$-critical type for all $\ell > \hat{\ell}$ or $\boldsymbol{C}_{\hat{\ell}}$ is the rightmost condition cluster ($\hat{\ell} = L$).

Note that if θ_{j-1}, θ_j and θ_{j+1} are $\hat{\ell}$-critical types such that $\theta_j = \overline{\boldsymbol{C}}_\ell$ and neither θ_{j-1} nor θ_{j+1} belong to a condition cluster, then we have

$$\theta_j + b - \frac{\theta_{j-1} + (\theta_j - (\overline{\boldsymbol{C}}_\ell - \underline{\boldsymbol{C}}_\ell))}{2} = \frac{\theta_{j+1} + \theta_j}{2} - \theta_j - b, \tag{2}$$

which is equivalent to

$$\theta_{j+1} - \theta_j = \theta_j - \theta_{j-1} + 4b + (\overline{\boldsymbol{C}}_\ell - \underline{\boldsymbol{C}}_\ell). \tag{3}$$

This is the standard arbitrage condition in the CS uniform quadratic example extended to the case where the $\hat{\ell}$-critical type θ_j is the upper endpoint of a condition cluster. If θ_{j-1} belongs to the condition cluster $\boldsymbol{C}_{\ell-1}$, replace θ_{j-1} by $\overline{\boldsymbol{C}}_{\ell-1}$ in the above expression, and if θ_{j+1} belongs to the condition cluster $\boldsymbol{C}_{\ell+1}$, replace θ_{j+1} by $\underline{\boldsymbol{C}}_{\ell+1}$.

Consider replacing the condition cluster $\boldsymbol{C}_{\hat{\ell}}$ by its λ translation (for notational convenience also denoted by $\boldsymbol{C}_{\hat{\ell}}$) for values $\lambda > 0$ that make it possible to

(a) maintain local equilibrium for types in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ (if $\hat{\ell} > 1$, and in the range $(0, \underline{\boldsymbol{C}}_{\hat{\ell}})$ otherwise) (this is achieved by choosing λ sufficiently small and increasing the length of each communication interval in this range by λ divided by the number of communication intervals in this range),

(b) maintain local equilibrium in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ and preserve indifference for all types θ such that $\theta = \overline{\boldsymbol{C}}_\ell$ with $\hat{\ell} < \ell < \tilde{\ell}$ (by condition (3), this is achieved by choosing λ sufficiently small and reducing the sizes of communication intervals in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ all by λ divided by the number of communication intervals in this range).

For each λ, denote the strategy that maintains local equilibrium for types $\theta > \overline{\boldsymbol{C}}_{\hat{\ell}-1}$ by f^λ.

Note that if, prior to the λ translation of $\boldsymbol{C}_{\hat{\ell}}$, type $\overline{\boldsymbol{C}}_{\hat{\ell}}$ prefers inducing an action less than $y^+(\boldsymbol{C}_{\hat{\ell}}, f^{\mathcal{C}_1})$ that is available given the profile $f^{\mathcal{C}_1}$, as postulated, it has to be the case that $\left|I^+(\boldsymbol{C}_{\hat{\ell}}, f^{\mathcal{C}_1})\right| > \left|I^-(\boldsymbol{C}_{\hat{\ell}}, f^{\mathcal{C}_1})\right|$. As a consequence of replacing $\boldsymbol{C}_{\hat{\ell}}$ by its λ translation and maintaining local equilibria in the ranges specified above, the lengths of communication intervals in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ increase and the lengths of communication intervals in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ decrease. It is easily checked that for all λ between $\lambda = 0$ and the value of λ that equalizes $\left|I^+(\boldsymbol{C}_\ell, f^\lambda)\right|$ and $\left|I^-(\boldsymbol{C}_\ell, f^\lambda)\right|$ the local equilibria in the ranges $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ can be preserved, as described above. Hence by payoff continuity and the intermediate value theorem, there exists a value of λ for which we have an equilibrium in the auxiliary game that is obtained by restricting the type space to $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$, leaving all condition clusters $\boldsymbol{C}_\ell$ with $\ell \neq \hat{\ell}$ unchanged, and replacing $\boldsymbol{C}_{\hat{\ell}}$ by its λ-translation. Denote this value of λ by λ'. Monotonicity of type $\overline{\boldsymbol{C}}_{\hat{\ell}}$'s payoff differential from actions $y^+(\boldsymbol{C}_{\hat{\ell}}, f^\lambda)$ and $y^-(\boldsymbol{C}_{\hat{\ell}}, f^\lambda)$ implies that λ' is unique. By a similar argument there exists a unique value of λ such that local equilibria in the ranges $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ are preserved as above and, in addition, we have $\tilde{\theta} = \overline{\boldsymbol{C}}_{\tilde{\ell}}$. Denote this value of λ by λ''.

Define $\lambda_{\min} := \min\{\lambda', \lambda''\}$ and note that with the $\lambda_{\min}$ translation of $\boldsymbol{C}_{\hat{\ell}}$ we have $\tilde{\theta} \in \left[\underline{\boldsymbol{C}}_\ell, \overline{\boldsymbol{C}}_\ell\right]$. Let n_1 be the number of communication intervals in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and n_2 the number of communication intervals in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$. If we replace $\boldsymbol{C}_{\hat{\ell}}$ by its $\lambda_{\min}$ translation while preserving local equilibria in the ranges $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ as indicated above, this increases the length of each communication interval I_j, in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ by $\frac{\lambda_{\min}}{n_1}$ and lowers the length of each communication interval I_j in the range $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ by $\frac{\lambda_{\min}}{n_2}$.

We can decompose the replacement of $\boldsymbol{C}_{\hat{\ell}}$ by its $\lambda_{\min}$-translation and the corresponding preservation of local equilibria in the ranges $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ into $n_1 \cdot n_2$ steps of size $\frac{\lambda_{\min}}{n_1 \cdot n_2}$. Define $I_j(0) := I_j$. At the rth step, $r = 1, \ldots, n_1 \cdot n_2$,

(1) identify two intervals $I_{j'}(r) \subseteq (\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and $I_{j''}(r) \subseteq (\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ among those that have been established by step $r-1$ and which satisfy $|I_{j'}(r)| < |I_{j'} + \frac{\lambda_{\min}}{n_1}|$ and $|I_{j''}(r)| > |I_{j''} - \frac{\lambda_{\min}}{n_2}|$,

(2) increase the length of the former by $\frac{\lambda_{\min}}{n_1 \cdot n_2}$ by changing its right endpoint,

(3) reduce the length of the latter by the same amount by changing its left endpoint,

(4) replace all intervals $I_j(r) \subseteq (\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ with $j > j'$ by their $\frac{\lambda_{\min}}{n_1 \cdot n_2}$-translation,

(5) replace all intervals $I_j(r) \subseteq (\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ with $j < j''$ by their $\frac{\lambda_{\min}}{n_1 \cdot n_2}$-translation,

(6) replace the $\boldsymbol{C}_{\hat{\ell}}$ that resulted from step $r-1$ by its $\frac{\lambda_{\min}}{n_1 \cdot n_2}$-translation,

(7) have the sender send the same message in $I_j(r)$ that she sent in $I_j(r-1)$ for all j,

(8) have the receiver best respond to the the new sender strategy.

By Observation 2 we have a strict payoff improvement at every step.

Denote the contract that results from replacing $\boldsymbol{C}_{\hat{\ell}}$ by its $\lambda_{\min}$-translation by $\mathcal{C}_2$. Denote the strategy profile that results from preserving local equilibria in the ranges $(\overline{\boldsymbol{C}}_{\hat{\ell}-1}, \underline{\boldsymbol{C}}_{\hat{\ell}})$ and $(\overline{\boldsymbol{C}}_{\hat{\ell}}, \underline{\boldsymbol{C}}_{\tilde{\ell}})$ as described above while otherwise being identical with $f^{\mathcal{C}_1}$ by $f^{\mathcal{C}_2}$.

If $\lambda_{\min} = \lambda'$, identify the maximal ℓ such that type $\overline{\boldsymbol{C}}_\ell$ prefers inducing an action less than $a^+(\boldsymbol{C}_\ell, f^{\mathcal{C}_2})$ that is available given the profile $f^{\mathcal{C}_2}$, if there is such an ℓ. Otherwise we are done. Note that this ℓ necessarily satisfies $\ell < \hat{\ell}$. Make this ℓ the new $\hat{\ell}$ and repeat the construction that, starting with $\mathcal{C}_1$ and the strategy profile $f^{\mathcal{C}_1}$, gave us $\mathcal{C}_2$ and $f^{\mathcal{C}_2}$.

If instead $\lambda_{\min} = \lambda''$, identify the minimal $\ell > \hat{\ell}$ such that there is a critical type $\tilde{\theta}$ in the set $\left[\underline{\boldsymbol{C}}_\ell, \overline{\boldsymbol{C}}_\ell\right)$ (note that this ℓ, if it exists, is necessarily larger than $\tilde{\ell}$). If there is no such ℓ we are done. Make this ℓ the new $\tilde{\ell}$ and repeat the construction that, starting with $\mathcal{C}_1$ and the strategy profile $f^{\mathcal{C}_1}$, gave us $\mathcal{C}_2$ and $f^{\mathcal{C}_2}$.

Starting with any $\mathcal{C}_i$ and $f^{\mathcal{C}_i}$ obtained in this manner construct $\mathcal{C}_{i+1}$ and $f^{\mathcal{C}_{i+1}}$ using the same procedure. Since there are finitely many indices ℓ and at each step either $\hat{\ell}$ drops or $\tilde{\ell}$ rises, this process terminates and that at that point we have an equilibrium with a strict payoff improvement.

Part II. If the equilibrium $e^{\mathcal{C}}$ induces at least two communication actions, then there is a condition clusters $\boldsymbol{C}$ and a critical type $\theta \neq 0, 1$ with $\boldsymbol{C} \cap \{\tilde{\theta}\} \neq \emptyset$:

Suppose for contradiction that the equilibrium $e^{\mathcal{C}}$ induces at least two communication actions, and that for all critical types $\tilde{\theta} \neq 0, 1$ and all condition clusters $\boldsymbol{C}$, it is the case that $\boldsymbol{C} \cap \{\tilde{\theta}\} = \emptyset$. Let $n > 1$ be the number of communication intervals in $e^{\mathcal{C}}$. Then any condition cluster $\boldsymbol{C}$ satisfies either $0 \in \boldsymbol{C}$ or $1 \in \boldsymbol{C}$, and there is a critical type $\theta_1 \in (0, 1)$.

Consider the case where $0 \in \boldsymbol{C}$ for a condition cluster $\boldsymbol{C}$. Let the contract $\mathcal{C}'$ only differ from $\mathcal{C}$ by replacing the condition cluster $\boldsymbol{C}$ by its $(\theta_1 - \overline{\boldsymbol{C}})$-translation, $\boldsymbol{C}'$. Evidently, the game $\Gamma^{\mathcal{C}'}$ has an equilibrium $e^{\mathcal{C}'}$ in which types $\theta \in (0, \theta_1 - \overline{\boldsymbol{C}})$ send the message sent by types in (θ_1, θ_2) in equilibrium $e^{\mathcal{C}}$, and all other types behave as they did before in equilibrium $e^{\mathcal{C}}$. The sender's expected payoff in the equilibrium $e^{\mathcal{C}'}$ is the same as in $e^{\mathcal{C}}$, type $\theta_1 - \overline{\boldsymbol{C}}$ strictly prefers the action that is induced by types in $(0, \theta_1 - \overline{\boldsymbol{C}})$ to all other equilibrium actions and type θ_1 strictly prefers the action that is induced by types in the communication interval that is bounded below by θ_1 to all other equilibrium actions.

Since the incentive constraints of types $\underline{\boldsymbol{C}}' = \theta_1 - \overline{\boldsymbol{C}}$ and $\overline{\boldsymbol{C}}' = \theta_1$ in the new equilibrium $e^{\mathcal{C}'}$ are slack, for any sufficiently small $\lambda > 0$ we can replace contract $\mathcal{C}'$ by a contract $\mathcal{C}^\lambda$ that only differs from $\mathcal{C}'$ by replacing the condition cluster $\boldsymbol{C}'$ by its λ-translation, $\boldsymbol{C}^\lambda$, so that the game $\Gamma^{\mathcal{C}^\lambda}$ has an equilibrium $e^{\mathcal{C}^\lambda}$, in which, relative to $e^{\mathcal{C}'}$, the length of the first communication interval increases by λ and the lengths of all the remaining communication intervals are reduced by $\frac{\lambda}{n-1}$. Combining this with the fact that in $e^{\mathcal{C}}$, and therefore in $e^{\mathcal{C}'}$, the first is the smallest communication interval, repeated application of Observation 1 (as before) implies that for any sufficiently small $\lambda > 0$ the sender's expected payoff from $e^{\mathcal{C}^\lambda}$ strictly exceeds that from $e^{\mathcal{C}}$. It follows that $e^{\mathcal{C}}$ cannot have been optimal.

For the case where $1 \in \boldsymbol{C}$ for a condition cluster $\boldsymbol{C}$, consider the contract $\mathcal{C}''$ that only differs from $\mathcal{C}$ by replacing the condition cluster $\boldsymbol{C}$ by its $-(\underline{\boldsymbol{C}} - \theta_{n-1})$-translation, $\boldsymbol{C}''$. In this case, the game $\Gamma^{\mathcal{C}''}$ has an equilibrium $e^{\mathcal{C}''}$ in which types $\theta \in (1 - (\underline{\boldsymbol{C}} - \theta_{n-1}), 1)$ send the message sent by types in $(\theta_{n-1}, \underline{\boldsymbol{C}})$ in equilibrium $e^{\mathcal{C}}$ and all other types behave as they did before in equilibrium $e^{\mathcal{C}}$. Similar to the previous case, the incentive constraints of types $\underline{\boldsymbol{C}}''$ and $\overline{\boldsymbol{C}}''$ are slack, $\left(\overline{\boldsymbol{C}}'', 1\right] = (1 - (\underline{\boldsymbol{C}} - \theta_{n-1}), 1]$ is the largest communication interval, and therefore for sufficiently small $\lambda > 0$ one can increase equilibrium payoffs by replacing $\boldsymbol{C}''$ by its λ translation. $\square$

Proof of Corollary 1. Part 1. is an immediate consequence of Proposition 6. For Part 2., note that it suffices to prove the claim for P and P' that are adjacent to each other. If P' is a condition of $\mathcal{C}$, or both P and P' are communication intervals, the result is an immediate consequence of our assumptions on the payoff functions U^i, $i = S, R$.

Suppose, therefore, that P' is a communication interval and P a condition of $\mathcal{C}$ with $\inf(P') \geq \sup(P)$. For each $P \in \mathcal{P}$ use $x(P)$ to denote the sender-preferred action given P and $y(P)$ to denote the receiver preferred action given P. In order to derive a contraction, suppose that we have that $a(P') \leq a(P)$ – i.e., $y(P') \leq x(P)$. The cross-partial condition implies that $x(P') > x(P)$. $\int_{P'} U^S(x,\theta)d\theta$ is an integral over strictly concave functions and, therefore, itself strictly concave. This implies that $\int_{P'} U^S(x,\theta)d\theta$ is strictly increasing for all $x < x(P')$. Therefore, since $y(P') \leq x(P) < x(P')$, and

$$\frac{d}{dx}\left(\int_P U^S(x,\theta)d\theta\right)\Bigg|_{x=x(P)} = 0,$$

we obtain that for sufficiently small $\epsilon > 0$,

$$\int_P U^S(x(P),\theta)d\theta + \int_{P'} U^S(y(P'),\theta)d\theta < \int_{P\cup P'} U^S(x(P)+\epsilon,\theta)d\theta.$$

This, however, implies that there exists $\epsilon > 0$ such that the sender would strictly prefer to have a single condition $\overline{P \cup P'}$ with instruction $x(P) + \epsilon$ (of course, a further improvement could be achieved by replacing the instruction $x(P) + \epsilon$ by the instruction $x\left(\overline{P \cup P'}\right)$ if they differ). This gives us the desired contradiction. $\square$

Proof of Lemma 3. Slightly abusing notation, also use J to denote the set $\{1, \ldots, J\}$. If $\{j \in J | \overline{D}_j < \theta_1\}$ is nonempty and, thus, j' is well-defined for $j \leq j'$, let $t_{2j} := \overline{D}_j$ and $t_{2j-1} := \underline{D}_j$. In addition, let $t_{2j'+1} := \min\{\theta_1, \underline{D}_{j''}\}$ if j'' is well-defined, and $t_{2j'+1} := \theta_1$ otherwise. Define

$$X_{j'} := \sum_{j=1}^{j'} t_{2j} - t_{2j-1} \quad \text{and} \quad Y_{j'} := \sum_{j=1}^{j'} t_{2j}^2 - t_{2j-1}^2.$$

Note that

$$\begin{aligned} y_1 &= \frac{t_1}{t_{2j'+1} - X_{j'}}\frac{t_1}{2} + \sum_{j=1}^{j'} \frac{t_{2j+1} - t_{2j}}{t_{2j'+1} - X_{j'}}\frac{t_{2j+1} + t_{2j}}{2} \\ &= \frac{t_{2j'+1}^2 - \sum_{j=1}^{j'} t_{2j}^2 - t_{2j-1}^2}{2(t_{2j'+1} - X_{j'})} \\ &= \frac{t_{2j'+1}^2 - Y_{j'}}{2(t_{2j'+1} - X_{j'})}, \end{aligned}$$

and

$$\tilde{y}_1 = \frac{t_{2j'+1} - X_{j'}}{2}.$$

Showing that

$$t_{2j'+1} - y_1 \geq t_{2j'+1} - X_{j'} - \tilde{y}_1$$

is equivalent to showing that

$$\begin{aligned}
t_{2j'+1} - \frac{t_{2j'+1}^2 - Y_{j'}}{2(t_{2j'+1} - X_{j'})} &\geq \frac{t_{2j'+1} - X_{j'}}{2} \quad \Leftrightarrow \\
2t_{2j'+1}(t_{2j'+1} - X_{j'}) - (t_{2j'+1}^2 - Y_{j'}) &\geq (t_{2j'+1} - X_{j'})^2 \quad \Leftrightarrow \\
Y_{j'} &\geq X_{j'}^2.
\end{aligned}$$

To prove that $Y_{j'} \geq X_{j'}^2$, we proceed by induction. Since $t_2^2 - t_1^2 = (t_2 - t_1)(t_2 + t_1) \geq (t_2 - t_1)^2$, the claim holds for $j' = 1$. Suppose, it holds for j'. We show that it then also holds for $j' + 1$.

$$\begin{aligned}
& (X_{j'+1})^2 \\
= & \left(\sum_{j=1}^{j'+1} t_{2j} - t_{2j-1}\right)^2 \\
= & \left(\sum_{j=1}^{j'} t_{2j} - t_{2j-1} + t_{2(j'+1)} - t_{2(j'+1)-1}\right)^2 \\
= & \left(\sum_{j=1}^{j'} t_{2j} - t_{2j-1}\right)^2 + 2\left(\sum_{j=1}^{j'} t_{2j} - t_{2j-1}\right)\left(t_{2(j'+1)} - t_{2(j'+1)-1}\right) + \left(t_{2(j'+1)} - t_{2(j'+1)-1}\right)^2 \\
< & \left(\sum_{j=1}^{j'} t_{2j} - t_{2j-1}\right)^2 + 2t_{2(j'+1)-1}\left(t_{2(j'+1)} - t_{2(j'+1)-1}\right) + \left(t_{2(j'+1)} - t_{2(j'+1)-1}\right)^2 \\
= & \left(\sum_{j=1}^{j'} t_{2j} - t_{2j-1}\right)^2 + t_{2(j'+1)}^2 - t_{2(j'+1)-1}^2 \\
\leq & \sum_{j=1}^{j'} t_{2j}^2 - t_{2j-1}^2 + t_{2(j'+1)}^2 - t_{2(j'+1)-1}^2 \quad \text{(by the induction hypothesis)} \\
= & Y_{j'+1}.
\end{aligned}$$

□

Proof of Proposition 7. For any condition C, use $x(C)$ to denote the sender's optimal instruction given that condition. Non-emptiness of the optimal contract follows from Proposition 1. Consider any contract $\mathcal{C}$ with a condition $C = \bigcup_{j=1}^{J} D_j$ and sender-optimal action $x(C)$, where the sets $D_j \subset [0,1]$, $j = 1, \ldots J$, are closed intervals with $\overline{D}_j < \underline{D}_{j+1}$ for $j < J$. We show that whenever $J > 1$ the contract $\mathcal{C}$ can be improved upon. Proposition 4 implies that for $b > \frac{1}{4}$, there cannot be more than two communication actions.

By Observation 1, any equilibrium with a single communication action can be improved upon, by replacing the condition C by the condition $\tilde{C} := \left[1 - \sum_{j=1}^{J} \overline{D}_j - \underline{D}_j, 1\right]$ and replacing $x(C)$ by $x(\tilde{C})$ – i.e., by consolidating the condition, moving it the to the right endpoint of the type space, and adjusting the instruction accordingly.

Suppose there is an equilibrium with two communication actions. Denote the lower of the two communication actions by y_1, the higher of the two communication actions by y_2, and the critical type by θ_1. Suppose that there exists $\hat{\jmath} \in J$ such that either $\overline{D}_{\hat{\jmath}} < \theta_1 < \underline{D}_{\hat{\jmath}+1}$, $\theta_1 < \underline{D}_1$, or $\overline{D}_J < \theta_1$. Replace the contract $\mathcal{C}$ by the contract $\tilde{\mathcal{C}}$ with condition $\tilde{C}$ and instruction $x(\tilde{C})$, where

$$\tilde{C} = \left[\theta_1 - \sum_{j=1}^{\hat{\jmath}} \overline{D}_j - \underline{D}_j, \theta_1 + \sum_{j=\hat{\jmath}+1}^{J} \overline{D}_j - \underline{D}_j\right].$$

Let $\tilde{y}_1$ be the receiver's best reply to beliefs concentrated on $[0, \underline{\tilde{C}}]$ and $\tilde{y}_2$ be the receiver's best reply to beliefs concentrated on $[\overline{\tilde{C}}, 1]$.

The sender's payoff conditional on the event $\tilde{C}$ exceeds the sender's payoff conditional on the event C in the original contract by Observation 1. If we let the receiver take action $\tilde{y}_1$ for types in $[0, \underline{\tilde{C}}]$ and action $\tilde{y}_2$ for types in $[\overline{\tilde{C}}, 1]$, then the sender's payoff conditional on the event $[0,1] \setminus \tilde{C}$ exceeds the payoff from the event $[0,1] \setminus C$ under the original contract, again by Observation 1.

If $\overline{D}_{\hat{\jmath}} < \theta_1 < \underline{D}_{\hat{\jmath}+1}$ for some $\hat{\jmath}$, then

$$\tilde{y}_2 - (\underline{\tilde{C}} + b) \geq y_2 - (\theta_1 + b) = \theta_1 + b - y_1 \geq \underline{\tilde{C}} + b - \tilde{y}_1.$$

The inequality on the left follows because $\tilde{y}_2 \geq y_2$ and $\underline{\tilde{C}} \leq \theta_1$; the equality in the middle is the usual arbitrage condition; and, the inequality on the right is implied by Lemma 3. As a result, all types in $[0, \underline{\tilde{C}}]$ prefer action $\tilde{y}_1$ to action $\tilde{y}_2$.

If, in addition, type $\overline{\tilde{C}}$ prefers action $\tilde{y}_2$ to action $\tilde{y}_1$, then these actions are equilibrium actions in the communication game that is induced by the condition $\tilde{C}$. Suppose, instead, that type $\overline{\tilde{C}}$ strictly prefers action $\tilde{y}_1$ to action $\tilde{y}_2$. Then, since $b > 0$, it must be the case that the length of the interval $[\overline{\tilde{C}}, 1]$ exceeds that of the interval $[0, \underline{\tilde{C}}]$. Hence, there exists $\delta > 0$ such that type $\overline{\tilde{C}} + \delta$ is indifferent between $y(0, \underline{\tilde{C}} + \delta)$ and $y(\overline{\tilde{C}} + \delta, 1)$. Therefore we have a two-step equilibrium in the communication game that is induced by the contract with condition $[\underline{\tilde{C}} + \delta, \overline{\tilde{C}} + \delta]$. Note that in this equilibrium, the communication intervals are of more equal length. Therefore, by Observation 2, the sender's payoff in this equilibrium – which is the payoff conditional on the event $[0,1] \setminus (\tilde{C} + \delta)$ – exceeds that from the event $[0,1] \setminus \tilde{C}$, where we had the receiver take action $\tilde{y}_1$ for types in $[0, \underline{\tilde{C}}]$ and action $\tilde{y}_2$ for types in $[\overline{\tilde{C}}, 1]$.

If $\theta_1 < \underline{D}_1$, then

$$\tilde{y}_2 - (\underline{\tilde{C}} + b) = \tilde{y}_2 - (\theta_1 + b) \geq y_2 - (\theta_1 + b) = \theta_1 + b - y_1.$$

As a result, all types in $[0, \underline{\tilde{C}}] = [0, \theta_1]$ prefer action $\tilde{y}_1$ to action $\tilde{y}_2$. If, in addition, type $\overline{\tilde{C}}$ prefers action $\tilde{y}_2$ to action $\tilde{y}_1$, then these actions are equilibrium actions in the communication game that is induced by the condition $\tilde{C}$. In that case, the sender strictly prefers the contract with condition $\tilde{C}$ and instruction $x(\tilde{C})$ to the original contract for the same reasons as given before. Suppose, instead, that type $\overline{\tilde{C}}$ strictly prefers action $\tilde{y}_1$ to action $\tilde{y}_2$. Then, as before, we can find a δ such that the sender strictly prefers the contract with condition $\tilde{C} + \delta$ and instruction $x(\tilde{C} + \delta)$ to the original contract.

If $\overline{D}_J < \theta_1$, then the exact same argument that we used for the case in which $\overline{D}_{\hat{\jmath}} < \theta_1 < \underline{D}_{\hat{\jmath}}$ for some $\hat{\jmath}$ implies that we can find an alternative contract (and equilibrium in the associated communication game) that the sender strictly prefers to the original contract.

It remains to consider the case in which there exists a $\hat{\jmath} \in J$ such that $\theta_1 \in D_{\hat{\jmath}}$. Note that it is impossible to have either $\theta_1 \in D_1$ if $\underline{D}_1 = 0$, or $\theta_1 \in D_J$ if $\overline{D}_J = 1$; in either case, there would not be two communication actions to make type θ_1 indifferent. Hence, $0 < \underline{D}_{\hat{\jmath}}$ and $\overline{D}_{\hat{\jmath}} < 1$.

Replace the contract $\mathcal{C}$ by the contract $\tilde{\mathcal{C}}$ with condition $\tilde{C}$ and instruction $x(\tilde{C})$ where

$$\tilde{C} = \left[\underline{D}_{\hat{\jmath}} - \sum_{j=1}^{\hat{\jmath}-1} \overline{D}_j - \underline{D}_j, \overline{D}_{\hat{\jmath}} + \sum_{j=\hat{\jmath}+1}^{J} \overline{D}_j - \underline{D}_j \right].$$

Let $\tilde{y}_1$ be the receiver's best reply to beliefs concentrated on $[0, \underline{\tilde{C}}]$ and $\tilde{y}_2$ be the receiver's best reply to beliefs concentrated on $[\overline{\tilde{C}}, 1]$. Lemma 3 implies that

$$\tilde{y}_2 - (\underline{\tilde{C}} + b) \geq y_2 - (\underline{D}_{\hat{\jmath}} + b) \geq \underline{D}_{\hat{\jmath}} + b - y_1 \geq \underline{\tilde{C}} + b - \tilde{y}_1.$$

With this in hand, we can use the exact same argument that we employed for the case in which $\overline{D}_{\hat{\jmath}} < \theta_1 < \underline{D}_{\hat{\jmath}+1}$ for some $\hat{\jmath}$ to argue that there exists a $\delta \geq 0$ and a contract with condition $[\underline{\tilde{C}} + \delta, \overline{\tilde{C}} + \delta]$ that the sender strictly prefers to the contract $\tilde{C}$. □

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