Estimation of Dynamic Games With Weak Assumptions on Payoff Type Information

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Abstract

We study the characterization of the identified set of parameters in infinite time horizon dynamic games, with only imposing weak assumptions on the information structure. In particular, we impose an assumption on the minimum amount of information the agents have about the payoff types. Our goal is to obtain a sharp identified set of parameters consistent with the observed distribution of actions and the minimum amount of potentially available information. We characterize the sharp identified set by extending the notion of Bayes Correlated Equilibrium in Bergemann and Morris (2016) to a dynamic setting. We propose a tractable estimation method building on this characterization result. Monte Carlo exercises demonstrate that our structural parameter estimates are inconsistent when the information structures on payoff types are misspecified. We then show that the identified sets obtained by the proposed approach contain the true parameter values without specifying the information structure on payoff types.

Keywords: Dynamic games, Information structure, Markov Perfect Bayesian Nash Equilibrium, Bayes Correlated Equilibrium, Partial identification

JEL Codes: C57, C73, L13

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1 Introduction

Dynamic games are essential tools in empirical industrial organization, e.g., entry/exit model and capacity competition model (Aguirregabiria et al., 2021). Dynamics matter because firms can make entry, exit, or change of production capacity decisions potentially every period, and the costs of these options naturally depend on their past choices or current states. A common strategy for solving dynamic games is to impose a simple information structure. Agent draws additively separable i.i.d. payoff types every period, which is supposed to be the only private information. The simple information structure enables researchers to solve dynamic games by solving many independent single-agent dynamic discrete choice problems. Structural parameters are then estimated to match the implied probabilities of taking actions conditional on payoff-relevant states to those in the data.

We propose a framework for estimating the structural parameters of dynamic games, leaving the information structure on payoff types unspecified. The information structure covers the cases where players know their payoff types and may receive signals that inform them about others' payoff types. In static games, Magnolfi and Roncoroni (2022) estimates the structural parameters, leaving the information structure unspecified. We extend their framework to dynamic games. We adopt Markov Perfect Bayesian Nash Equilibrium (MPBNE, Aguirregabiria et al., 2021), as an equilibrium concept, but with a general information structure on payoff types. We assume that, for the true structural parameters and information structure, 1) at least one equilibrium exists, and 2) a single stationary MPBNE play generates the data.

We develop a sharp identified set of the structural parameters for dynamic games, where we leave the information structure on payoff types unspecified in constructing the identified set. To make the approach computationally tractable, we introduce *Markov Perfect Bayes Correlated Equilibrium* (MPBCE) by adopting the notion of Bayes Correlated Equilibrium (BCE, Bergemann and Morris, 2013, 2016) to dynamic games, restricting agents' strategies to Markov strategies. We show that the sharp identified set under *weak assumptions on payoff type information* can be obtained by instead computing the identified set under MPBCE. In static games, Magnolfi and Roncoroni (2022) shows that the sharp identified set under weak assumptions on information can be obtained by calculating the identified set under BCE. The idea of our proof is analogous to Magnolfi and Roncoroni (2022) except that we need to deal with the terms associated with the continuation value. An additional piece we need to show there is the equality of the continuation value implied by the MPBCE and that of the corresponding MPBNE, which follows using Bergemann and Morris (2016)'s approach of selecting an information structure and an MPBNE strategy profile so that the joint distribution of actions and payoff types are the same as that of MPBCE.

We convert the problem of calculating the sharp identified set under weak assumptions on payoff type information to computing the identified set under MPBCE because it is impractical to solve for the MPBNE with various information structures on payoff types. In static games, the identified set under BCE can be calculated using linear programming (Magnolfi and Roncoroni, 2022). However, in our case, the continuation value, which involves the expected values of payoff types conditional on the actions chosen, complicates the situation. The linear constraints in Magnolfi and Roncoroni (2022) become quadratic constraints in our case. The quadratic constraints make the problem intractable because the problem of solving a system of equations with many non-convex quadratic constraints is NP-hard in general (Park and Boyd, 2017). We devise a constrained minimization problem that can be solved using linear programming by adding nuisance parameters for the values of the expected values of payoff types conditional on states.

In Monte Carlo experiments with a duopoly capacity competition game, we demonstrate that our structural parameter estimates are inconsistent when the information structures are misspecified. We use a simplified version of the quantity competition model in Besanko and Doraszelski (2004). There are two firms in a market of homogeneous products. Firms choose the next period's production capacity between "low" and "high" for each period, where changing the capacity from the current period is costly. Firms want to avoid their capacity to be both high because the competitive market results in a low product price, hurting their profits. Thus, when firms have a signal about their competitor, they try to avoid choosing high capacity if the competitor is likely to choose high capacity. Consequently, there would be an upward bias in the parameter estimate of the upgrading cost when we assume that firms only know their payoff types, but the truth is they have the opponent's payoff type information. We need to set the parameter value for the upgrading cost to be high to justify the observed firm choices of avoiding increasing their capacity because of the information about the opponent's payoff types. In contrast, the identified sets obtained by the proposed approach contain the true parameter values without specifying the information structure on payoff types.

Our research contributes to the literature on estimating dynamic games in empirical industrial organization. There is an extensive literature of estimating static games with a specific information structure (e.g., Bresnahan and Reiss, 1991; Berry, 1992; Tamer, 2003; Rysman, 2004; Bajari et al., 2010), and, recently, several researchers started to discuss about "weak assumptions on information" in static games (Magnolfi and Roncoroni, 2022; Syrgkanis et al., 2021; Bergemann et al., 2022). Similarly in dynamic games, researchers have been estimated dynamic games with a specific information structure on payoff types (e.g., Rust, 1994; Ericson and Pakes, 1995; Pakes et al., 2007; Aguirregabiria and Mira, 2007; Bajari et al., 2007; Gallant et al., 2018). We aim to start the discussion of weak assumptions on information in dynamic games. We note that our research is different from those that estimate dynamic games with other equilibrium concepts to reduce computational burden (Weintraub et al., 2008; Benkard et al., 2015; Ifrach and Weintraub, 2016) or to relax the assumption of Markov strategy (Fershtman and Pakes, 2012; Aguirregabiria and Magesan, 2019).

2 Model

2.1 Dynamic Games With Information Structure on Payoff Shocks

We consider discrete time and infinite time horizon dynamic games. Game is played by a finite set of players indexed by $i = 1, 2, \ldots, N$. The following primitives, which we formally introduce below, are the same across markets and periods: the number of players N; set of actions of all players $\mathcal{A}$; set of states of all players $\mathcal{X}$; flow payoff function $\bar{\pi}$; discount factor β; information structure on payoff types S; joint distribution of private payoff types for all players F_ε; and state transition function given all players' actions G. The game structure is common knowledge among players.

Every period $t = 1, 2, \ldots$, player i observes the state s_{it} and chooses an action a_{it} from a finite set $\mathcal{A}_i = \{1, \ldots, A_i\}$ to maximize the expected discounted payoff:

$$\mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau \bar{\pi}(a_{t+\tau}, s_{it+\tau}) \middle| a_{it}, s_{it} \right],$$

where $\bar{\pi}(\cdot)$ is the flow payoff function, $a_t = (a_{1t}, \ldots, a_{Nt}) \in \mathcal{A} = \{A_1, \ldots, A_L\}, L < \infty$ is a vector of actions for all players, and $\beta \in [0, 1)$ is a discount factor.

Player's state s_{it} contains three elements: observable public states x_t, unobservable private payoff types ε_{it}, and unobservable private signals z_{it}. Observable public states $x_t = (x_{1t}, \ldots, x_{Nt}) \in \mathcal{X} = \{X_1, \ldots, X_K\}, K < \infty$, include the observable states for all players. Unobservable private payoff types for player i at time t are denoted as $\varepsilon_{it} = (\varepsilon_{1,i,t}, \ldots, \varepsilon_{A,i,t})$. Unobservable private payoff types for all players $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt}) \sim F_\varepsilon(\varepsilon; \theta_\varepsilon)$ are i.i.d. across markets and periods, where $\mathcal{E}$ is the support of ε_t, and the cdf $F_\varepsilon(\varepsilon; \theta_\varepsilon)$ is parameterized by a finite dimensional vector $\theta_\varepsilon \in \Theta_\varepsilon$.

Public states x_t and private payoff types ε_{it} enter into the payoff function, and the payoff function is additively separable in the payoff types:

$$\bar{\pi}(a_t, s_{it}) = \pi(a_t, x_t) + \varepsilon_{a_{it},i,t}.$$

The payoff function net of payoff types, $\pi(a_t, x_t; \theta_\pi)$, is parameterized by a finite dimensional vector $\theta_\pi \in \Theta_\pi$. We do not consider the additive separability assumption necessary for the characterization result, but it makes the estimation procedure more manageable.

Unobservable private signals for player i at time t are denoted as $z_{it} \in \mathcal{Z}_i$. The signals are intended to cover the cases where the players know their payoff types and may receive noisy signals that inform them about the others' payoff types. Thus, we restrict the possible values of z_{it}, $\mathcal{Z}_i$, to a set of vectors that contain ε_{it} as the first element, i.e., $z_{it} = (\varepsilon_{it}, \tilde{z}_{it})$, where $\tilde{z}_{it} \in \tilde{\mathcal{Z}}_i$ is a component of the signal that may inform the player i about the opponents' payoff type. The possible values of $\tilde{z}_{it}$, $\tilde{\mathcal{Z}}_i$, determines the possible values of z_{it} as $\mathcal{Z}_i = \mathcal{E} \times \tilde{\mathcal{Z}}_i$. Signals $\tilde{z}_{it}$ for all players $\tilde{z}_t = (\tilde{z}_{1t}, \ldots, \tilde{z}_{Nt}) \in \tilde{\mathcal{Z}} = \bigtimes_{i=1}^{N} \tilde{\mathcal{Z}}_i$ are created from a fixed function $\tilde{Z} : \mathcal{X} \times \mathcal{E} \mapsto \Delta(\tilde{\mathcal{Z}})$, where $\Delta(\tilde{\mathcal{Z}})$ is the all possible distributions over $\tilde{\mathcal{Z}}$. We denote the conditional distribution of players' signals $\tilde{z}_t$ given the public states x_t and private payoff types ε_t as $\tilde{Z}(\tilde{z}_t|x_t, \varepsilon_t)$. The conditional distribution $\tilde{Z}(\cdot|x_t, \varepsilon_t)$ determines the conditional distribution of unobserved private signals for all players $z_t = (z_{1t}, \ldots, z_{Nt}) \in \mathcal{Z} = \bigtimes_{i=1}^{N} \mathcal{Z}_i$ given (x_t, ε_t) since the first element of each unobserved private signal is deterministic given the payoff types:

$$\begin{aligned} Z(z_t|x_t, \varepsilon_t) &= Z(e_{it}, \tilde{z}_{it}|x_t, \varepsilon_t) \\ &= \mathbb{1}(e_{it} = \varepsilon_{it})\tilde{Z}(\tilde{z}_t|x_t, \varepsilon_t), \end{aligned}$$

where we denote the conditional distribution of players' signals z_t given the public states x_t and private payoff types ε_t as $Z(z_t|x_t, \varepsilon_t)$.

Information structure on payoff types S is formally defined as $S = (\mathcal{Z}, Z)$, and the set $\mathcal{S}$ contains all such information structures that give players their payoff types as the first element of the signals. Players take their signals z_{it} into account in their strategy, although the signals do not directly enter into the payoff function.

For instance, the set of information structure on payoff types $\mathcal{S}$ defined here includes perfectly private information $\underline{S}$, complete information $\overline{S}$, and privileged information S^P. Perfectly private information is the case where players only know their own payoff types: for all i and t,

$$z_{it} = \varepsilon_{it} \text{ with probability } 1.$$

Perfectly private information is commonly adopted in the estimation of dynamic games.

Complete information is the case where payoff types of all players are observed by them: for all i and t,

$$z_{it} = \varepsilon_t \text{ with probability } 1.$$

Gallant et al. (2018) adopts complete information in their estimation of dynamic games of generic drug industry.

Privileged information is the case where some players $i \in \mathcal{N}^p$ know their opponents' type:

for all t,

$$z_{it} = \begin{cases} \varepsilon_t & i \in \mathcal{N}^p \\ \varepsilon_{it} & i \notin \mathcal{N}^p \end{cases} \text{ with probability } 1.$$

The examples above allow players to know their opponents' payoff types for sure if they can. We can also think of various information structures on payoff types that convey noisy information about their opponents' payoff types to the players.

Example. (Duopoly capacity competition model) We use a simplified version of the quantity competition model in Besanko and Doraszelski (2004) throughout the paper. There are two players or firms $i = 1, 2$ in a market of homogeneous products. Firms choose next period's production capacity between "low" ($a_{it} = 1$) and "high" ($a_{it} = 2$) for each period. Hence, the action space is $\mathcal{A} = \{(a_1, a_2) : (\text{low,low}), (\text{low,high}), (\text{high,low}), (\text{high,high})\}$. The state x_t is the firms' current capacities, which is determined by their previous choice, $x_t = a_{t-1} = (a_{1t-1}, a_{2t-1})$. We denote firm i's opponent firm as $-i$ in what follows.

The flow payoff for firm i at time t is

$$\bar{\pi}_{it} = p_t a_{it-1} - C(a_{it-1}, a_{it}) + \varepsilon_{a_{it},i,t},$$

where p_t is the price of the homogeneous product determined by the inverse demand function

$$p_t = b_0 - b_1(a_{1t-1} + a_{2t-1}),$$

the adjustment cost function C is defined as

$$C(a_{it-1}, a_{it}) = \begin{cases} 0 & \text{if } a_{it-1} = a_{it} \\ c_{12} & \text{if } a_{it-1} < a_{it} \\ c_{21} & \text{if } a_{it-1} > a_{it} \end{cases},$$

and the cost shocks $\varepsilon_{a,i,t} \in \mathcal{E}_0, |\mathcal{E}_0| < \infty$ are, for all $e \in \mathcal{E}_0$,

$$\Pr(\varepsilon_{a,i,t} = e) = \theta_{\varepsilon e} \in \left\{ \theta_{\varepsilon e} \in (0, 1] : \sum_{e \in \mathcal{E}_0} \theta_{\varepsilon e} = 1 \right\} \text{ for all } a, i, \text{ and } t.$$

The support of ε_t is $\mathcal{E} = (\mathcal{E}_0)^{2\times 2}$.

Thus, given competitor's state a_{-it-1}, the flow payoff can be rewritten as:

$$\bar{\pi}_{it} = b_0 a_{it-1} - b_1 a_{it-1}^2 - b_1 a_{it-1} a_{-it-1} - C(a_{it-1}, a_{it}) + \varepsilon_{a_{it},i,t}$$

or

$$\pi_{it}(a_t, x_t) = \pi_{it}(a_{1t}, a_{2t}, a_{1t-1}, a_{2t-1}) = b_0 a_{it-1} - b_1 a_{it-1}^2 - b_1 a_{it-1} a_{-it-1} - C(a_{it-1}, a_{it}).$$

The competitor's action a_{-it} is not included in i's flow payoff in this example.

The payoff parameters are $\theta_\pi = (b_0, b_1, c_{12}, c_{21})$ and the payoff type distribution parameters are $\theta_\varepsilon = \{\theta_{\varepsilon e}\}_{e \in \mathcal{E}_0}$. We prespecify the discount factor to be $\beta = 0.75$ and set the true parameter values to be $(b_0, b_1, c_{12}, c_{21}) = (10, 2.2, 2.9, 2.1)$, $\mathcal{E}_0 = \{1, 3, 6, 10\}$, and $(\theta_{\varepsilon 1}, \theta_{\varepsilon 3}, \theta_{\varepsilon 6}, \theta_{\varepsilon 10}) = (0.3, 0.3, 0.3, 0.1)$ in the Monte Carlo simulations below. We focus on estimating upgrade cost c_{12}, assuming the other parameters fixed at the true values.

We consider three information structures on payoff types. The first is perfectly private information $\underline{S}$, where firms only know their own payoff types: for $i = 1, 2$,

$$z_{it} = \varepsilon_{it} \text{ with probability } 1.$$

The second is complete information $\overline{S}$, where payoff types of all firms are observed by them: for $i = 1, 2$,

$$z_{it} = (\varepsilon_{it}, \varepsilon_{-i,t}) \text{ with probability } 1.$$

The third is privileged information S^P, where player $i = 1$ knows $i = 2$'s payoff types:

$$z_{it} = \begin{cases} (\varepsilon_{it}, \varepsilon_{-i,t}) & i = 1 \\ \varepsilon_{it} & i = 2 \end{cases} \text{ with probability } 1.$$

If the firms happen to choose high capacity simultaneously, their profits dramatically decrease in our parameter specification. Thus, the firms try to avoid upgrading when the opponent is likely to upgrade if they have the opponent's payoff type information. Consequently, when we assume firms only know their payoff types, the upgrade cost estimate needs to be high to justify firms avoiding upgrading because of the opponent's payoff type information. Therefore, we expect an upward bias when we assume firms only know their payoff types, but the truth is they have the opponent's payoff type information.

2.2 Strategy and Equilibrium

The timing of the game is

1. Agents privately observe $z_{it} = (\varepsilon_{it}, \tilde{z}_{it})$ and publicly observe x_t.

2. Agents simultaneously choose actions a_{it}.

3. Period payoffs realize $\bar{\pi}(a_t, x_t, \varepsilon_{it})$.

4. State evolves according to $F_{x,\varepsilon}(x_{t+1}, \varepsilon_{t+1}|a_t, x_t, \varepsilon_t) = G(x_{t+1}|a_t, x_t)F_\varepsilon(\varepsilon_{t+1})$,

where $G(x_{t+1}|a_t, x_t)$ is the probability of transitioning to state x_{t+1} from the current state x_t given the actions of the players a_t.

We restrict players' strategy to pure Markov strategies $\sigma_i : \mathcal{X} \times \mathcal{Z}_i \mapsto \mathcal{A}_i$ and adopt Markov Perfect Bayesian Nash Equilibrium (MPBNE) as a solution concept. We slightly modify the standard notion of MPBNE to accomodate the signals. We define the alternative specific value function $V_{a_{it}}(x_t, z_{it})$ as

$$
\begin{aligned}
& V_{a_{it}}(x_t, z_{it}) \\
= & \int_{\mathcal{Z}_{-i}} \sum_{a_{-it} \in \mathcal{A}_{-i}} \left\{ \pi_i(a_t, x_t, \varepsilon_{it}) + \beta \int_{\mathcal{X}} V_i(x_{t+1}) dG(x_{t+1}|x_t, a_t) \right\} \left\{ \prod_{j \neq i} \sigma_j(a_{jt}|x_t, z_{jt}) \right\} dZ(z_{-it}|x_t, z_{it}),
\end{aligned}
$$

where the value function $V_i(x_t)$ solves the integrated Bellman equation:

$$
V_i(x_t) \quad = \quad \int_{\mathcal{E}} \int_{\mathcal{Z}_i} \left\{ \max_{a_{it}} V_{a_{it}}(x_t, z_{it}) \right\} dZ(z_{it}|x_t, \varepsilon_t) dF_\varepsilon(\varepsilon_t).
$$

A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ is an MPBNE of game $\mathcal{G}(\theta, S)$, if, for every i, t, x_t, and z_{it} and $a_{it} \in \mathcal{A}$ with $\sigma_i(a_{it}|x_t, z_{it}) > 0$, then:

$$
V_{a_{it}}(x_t, z_{it}) \geq V_{a'_{it}}(x_t, z_{it}), \forall a'_{it} \in \mathcal{A}.
$$

Player i's policy function is the action that takes the maximum of the alternative specific value function $V_{a_{it}}(x_t, z_{it})$,

$$
\sigma_i(a_{it}|x_t, z_{it}) = \arg\max_{a_{it}} V_{a_{it}}(x_t, z_{it}).
$$

We neither discuss the existence nor uniqueness conditions of dynamic games with a general information structure on payoff types under the MPBNE.

Example. (Duopoly capacity competition model) Firms have pure Markov strategies

$$
\sigma_i(a_{it}|x_t, z_{it}) = \sigma_i(a_{it}|a_{it-1}, a_{-it-1}, z_{it})
$$

to decide actions a_{it} given the players' previous actions and the signals they receive. The alternative specific value function is

$$
\begin{aligned}
& V_{a_{it}}(x_t, z_{it}) \\
= & \int_{\mathcal{Z}_{-i}} \sum_{a_{-it}\in\mathcal{A}_{-i}} \left\{ \pi_i(a_t, x_t, \varepsilon_{it}) + \beta \int_{\mathcal{X}} V_i(x_{t+1}) dG(x_{t+1}|x_t, a_t) \right\} \left\{ \prod_{j\neq i} \sigma_j(a_{jt}|x_t, z_{jt}) \right\} dZ(z_{-it}|x_t, z_{it}) \\
= & \sum_{z_{-it}\in\mathcal{Z}_{-i}} \sum_{a_{-it}\in\mathcal{A}_{-i}} \{\pi_i(a_{it}, x_t) + \varepsilon_{a_{it},i,t} + \beta V_i(a_t)\} \, \sigma_{-i}(a_{-it}|z_{-it}, x_t) Z(z_{-it}|x_t, z_{it}) \\
= & \pi_i(a_{it}, x_t) + \varepsilon_{a_{it},i,t} + \beta \sum_{z_{-it}\in\mathcal{Z}_{-i}} \sum_{a_{-it}\in\mathcal{A}_{-i}} V_i(a_t) \sigma_{-i}(a_{-it}|z_{-it}, x_t) Z(z_{-it}|x_t, z_{it}),
\end{aligned}
$$

where the integrated Bellman equation is

$$
V_i(x_t) = \sum_{\varepsilon_t \in \mathcal{E}_0^{2\times 2}} \sum_{z_{it}\in\mathcal{Z}_i} \left\{ \max_{a_{it}\in\mathcal{A}_i} V_{a_{it}}(x_t, z_{it}) \right\} Z(z_{it}|x_t, \varepsilon_t) F(\varepsilon_t).
$$

3 Identification

3.1 MPBNE Predictions

Parameter θ and information structure on payoff types S characterize game $\mathcal{G}(\theta, S)$. Denote the set of all MPBNE strategy profiles for $\mathcal{G}(\theta, S)$ as $\mathrm{MPBNE}(\theta, S)$. An equilibrium $\sigma \in \mathrm{MPBNE}(\theta, S)$ induces joint conditional choice probability (CCP) predictions for all observable states $x \in \mathcal{X}$:

$$
q_\sigma(a|x) = \int_{\mathcal{E}} \int_{\mathcal{Z}} \left(\prod_{i=1}^{N} \sigma_i(a_i|x, z_i) \right) dZ(z|x, \varepsilon) dF_\varepsilon(\varepsilon).
$$

We define the MPBNE predictions as the set of predictions induced by all equilibria of the game $\mathcal{G}(\theta, S)$:

$$
Q_{\theta,S}^{\mathrm{MPBNE}} = \{\{q(\cdot|x) \in \Delta(\mathcal{A})\}_{x\in\mathcal{X}} : \exists \sigma \in \mathrm{MPBNE}(\theta, S) \text{ such that}, \forall x, q(\cdot|x) = q_\sigma(\cdot|x)\},
$$

where $\Delta(\mathcal{A})$ is the all possible distributions over the possbile values of players' actions $\mathcal{A}$.

3.2 Sharp Identified Set

We study the identified set for parameters θ from data on actions a and observable states x. Payoff types ε are unobservable to the econometrician. The setup is summarized in Assumption 1.

Assumption 1 (Observables)**.** *The econometrician observes the joint conditional choice probabilities (CCPs) given observable states, $P(a|x)$, for all actions $a \in \mathcal{A}$ and observable states*

$x \in \mathcal{X}$.

We assume that, for the true payoff parameters $\theta_0 \in \Theta$ and true information structure on payoff types $S_0 \in \mathcal{S}$, 1) at least one equilibrium in $\mathrm{MPBNE}(\theta_0, S_0)$ exists, and 2) the data are generated by a single stationary MPBNE play in game $\mathcal{G}(\theta_0, S_0)$. The properties of the data generating process are summarized by Assumption 2.

Assumption 2 (Data generating process)**.** *The set* $\mathrm{MPBNE}(\theta_0, S_0)$ *is non-empty and actions* a *are generated by a single stationary MPBNE play of the game* $\mathcal{G}(\theta_0, S_0)$*, so that* $\{P(\cdot|x)\}_{x\in\mathcal{X}} \in Q^{MPBNE}_{\theta_0,S_0}$.

Given the link between the game-theoretic model predictions and observables, we want to recover θ_0 without knowing (nor attempt to recover) the true information structure on payoff types S_0. Under Assumptions 1 and 2, the sharp identified set under *weak assumptions on payoff type information*, $\Theta_I^{\mathrm{MPBNE}}(\mathcal{S})$, is a set of parameters θ that yields joint CCP predictions $Q^{\mathrm{MPBNE}}_{\theta,S}$ that matches the observed joint CCPs $\{P(\cdot|x)\}_{x\in\mathcal{X}}$ for some $S \in \mathcal{S}$:

$$\Theta_I^{\mathrm{MPBNE}}(\mathcal{S}) = \left\{\theta \in \Theta : \exists S \in \mathcal{S} \text{ such that } \{P(\cdot|x)\}_{x\in\mathcal{X}} \in Q^{\mathrm{MPBNE}}_{\theta,S}\right\}.$$

Set $\Theta_I^{\mathrm{MPBNE}}(\mathcal{S})$ captures all the restrictions on parameters implied by assuming players know at least their own payoff types.

The prevalent approach in the literature on the estimation of dynamic games is to prespecify information structure S' and collect parameters θ that yield MPBNE predictions $Q^{\mathrm{MPBNE}}_{\theta,S'}$ that matches the observed joint CCPs:

$$\Theta_I^{\mathrm{MPBNE}}(S') = \left\{\theta \in \Theta : \{P(\cdot|x)\}_{x\in\mathcal{X}} \in Q^{\mathrm{MPBNE}}_{\theta,S'}\right\}.$$

This approach may yield inconsistent estimators when the information structure is misspecified, i.e., the prespecified information structure S' is not the same as the true information structure S_0.

3.3 Markov Perfect Bayes Correlated Equilibrium (MPBCE)

Consider the perfectly private information structure on payoff types $\underline{S}$, where players only know their own payoff types. Let $\Delta(\mathcal{A} \times \mathcal{E})$ be the all possible distributions over the possbile values of players' actions and payoff types $\mathcal{A} \times \mathcal{E}$. *Markov Perfect Bayes Correlated Equilibrium* (MPBCE) for game $\mathcal{G}(\theta, \underline{S})$ is a set of conditional distributions $\phi = \{\phi(\cdot|x) \in \Delta(\mathcal{A} \times \mathcal{E})\}_{x\in\mathcal{X}}$ that is

1. *Consistent with the prior*: for all $\varepsilon \in \mathcal{E}$ and $x \in \mathcal{X}$,

$$\sum_{a \in \mathcal{A}} \int_{e \leq \varepsilon} \phi(a, e|x) de = \int_{e \leq \varepsilon} dF_\varepsilon(e).$$

2. *Incentive compatibile*: for all i, t, $a_{it} \in \mathcal{A}_i$, $a_{it}^o \in \mathcal{A}_i$, $\varepsilon_{it} \in \mathcal{E}_i$, and $x_t \in \mathcal{X}$,

$$\int_{\mathcal{E}_{-i}} \sum_{a_{-it} \in \mathcal{A}_{-i}} \left(v_i^\phi(a_{it}, a_{-it}, \varepsilon_{it}, x_t) - v_i^\phi(a_{it}^o, a_{-it}, \varepsilon_{it}, x_t) \right) \phi(a_{it}, a_{-it}, \varepsilon_t | x_t) d\varepsilon_{-i} \geq 0,$$

   where

$$v_i^\phi(a_{it}, a_{-it}, \varepsilon_{it}, x_t) = \pi_i(a_{it}, a_{-it}, x_t) + \varepsilon_{a_{it}, i, t} + \beta \int_{\mathcal{X}} V_i^\phi(x_{t+1}) dG(x_{t+1}|x_t, a_t)$$

   and $V_i^\phi(\cdot)$ is a value function implied by the MPBCE ϕ as we formally introduce below.

Denote the set of all MPBCE as $\mathrm{MPBCE}(\theta)$.

An equilibrium $\phi \in \mathrm{MPBCE}(\theta)$ induces joint CCP predictions for all $x \in \mathcal{X}$:

$$q_\phi(a|x) = \int_{\mathcal{E}} \phi(a, \varepsilon|x) dF_\varepsilon(\varepsilon).$$

The set of predictions implied by $\mathrm{MPBCE}(\theta)$ is defined as

$$Q_\theta^{\mathrm{MPBCE}} = \{\{q(\cdot|x) \in \Delta(\mathcal{A})\}_{x \in \mathcal{X}} : \exists \phi \in \mathrm{MPBCE}(\theta) \text{ such that}, \forall x, q(\cdot|x) = q_\phi(\cdot|x)\}.$$

The identified set under MPBCE is defined as

$$\Theta_I^{\mathrm{MPBCE}} = \left\{ \theta \in \Theta : \{P(\cdot|x)\}_{x \in \mathcal{X}} \in Q_\theta^{\mathrm{MPBCE}} \right\}.$$

We show that the incentive compatibility constraints are quadratic constraints for ϕ given the joint CCPs $\{P(\cdot|x)\}_{x \in \mathcal{X}}$ from the data. We introduce the following vector and matrix notations to proceed: for $X \in \mathcal{X}$, $X_1, \ldots, X_K \in \mathcal{X}$, $A \in \mathcal{A}$, and $A_1, \ldots, A_L \in \mathcal{A}$,

$$\begin{aligned}
G(X,A) &= (G(X_1|X,A), \ldots, G(X_K|X,A))^\top, \\
V_i^\phi &= (V_i^\phi(X_1), \ldots, V_i^\phi(X_K))^\top, \\
P^X &= (P(A_1|X), \ldots, P(A_L|X))^\top, \\
\Pi_i^X &= (\pi_i(A_1,X), \ldots, \pi_i(A_L,X))^\top, \\
E_i^{\phi,X} &= (E_\phi[\varepsilon_{a_i,i}|a=A_1, x=X], \ldots, E_\phi[\varepsilon_{a_i,i}|a=A_L, x=X])^\top, \\
\Gamma^X &= (\Pr(x'=X_1|x=X), \ldots, \Pr(x'=X_K|x=X))^\top, \\
\Gamma &= \begin{pmatrix} (\Gamma^{X_1})^\top \\ \vdots \\ (\Gamma^{X_K})^\top \end{pmatrix}, \text{ and} \\
M_i^\phi &= ((P^{X_1})^\top(\Pi_i^{X_1} + E_i^{\phi,X_1}), \ldots, (P^{X_K})^\top(\Pi_i^{X_K} + E_i^{\phi,X_K}))^\top.
\end{aligned}$$

Then, the value function $V_i^\phi(\cdot)$ is determined by the integrated Bellman equation under the assumption of the perfectly private information $\underline{S}$:

$$\begin{aligned}
V_i^\phi(\cdot|x) &= \int_{\mathcal{E}} \max_{a_i \in \mathcal{A}} \left\{ \pi_i(a_i, a_{-i}, x) + \varepsilon_{a_i,i} + \beta \int_{\mathcal{X}} V_i^\phi(x') dG(x'|x, a_i, a_{-i}) \right\} dF_\varepsilon(\varepsilon_i) \\
&= \int_{\mathcal{E}} \max_{a_i \in \mathcal{A}} \left\{ \pi_i(a_i, a_{-i}, x) + \varepsilon_{a_i,i} + \beta G(x,a)^\top V_i^\phi \right\} dF_\varepsilon(\varepsilon_i) \\
&= (P^x)^\top(\Pi_i^x + E_i^{\phi,x} + \beta G(x,a)^\top V_i^\phi) \\
&= (P^x)^\top(\Pi_i^x + E_i^{\phi,x}) + \beta(\Gamma^x)^\top V_i^\phi.
\end{aligned}$$

Hence, by stacking the integrated Bellman equations for all $x \in \mathcal{X}$, we get

$$V_i^\phi = M_i^\phi + \beta\Gamma V_i^\phi \iff V_i^\phi = (I - \beta\Gamma)^{-1} M_i^\phi.$$

Note that $I - \beta\Gamma$ is invertible as follows. Since Γ is a stochastic matrix, the largest eigenvalue of Γ is smaller than or equal to one. The eigenvalues of $I - \beta\Gamma$ are given by $1 - \beta\gamma$, where γs are the eigenvalues of Γ. Thus, we have $1 - \beta\gamma > 0$ since $0 < \beta < 1$ and $\gamma \leq 1$, which implies that $I - \beta\Gamma$ is invertible.

Additionally, observe that

$$\begin{aligned}
M_i^\phi &= ((P^{X_1})^\top(\Pi_i^{X_1} + E_i^{\phi,X_1}), \ldots, (P^{X_K})^\top(\Pi_i^{X_K} + E_i^{\phi,X_K}))^\top \\
&= ((P^{X_1})^\top \Pi_i^{X_1} + E_\phi[\varepsilon_{a_i,i}|x=X_1], \ldots, (P^{X_K})^\top \Pi_i^{X_K} + E_\phi[\varepsilon_{a_i,i}|x=X_K])^\top,
\end{aligned}$$

where

$$E_\phi[\varepsilon_{a_i,i}|x=X] = \sum_{\varepsilon\in\mathcal{E}}\sum_{a\in\mathcal{A}} \varepsilon_{a_i,i}\phi(a_i, a_{-i}, \varepsilon_i, \varepsilon_{-i}|X).$$

Therefore, we have shown that $v_i^\phi(a_{it}, a_{-it}, \varepsilon_{it}, x_t)$ can be expressed as a linear combination of ϕ and conclude that the incentive compatibility constraints are quadratic constraints for ϕ.

3.4 Robust Prediction Property of MPBCE

We show that the robust prediction property of BCE (Bergemann and Morris, 2016) can be translated into the dynamic games in our setting. Let q be a shorthand notation for $\{q(\cdot|x)\}_{x\in\mathcal{X}}$.

Lemma 1. *For all $\theta \in \Theta$,*
1. If $q \in Q_\theta^{\mathrm{MPBCE}}$, then $q \in Q_{\theta,S}^{\mathrm{MPBNE}}$ for some $S \in \mathcal{S}$.
2. Conversely, for all $S \in \mathcal{S}$, $Q_{\theta,S}^{\mathrm{MPBNE}} \subseteq Q_\theta^{\mathrm{MPBCE}}$.

The idea of the proof is the same as Bergemann and Morris (2016) except that we need to deal with the continuation value. The continuation value complicates the proof of part 1 of Lemma 1. An additional piece we need to show there is the equality of the value function of the MPBNE σ, V_i, and the value function of the MPBCE ϕ, V_i^ϕ, which follows because we are selecting the information structure on payoff types and MPBNE strategy so that the resulting joint distribution of actions and payoff types conditional on states is the same as that of the MPBCE.

Proof. 1. Consider $q \in Q_\theta^{\mathrm{MPBCE}}$. Then, $\exists\phi \in \mathrm{MPBCE}(\theta)$ such that $q = q_\phi$. The goal is to show that $\exists S \in \mathcal{S}, \exists\sigma$ such that $q_\sigma = q_\phi$ and $q_\sigma \in Q_{\theta,S}^{\mathrm{MPBNE}}$. Take any $x \in \mathcal{X}$. Construct S so that $\tilde{\mathcal{Z}} = \mathcal{A}$ and $\tilde{Z}$ is a probability kernel $\left\{\tilde{Z}(\cdot|x,\varepsilon) : \varepsilon \in \mathcal{E}\right\}$ such that

$$\int_E \tilde{Z}(a|x,\varepsilon)dF_\varepsilon(\varepsilon) = \phi(a, E|x), \forall E \in \left\{E \in \mathcal{B}(\mathcal{E}) : \int_E dF_\varepsilon > 0, \forall a \in \mathcal{A}\right\}.$$

Also, $\forall\varepsilon_i, \forall\tilde{z}_i$, take

$$\sigma_i(a_i|x, \varepsilon_i, \tilde{z}_i) = \begin{cases} 1 & a_i = \tilde{z}_i \\ 0 & a_i \neq \tilde{z}_i \end{cases}.$$

We first show that $q_\sigma = q_\phi$ under these S and σ. For all $a \in \mathcal{A}$,

$$\begin{aligned} q_\sigma(a|x) &= \int_{\mathcal{E}} \int_{\tilde{\mathcal{Z}}} \left(\prod_{i=1}^{N} \sigma_i(a_i|x, \varepsilon_i, \tilde{z}_i) \right) d\tilde{Z}(\tilde{z}|x, \varepsilon) dF_\varepsilon(\varepsilon) \\ &= \int_{\mathcal{E}} \tilde{Z}(a|x, \varepsilon) dF_\varepsilon(\varepsilon) \\ &= \int_{\mathcal{E}} \phi(a, \varepsilon|x) d\varepsilon \\ &= q_\phi(a|x). \end{aligned}$$

We next show that the incentive compatibility conditions of MPBCE guarantee that such σ is an MPBNE of $\mathcal{G}(\theta, S)$ given $q_\sigma = q_\phi$. For $\sigma_i(a_i|x, \varepsilon_i, \tilde{z}_i) > 0$ or $\tilde{z}_i = a_i$,

$$\begin{aligned} & V_{a_i}(x, z_i) - V_{a_i^o}(x, z_i) \\ =\; & \int_{\mathcal{Z}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} (v_i - v_i^o) \left(\prod_{j \neq i} \sigma_j(a_j|x, z_j) \right) dZ(z_{-i}|x, z_i) \\ =\; & \int_{\mathcal{E}_{-i}} \int_{\tilde{\mathcal{Z}}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} (v_i - v_i^o) \left(\prod_{j \neq i} \sigma_j(a_j|x, \varepsilon_j, \tilde{z}_j) \right) d\tilde{Z}(\tilde{z}_{-i}|x, \varepsilon_i, \varepsilon_{-i}, \tilde{z}_i) dF_{\varepsilon_{-i}}(\varepsilon_{-i}|x, \varepsilon_i, \tilde{z}_i) \\ =\; & \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} (v_i - v_i^o) \tilde{Z}(a_{-i}|x, \varepsilon_i, \varepsilon_{-i}, \tilde{z}_i) dF_{\varepsilon_{-i}}(\varepsilon_{-i}|x, \varepsilon_i, \tilde{z}_i) \\ =\; & \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} (v_i - v_i^o) \tilde{Z}(a_{-i}|x, \varepsilon_i, \varepsilon_{-i}, a_i) dF_{\varepsilon_{-i}}(\varepsilon_{-i}|x, \varepsilon_i, a_i) \\ =\; & \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} (v_i - v_i^o) \phi(a_{-i}, \varepsilon_{-i}|a_i, \varepsilon_i, x) d\varepsilon_{-i}, \end{aligned}$$

where

$$\begin{aligned} v_i &= v_i(a_i, a_{-i}, \varepsilon_i, x) = \pi_i(a_i, a_{-i}, x) + \varepsilon_{a_i, i} + \beta \int_{\mathcal{X}} V_i(x') dG(x'|x, a_i, a_{-i}) \text{ and} \\ v_i^o &= v_i(a_i^o, a_{-i}, \varepsilon_i, x). \end{aligned}$$

So, if $V_i = V_i^\phi$ holds, then $v_i(a_i, a_{-i}, \varepsilon_i, x) = v_i^\phi(a_i, a_{-i}, \varepsilon_i, x)$ and $V_{a_i}(x, z_i) - V_{a_i^o}(x, z_i) \geq 0$ follow by the incentive compatibility conditions of MPBCE. Notice that the alternative specific value function under the current S and σ is

$$\begin{aligned} V_{a_i}(x, z_i) &= \int_{\mathcal{Z}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \left\{ \prod_{j \neq i} \sigma_j(a_j|x, z_j) \right\} dZ(z_{-i}|x, z_i) \\ &= \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_{-i}, \varepsilon_{-i}|a_i, \varepsilon_i, x) d\varepsilon_{-i} \end{aligned}$$

for $\tilde{z}_i = a_i$. Thus, the integrated Bellman equation is

$$\begin{aligned}
V_i(x) &= \int_{\mathcal{E}} \int_{\mathcal{Z}_i} \left\{ \max_{a_i} V_{a_i}(x, z_i) \right\} dZ(z_i|x,\varepsilon) dF_\varepsilon(\varepsilon) \\
&= \int_{\mathcal{E}} V_{a_i}(x, \varepsilon_i, a_i) \tilde{Z}(a_i|x,\varepsilon) dF_\varepsilon(\varepsilon) \\
&= \int_{\mathcal{E}} \left\{ \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_{-i}, \varepsilon_{-i}|a_i, \varepsilon_i, x) d\varepsilon_{-i} \right\} \tilde{Z}(a_i|x,\varepsilon) dF_\varepsilon(\varepsilon) \\
&= \int_{\mathcal{E}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_{-i}|a_i, \varepsilon_i, x) \phi(a_i, \varepsilon|x) d\varepsilon \\
&= \int_{\mathcal{E}_i} \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_{-i}|a_i, \varepsilon_i, x) \phi(a_i, \varepsilon_i, \varepsilon_{-i}|x) d\varepsilon_{-i} d\varepsilon_i \\
&= \int_{\mathcal{E}_i} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_{-i}|a_i, \varepsilon_i, x) \phi(a_i, \varepsilon_i|x) d\varepsilon_i \\
&= \int_{\mathcal{E}_i} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_i, a_{-i}, \varepsilon_i|x) d\varepsilon_i \\
&= \int_{\mathcal{E}_i} \int_{\mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a_i, a_{-i}, \varepsilon_i, \varepsilon_{-i}|x) d\varepsilon_{-i} d\varepsilon_i \\
&= \int_{\mathcal{E}} \sum_{a_{-i} \in \mathcal{A}_{-i}} v_i(a_i, a_{-i}, \varepsilon_i, x) \phi(a, \varepsilon|x) d\varepsilon \\
&= (q_\sigma(x))^\top (\Pi_i^x + E_i^{\phi,x} + \beta G(x,a)^\top V_i).
\end{aligned}$$

As $q_\sigma = q_\phi$, the value function of the MPBNE σ, V_i, equals the value function of the MPBCE ϕ, V_i^ϕ. Therefore, σ is an MPBNE of $\mathcal{G}(\theta, S)$.

2. Consider $q \in Q_{\theta,S}^{\mathrm{MPBNE}}$. Then, $\exists \sigma \in \mathrm{MPBNE}(\theta, S)$ such that $q = q_\sigma$. Choose $\phi \in \mathrm{MPBCE}(\theta)$ as

$$\phi(a, E|x) = \int_E \int_{\tilde{\mathcal{Z}}} \left(\prod_{i=1}^N \sigma_i(a_i|x, \varepsilon_i, \tilde{z}_i) \right) d\tilde{Z}(\tilde{z}|x,\varepsilon) dF_\varepsilon(\varepsilon),$$

for all $a \in \mathcal{A}$, $E \in \mathcal{B}(\mathcal{E})$, and $x \in \mathcal{X}$. Then, for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$,

$$\begin{aligned}
q_\sigma(a|x) &= \int_{\mathcal{E}} \int_{\tilde{\mathcal{Z}}} \left(\prod_{i=1}^N \sigma_i(a_i|x, \varepsilon_i, \tilde{z}_i) \right) d\tilde{Z}(\tilde{z}|x,\varepsilon) dF_\varepsilon(\varepsilon) \\
&= \int_{\mathcal{E}} \phi(a, \varepsilon|x) d\varepsilon \\
&= q_\phi(a|x).
\end{aligned}$$

Thus,

$$q = q_\sigma = q_\phi \in Q_\theta^{\mathrm{MPBCE}}.$$

$\square$

3.5 Characterization Result

We characterize the sharp identified set under weak assumptions on payoff type information using an identified set under MPBCE. The following proposition shows that the sharp identified set under weak assumptions on payoff type information can be obtained by instead computing the identified set under MPBCE. The idea of the proof of using the robust prediction property of MPBCE in Lemma 1 is analogous to Magnolfi and Roncoroni (2022).

Proposition 1. *Under Assumptions 1 and 2, $\Theta_I^{\mathrm{MPBCE}} = \Theta_I^{\mathrm{MPBNE}}(\mathcal{S})$, which implies that the identified set under MPBCE contains the true parameter value, $\theta_0 \in \Theta_I^{\mathrm{MPBCE}}$.*

Proof. Prove $\Theta_I^{\mathrm{MPBNE}}(\mathcal{S}) \subseteq \Theta_I^{\mathrm{MPBCE}}$: Consider $\theta \in \Theta_I^{\mathrm{MPBNE}}(\mathcal{S})$. Then, $\exists S \in \mathcal{S}$ such that $\{P(\cdot|x)\}_{x\in\mathcal{X}} \in Q_{\theta,S}^{\mathrm{MPBNE}}$. Lemma 1 yields $Q_{\theta,S}^{\mathrm{MPBNE}} \subseteq Q_\theta^{\mathrm{MPBCE}}$, which implies $\theta \in \Theta_I^{\mathrm{MPBCE}}$.

Prove $\Theta_I^{\mathrm{MPBCE}} \subseteq \Theta_I^{\mathrm{MPBNE}}(\mathcal{S})$: Consider $\theta \in \Theta_I^{\mathrm{MPBCE}}$. Then, $\exists\phi$ such that $q_\phi \in Q_\theta^{\mathrm{MPBCE}}$. Lemma 1 yields, $\exists S \in \mathcal{S}$, $q_\phi \in Q_{\theta,S}^{\mathrm{MPBNE}}$, which implies $\theta \in \Theta_I^{\mathrm{MPBNE}}(\mathcal{S})$.

Prove $\theta_0 \in \Theta_I^{\mathrm{MPBCE}}$: By Assumption 2, $\{P(\cdot|x)\}_{x\in\mathcal{X}} \in Q_{\theta_0,S_0}^{\mathrm{MPBNE}}$. Additionally, Lemma 1 yields $Q_{\theta_0,S_0}^{\mathrm{MPBNE}} \subseteq Q_{\theta_0}^{\mathrm{MPBCE}}$, which implies $\theta_0 \in \Theta_I^{\mathrm{MPBCE}}$.

$\square$

4 Monte Carlo Simulation Results

In Monte Carlo simulation exercises, we use the duopoly capacity competition model with the three information structures on payoff types: perfectly private information $\underline{S}$, complete information $\overline{S}$, and privileged information S^P. We set the structural parameters $(b_0, b_1, c_{12}, c_{21}) = (10, 2.2, 2.9, 2.1)$, $\mathcal{E}_0 = \{1, 3, 6, 10\}$, and $(\theta_{\varepsilon 1}, \theta_{\varepsilon 3}, \theta_{\varepsilon 6}, \theta_{\varepsilon 10}) = (0.3, 0.3, 0.3, 0.1)$. For each of the information structures, we simulate the joint CCPs from the duopoly capacity competition model using a joint value function iteration approach as follows.

1. Start from guesses about firm i's value function $V_i^0(a_i, a_{-i})$ and firm $-i$'s value function $V_{-i}^0(a_{-i}, a_i)$ and strategy $\sigma_{-i}^0(a'_{-i}|a_{-i}, a_i, z_{-i})$.

2. For $r = 0, 1, \ldots$, update firm i's value function:

$$\begin{aligned} V_i^{r+1}(a_i, a_{-i}) &= \sum_{\varepsilon \in \mathcal{E}_0^{2\times 2}} \sum_{z_i \in \mathcal{Z}_i} \max_{a_i' \in \{1,2\}} \Bigg\{ \pi_i(a_i', a_i, a_{-i}) + \varepsilon_{a_i', i} \\ &\quad + \beta \sum_{z_{-i} \in \mathcal{Z}_{-i}} \sum_{a_{-i}' \in \{1,2\}} V_i^r(a_i', a_{-i}') \sigma_{-i}^r(a_{-i}' | a_{-i}, a_i, z_{-i}) Z(z_{-i} | a_i, a_{-i}, z_i) \Bigg\} \\ &\quad \times Z(z_i | a_i, a_{-i}, \varepsilon_i, \varepsilon_{-i}) F(\varepsilon). \end{aligned}$$

3. Update firm i's policy function:

$$\begin{aligned} \sigma_i^{r+1}(a_i' | a_i, a_{-i}, z_i) &= \mathbb{1} \Bigg[a_i' = \arg\max_{a \in \{1,2\}} \Bigg\{ \pi_i(a, a_i, a_{-i}) + \varepsilon_{a,i} \\ &\qquad + \beta \sum_{z_{-i} \in \mathcal{Z}_{-i}} \sum_{a_{-i}' \in \{1,2\}} V_i^{r+1}(a, a_{-i}') \sigma_{-i}^r(a_{-i}' | a_{-i}, a_i, z_{-i}) Z(z_{-i} | a_i, a_{-i}, z_i) \Bigg\} \Bigg]. \end{aligned}$$

4. Update firm $-i$'s value function $V_{-i}^{r+1}(a_{-i}, a_i)$ using $V_i^{r+1}(a_i, a_{-i})$ and $\sigma_i^{r+1}(a_i' | a_i, a_{-i}, z_i)$.

5. Update firm $-i$'s policy function $\sigma_{-i}^{r+1}(a_{-i}' | a_{-i}, a_i, z_{-i})$ using $V_{-i}^{r+1}(a_{-i}, a_i)$ and $\sigma_i^{r+1}(a_i' | a_i, a_{-i}, z_i)$.

6. Iterate until $\max\{\|V_i^{r+1} - V_i^r\|, \|V_{-i}^{r+1} - V_{-i}^r\|\} < tol$, where $\|\cdot\|$ is a max norm and tol is set to 1.0×10^{-15}.

7. Calculate joint CCPs from the final policy function $\sigma_i(a_i' | a_i, a_{-i}, z_i)$ and $\sigma_{-i}(a_{-i}' | a_{-i}, a_i, z_{-i})$ obtained from the iteration:

$$\begin{aligned} & P(a_i', a_{-i}' | a_i, a_{-i}) \\ = & \sum_{\varepsilon \in \mathcal{E}_0^{2\times 2}} \sum_{z_{-i} \in \mathcal{Z}_{-i}} \sum_{z_i \in \mathcal{Z}_i} \sigma_i(a_i' | a_i, a_{-i}, z_i) \sigma_{-i}(a_{-i}' | a_{-i}, a_i, z_{-i}) Z(z_i, z_{-i} | a_i, a_{-i}, \varepsilon_i, \varepsilon_{-i}) F(\varepsilon). \end{aligned}$$

Table 1 tabulates the simulated joint CCPs for the three information structures on payoff types. The firms avoid the actions being both high (High/High) when they have signals on the opponent's payoff types. For example, for a state with both firms having a low capacity (Low/Low), the probability of the firms with perfectly private information $\overline{S}$ both selecting high capacity is 13%, while that with complete information $\overline{S}$ is 3%.

We focus on estimating upgrade cost c_{12}, assuming the other parameters are fixed at the true values. We calculate identified sets for the upgrade cost c_{12} 1) under the MPBNE with prespecified information structures and 2) under the MPBCE. The identified sets under the MPBCE correspond to the sharp identified sets under weak assumptions on payoff type information.

4.1 Identified Set Under MPBNE With Prespecified Information Structure

We calculate identified sets for upgrade cost c_{12} under the MPBNE with three prespecified information structures: perfectly private information $S' = \underline{S}$, complete information $S' = \overline{S}$, and privileged information $S' = S^P$. Given a candidate parameter value for θ, we calculate the joint CCP predictions $q_\sigma(a|x;\theta)$ using the joint value function iteration approach above. We then collect parameters θ that yields joint CCP predictions $q_\sigma(a|x;\theta)$ that matches the observed joint CCPs $P(a|x)$ best in terms of L_1-norm:

$$\Theta_I^{\mathrm{MPBNE}}(S') = \arg\min_\theta \sum_{x\in\mathcal{X}} \sum_{a\in\mathcal{A}} |P(a|x) - q_\sigma(a|x;\theta)|.$$

We search for the parameter space of upgrade cost c_{12} from 0 to 6 in 0.01 increments, $\{0.00, 0.01, 0.02, \ldots, 5.98, 5.99, 6.00\}$.

We note that we can only get at most one MPBNE prediction for each parameter value using this approach. We need to obtain all possible MPBNE predictions for each parameter value for a complete analysis. Additionally, we observe that MPBNE does not exist for some parameter values. Thus, care must be taken in interpreting the results. We plan to update the way of solving the identified set under the MPBNE to incorporate the possibility of multiple equilibria. For instance, Bajari et al. (2007)'s approach can handle the possibility of multiple equilibria.

Table 2 shows the identified sets under the MPBNE with three prespecified information structures. Three identified sets are calculated for each of the three data generating processes with different information structures. The three diagonal elements are the identified sets when the information structures are correctly specified. The identified sets in the diagonal contain true parameter value 2.90: perfectly private information $\underline{S}$, $[2.32, 2.98]$; complete information $\overline{S}$, $[2.69, 3.09]$; and privileged information S^P, $[2.78, 3.14]$. In contrast, the first row demonstrates that there is an upward bias when we assume perfectly private information $\underline{S}$, but the true information structure gives firms signals about their opponent's payoff types: e.g., the identified set is $[3.45, 3.50]$ when we assume perfectly private information $\underline{S}$, but the truth is complete information $\overline{S}$.

4.2 Identified Set Under MPBCE

We calculate identified sets for the upgrade cost c_{12} under the MPBCE or the sharp identified sets under weak assumptions on payoff type information. The linear constraints for BCE in Magnolfi and Roncoroni (2022) become quadratic constraints for MPBCE in our case. The quadratic constraints make the problem intractable because the problem of solving a system of equations with many non-convex quadratic constraints is NP-hard in general (Park and Boyd, 2017).

Furthermore, the convexification idea of Magnolfi and Roncoroni (2022) does not work since the conversion of three or more non-convex quadratic constraints to computationally tractable form requires restrictive conditions or is not possible (Yildiran, 2009; Dey et al., 2021).

However, the constraints are linear in MPBCE ϕ given the expected values of payoff types conditional on states, $E[\varepsilon_{a_i,i}|x = X_k] = \xi_k$ for all $k = 1, \ldots, K$. Thus, we introduce $\xi = (\xi_1, \ldots, \xi_K)^\top$ as additional nuisance parameters and search ξ so that they justify the estimated MPBCE ϕ with small errors $s = (s_1, \ldots, s_K)^\top$:

$$\forall k, \left| \sum_{\varepsilon \in \mathcal{E}} \sum_{a \in \mathcal{A}} \varepsilon_{a_i,i} \phi(a_i, a_{-i}, \varepsilon | X_k) - \xi_k \right| \leq s_k.$$

We discretize the support of payoff types $\mathcal{E}$ to a finite dimension if the support is infinite. Let $r = (r_1, \ldots, r_K)^\top$ and $\lambda > 0$ be a coefficient that controls the penalty on the approximation errors of the expected values of payoff types conditional on states relative to that on the joint CCPs. Given θ, we solve

$$\mathcal{Q}(\theta) = \min_{\xi} \left(\min_{\phi, q, r, s} \sum_{k=1}^{K} r_k + \lambda \sum_{k=1}^{K} s_k \right)$$

subject to:

$$\begin{cases} \forall a, x, q(a|x) = \sum_{\varepsilon \in \mathcal{E}} \phi(a, \varepsilon | x), \\ \forall \varepsilon, x, \sum_{a \in \mathcal{A}} \phi(a, \varepsilon | x) = f_\varepsilon(\varepsilon | x; \theta), \\ \forall x, \sum_{\varepsilon \in \mathcal{E}} \sum_{a \in \mathcal{A}} \phi(a, \varepsilon | x) = 1, \\ \forall i, t, a_{it}, a^o_{it}, \varepsilon_{it}, x_t, \\ \sum_{\varepsilon_{-i} \in \mathcal{E}_{-i}} \sum_{a_{-i} \in \mathcal{A}_{-i}} \left(v^\phi_i(a_{it}, a_{-it}, \varepsilon_{it}, x_t; \theta) - v^\phi_i(a^o_{it}, a_{-it}, \varepsilon_{it}, x_t; \theta) \right) \phi(a_{it}, a_{-it}, \varepsilon_t | x_t) \geq 0, \\ \forall k, r_k \geq 0, \|P(\cdot | X_k) - q(\cdot | X_k)\|_1 \leq r_k, \text{ and} \\ \forall k, s_k \geq 0, |\sum_{\varepsilon \in \mathcal{E}} \sum_{a \in \mathcal{A}} \varepsilon_{a_i,i} \phi(a_i, a_{-i}, \varepsilon | X_k) - \xi_k| \leq s_k, \end{cases}$$

where

$$\begin{aligned} v^\phi_i(a_{it}, a_{-it}, \varepsilon_{it}, x_t; \theta) &= \pi_i(a_{it}, a_{-it}, x_t; \theta) + \varepsilon_{a_{it},i,t} + \beta G(x, a)^\top V^\phi_i, \\ V^\phi_i &= (I - \beta \Gamma)^{-1} M^\phi_i, \text{ and} \\ M^\phi_i &= ((P^{X_1})^\top \Pi^{X_1}_i + \xi_1, \ldots, (P^{X_K})^\top \Pi^{X_K}_i + \xi_K)^\top. \end{aligned}$$

We can solve this constrained minimization problem using linear programming since the objective function and the constraints are all linear in the arguments. The objective function becomes small when the joint CCP predictions are close to the observed joint CCPs and approximation errors of the expected values of payoff types conditional on states are small. We implement the minimization for a grid of θ and then collect

$$\hat{\Theta}_I^{\text{MPBCE}} = \{\theta : \mathcal{Q}(\theta) \leq \tau\}$$

as an estimator of the identified set for some small $\tau \geq 0$. We search for c_{12} from 2 to 4 in 0.01 increments, $\{2.00, 2.01, 2.02, \ldots, 3.98, 3.99, 4.00\}$. We set $\lambda = 1.0 \times 10^5$ and explore the identified sets for the following three thresholds: $\tau = 0.01, 0.005, 0.001$. The minimization problem for ξ is tricky because of the existence of many local minima. We start the minimization algorithm for ξ from two initial values for each grid of θ: 1) the true values of ξ, for all $k = 1, \ldots, K$, $\xi_k = E[\varepsilon_{a_i,i} | x = X_k]$ for $\theta = \theta_0$; 2) the values of ξ that are obtained in the previous grid value.

Table 3 shows the identified sets under the MPBCE. The identified sets under the MPBCE contain the true parameter value $c_{12} = 2.90$: range from 2.1 to 3.1 for the case where the underlying information structure is perfectly private information $\underline{S}$; from 2.6 to 3.5 for complete information $\overline{S}$; from 2.6 to 3.6 for privileged information S^P. The identified sets under MPBCE are wider than those under the MPBNE with correct information structure specifications. The wider identified sets are expected because the identified sets under MPBCE capture the parameter values that can yield observed joint CCPs with information structures different from the correctly specified ones.

We consider some parameter values in the sharp identified set under weak assumptions on payoff type information are not captured in the identified sets under MPBCE calculated in Table 3. For example, when the threshold is $\tau = 0.001$, $c_{12} = 2.51$ is the only value excluded from the identified set between 2.20 and 3.03 for the case where the underlying information structure is perfectly private information. We suspect that this is because the minimization with respect to ξ for $c_{12} = 2.51$ is stuck to a local minimum. We may want to try more initial values for ξ to alleviate the concern caused by many local minima.

5 Discussion

We plan to work on inference procedures of the proposed estimation method. We conjecture that Chernozhukov et al. (2007)'s moment inequality approach works in our setting similarly to Magnolfi and Roncoroni (2022) and Syrgkanis et al. (2021). We also plan to extend the characterization result of the sharp identified set under weak assumptions on information using BCE to multi-stage games. We build on the recent development of the robust prediction property of BCE in multi-stage games (Makris and Renou, 2021). Additionally, the number of inequalities we need to handle in computing the identified set under MPBCE increases with the dimension of MPBCE, which may result in a longer calculation time. We will explore feasible empirical applications taking this feature into account.

There is an important remaining question. What are counterfactuals of interest, leaving information structure unspecified in structural parameter estimation? Magnolfi and Roncoroni (2022), Syrgkanis et al. (2021), and Bergemann et al. (2022) discuss this question in static games. We are planning to incorporate their ideas into dynamic games.

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Table 1. Joint CCPs: Duopoly Capacity Competition Model Example

(a) Perfectly Private Information

	Action $i = 1/i = 2$			
State $i = 1/2$	Low/Low	Low/High	High/Low	High/High
Low/Low	0.409	0.230	0.230	0.129
Low/High	0.131	0.598	0.048	0.221
High/Low	0.131	0.048	0.598	0.221
High/High	0.072	0.197	0.197	0.532

(b) Complete Information

	Action $i = 1/i = 2$			
State $i = 1/2$	Low/Low	Low/High	High/Low	High/High
Low/Low	0.409	0.279	0.279	0.032
Low/High	0.115	0.656	0.113	0.115
High/Low	0.115	0.113	0.656	0.115
High/High	0.032	0.279	0.279	0.409

(c) Privileged Information

	Action $i = 1/i = 2$			
State $i = 1/2$	Low/Low	Low/High	High/Low	High/High
Low/Low	0.409	0.295	0.230	0.064
Low/High	0.115	0.672	0.064	0.147
High/Low	0.131	0.097	0.598	0.172
High/High	0.048	0.262	0.221	0.467

Table 2. Identified Sets Under MPBNE With Prespecified Information Structures

Assumed	True Information Structure		
Info. Str.	$S_0 = \underline{S}$	$S_0 = \overline{S}$	$S_0 = S^P$
$S' = \underline{S}$	$[2.32, 2.98]$	$[3.45, 3.50]$	$[3.20, 3.21], [3.26, 3.28]$
$S' = \overline{S}$	$[2.11, 2.57]$	$[2.69, 3.09]$	$[2.11, 2.57]$
$S' = S^P$	$[2.67, 3.14]$	$[2.78, 3.14]$	$[2.78, 3.14]$

Notes. The parameter of interest is upgrading cost c_{12}. The true parameter value is $c_{12} = 2.90$.

Table 3. Identified Sets Under MPBCE

Threshold	True Information Structure $S_0 = \underline{S}$	$S_0 = \overline{S}$	$S_0 = S^P$
$\tau = 0.01$	$[2.14, 3.03], \{3.17\}$	$[2.66, 3.48], [3.51, 3.56], \{3.58\}$	$[2.63, 3.56], [3.58, 3.65], \{3.67\}$
$\tau = 0.005$	$[2.15, 3.03]$	$[2.67, 3.48], [3.51, 3.56], \{3.58\}$	$[2.64, 3.56], [3.58, 3.62]$
$\tau = 0.001$	$[2.20, 2.50], [2.52, 3.03]$	$[2.67, 3.48], [3.51, 3.53], \{3.55\}$	$[2.65, 2.66], [2.68, 3.56], [3.58, 3.59]$

Notes. The parameter of interest is upgrading cost c_{12}. The true parameter value is $c_{12} = 2.90$.