Appendix to "Informationally Efficient Climate Policy"

This appendix contains numerical detals, proofs, and additional formal analysis.

A Numerical Details

There is no broad agreement on a distribution to use for climate change impacts. I calibrate the distribution for aggregate impacts $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ to Pindyck (2019). In 2016, he asked around 1,000 climate scientists and economists to report their subjective percentiles for the percentage reduction in GDP that climate change will cause in fifty years, assuming that no additional emission controls are enacted before then. He fit four distributions to the results and found that a lognormal distribution produced the highest corrected R^2. The location parameter for his fitted lognormal distribution is -2.446 and the scale parameter is 1.476. His distribution describes a parameter (which he labels ϕ) that is equal to $T_{2065} \sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$. Using this estimate and treating T_{2065} as known, $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ is lognormally distributed with location parameter $-2.446 - \ln(T_{2065})$ and scale parameter 1.476.

The value for T_{2065} should be the temperature that experts would have expected to hold based on their information in 2016. The IPCC's AR5 summarizes knowledge around that time. Hausfather and Peters (2020) suggest that a no-additional-emission-controls scenario is consistent with SSP4–6.0 from the IPCC's AR6. So I consider RCP 6.0 in the IPCC's AR5. There, the mean of the CMIP5 models is for 2.2 °C of warming in 2046–2065 relative to a 1986–2005 reference period, which in turn is on average 0.61 °C warmer than over 1850–1900 (Collins et al., 2013, Table 12.2). I therefore fix $T_{2065} = 2.2 + 0.61 = 2.81$ °C. This implies a location parameter for $\sum_{i=1}^{N} \kappa_i[\bar{\zeta}_i + \zeta_i]$ of -3.48.

In order to determine either the emission tax or damage charges, it remains to calibrate r, α, and C_0. I take r to be the policymaker's consumption discount rate. According to World Bank date, average growth in global output per capita was 1.85% per year over 2000-2019. Choosing an annual utility discount rate of 1.5% and a coefficient of relative risk aversion of 1 to match the log utility specification, the Ramsey rule implies that $r = 0.015 + 1 * 0.0185 = 0.0335$, or 3.4% per year.

The parameter α, or the "transient climate response to cumulative carbon emissions", is 1.6/1000 °C/Gt C, from the central value in Matthews et al. (2009). This value is consistent with Collins et al. (2013) and Dietz and Venmans (2019) and is the same as used in Rudik (2020).

I calibrate initial consumption C_0 to World Bank data. In 2021, global output was $86.7 trillion in year 2015 US dollars. Converting to year 2021 US dollars using World Bank deflators, I set $C_0 = 97,975$ billion dollars.

Now consider the calculations underlying Figure 3. Substituting (5) into (6) yields the probability that the damage charge Δ_t is constrained by the deposit D from reaching its first-best value of $C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]$:

$$Pr(\Delta_t < C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]) = 1 - F(rD/[C_0\alpha]),$$

where $F(\cdot)$ is the cumulative density function for $\tilde{\zeta}_t+\tilde{\lambda}_t$. In the absence of either measurement error in the aggregate signal or stochasticity in climate impacts (i.e., if $\tilde{\omega}^2 = 0$ and $\sigma^2 = 0$), $F(\cdot)$ is completely determined by the lognormal distribution of $\sum_{i=1}^N \kappa_i[\bar{\zeta}_i+\zeta_i]$ defined above.

A.1 Additional Parameters for the Evolution of Beliefs

To analyze the evolution of beliefs in Figure 1, I consider a normal distribution that has the same mean and, when $\Gamma = 0$, variance as the lognormal distribution for $\sum_{i=1}^N \kappa_i[\bar{\zeta}_i + \zeta_i]$ described above:

$$\sum_{i=1}^N \kappa_i\bar{\zeta}_i = \exp(-3.48 + 1.48^2/2) = 0.0916,$$

and

$$\tau_0^2 \sum_{i=1}^N \kappa_i^2 = [\exp(1.48^2) - 1] \exp(-2 * 3.48 + 1.48^2) = 0.0658.$$

The κ_i are drawn from a symmetric Dirichlet distribution with concentration parameter 1. The regulator and all agents know the values of the κ_i. I fix $N = 10$ in the base specification

Now consider the sources of noise. First, I assume that $\sigma^2 \sum_{i=1}^N \kappa_i^2$ is the same as the prior variance $\tau_0^2 \sum_{i=1}^N \kappa_i^2$, so that

$$\sigma^2 = \frac{0.0658}{\sum_{i=1}^N \kappa_i^2}.$$

Second, in the base specification, I assume that aggregate measurement error has twice the variance as aggregate stochasticity:

$$\tilde{\omega}^2 = 2 * \sigma^2 \sum_{i=1}^N \kappa_i^2.$$

Third, I assume that sectoral measurement errors have half the variance as sectoral stochasticity:

$$\omega^2 = 0.5 * \sigma^2.$$

For the purposes of simulating beliefs, I assume that the true value of $\sum_{i=1}^N \kappa_i\zeta_i$ is the same as $\sum_{i=1}^N \kappa_i\bar{\zeta}_i$, so that the full-information optimal tax is twice the prior tax. I draw the ζ_i from a multivariate normal distribution with mean vector equal to $\sum_{i=1}^N \kappa_i\bar{\zeta}_i$ and I then adjust the draws ex post by adding a constant to ensure that $\sum_{i=1}^N \kappa_i\zeta_i = \sum_{i=1}^N \kappa_i\bar{\zeta}_i$. I draw 1 million trajectories for the random variables conditional on these ζ_i. These trajectories yield 1 million trajectories for $\hat{\mu}_t$ and $\tilde{\mu}_t$ from equations (3) and (4), respectively. Averaging over these trajectories yields the trajectories $E_0[\hat{\mu}_t|\boldsymbol{\zeta}]$ and $E_0[\tilde{\mu}_t|\boldsymbol{\zeta}]$ despicted in the figure.

B Taxing the Stock of Carbon

Assume informational efficiency, so that all agents observe all $\zeta_{it} + \lambda_{it}$. Denote firm i's cumulative emissions from time 0 to time $t-1$ as M_{it}. In period t, the regulator taxes M_{it} at rate $\tilde{\nu}_t$.

Final-good firms' problem is as in Appendix E. I here assume that all actors observe all signals in the economy and that firms discount the future at rate r.

The representative intermediate-good producer in sector i solves the following Bellman equation:

$$J(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = \max_{L_{it}, e_{it}, R_{it}} \Bigg\{ \hat{E}_t \left[p_{it} \exp[-\zeta_{it} T_t] \right] L_{it} Y^{it}(e_{it}) - w_{it} L_{it} - \tilde{\nu}_t M_{it} - p^R_t R_{it} + \frac{1}{1+r} \hat{E}_t[J(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] \Bigg\},$$

with each control weakly positive. At an interior solution, the first-order condition for emissions is

$$\hat{E}_t \left[p_{it} \exp[-\zeta_{it} T_t] \right] L_{it} Y^{it\prime}(e_{it}) = -\frac{1}{1+r} \hat{E}_t[J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})],$$

and the first order condition for carbon removal is

$$p^R_t = -\frac{1}{1+r} \hat{E}_t[J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})].$$

Substitute for p_{it} and then C_t in the first-order conditions as in Appendix E:

$$-\frac{1}{1+r} \hat{E}_t[J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0, \tag{A-1}$$

$$-\frac{1}{1+r} \hat{E}_t[J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})] = \frac{c_t'(R_t)}{1 - c_t(R_t)} C_0. \tag{A-2}$$

The envelope theorem yields:

$$J_1(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = -\tilde{\nu}_t + \frac{1}{1+r} \hat{E}_t[J_1(M_{it} + e_{it} - R_{it}, \hat{\mu}_{t+1}, \hat{\Omega}_{t+1})].$$

Recursively substituting, we find:

$$J_1(M_{it}, \hat{\mu}_t, \hat{\Omega}_t) = -\sum_{s=0}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\nu}_{t+s}].$$

Advancing by one timestep, substituting into (A-1) and (A-2), and applying the law of iterated expectations, an interior solution must satisfy:

$$\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\nu}_{t+s}] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0,$$
$$\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\nu}_{t+s}] = \frac{c_t'(R_t)}{1-c_t(R_t)} C_0.$$

Now set $\tilde{\nu}_t = C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t]$. The foregoing conditions become:

$$\alpha \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\zeta}_{t+s} + \tilde{\lambda}_{t+s}] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})},$$
$$\alpha \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \hat{E}_t[\tilde{\zeta}_{t+s} + \tilde{\lambda}_{t+s}] = \frac{c_t'(R_t)}{1-c_t(R_t)},$$

which in turn are equivalent to

$$\alpha \frac{1}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \hat{\mu}_t\right] = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})},$$
$$\alpha \frac{1}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \hat{\mu}_t\right] = \frac{c_t'(R_t)}{1-c_t(R_t)}.$$

Once we adjust for the possibility of corner solutions, these conditions are the same as the conditions for welfare maximization in (1) and (2). Therefore this choice of $\tilde{\nu}_t$ must be the optimal choice for a regulator who can commit to a choice rule at time 0.

C Proof of Proposition 1

Consider the zero-mean random vector

$$s_t \triangleq \begin{bmatrix} \sum_{k=1}^{N} \kappa_k \zeta_k \\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j} + \lambda_{1j}] - \bar{\zeta}_1 \\ \vdots \\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj} + \lambda_{Nj}] - \bar{\zeta}_N \\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \end{bmatrix}.$$

Observe that

$$\hat{\mu}_t \triangleq E\left[\sum_{k=1}^{N} \kappa_k \zeta_k \,\middle|\, \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j} + \lambda_{1j}] - \bar{\zeta}_1, ..., \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj} + \lambda_{Nj}] - \bar{\zeta}_N, \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j + \tilde{\lambda}_j] - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k\right].$$

Let Ψ_t indicate the $(N+1)\times(N+1)$ covariance matrix of the final $(N+1)\times 1$ elements of s_t and Σ_t indicate the $1\times(N+1)$ vector of covariances between $\sum_{k=1}^N \kappa_k\zeta_k$ and the other elements of s_t, so that

$$\Sigma_t \triangleq \left[\kappa_1\tau_0^2+(1-\kappa_1)\Gamma\tau_0^2,\quad \ldots,\quad \kappa_N\tau_0^2+(1-\kappa_N)\Gamma\tau_0^2,\quad \tau_0^2\sum_{k=1}^N\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^N\sum_{j=k+1}^N\kappa_k\kappa_j\right].$$

From the projection theorem,

$$\hat{\mu}_t = \Sigma_t\Psi_t^{-1}\begin{bmatrix}\frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{1j}+\lambda_{1j}]-\bar{\zeta}_1\\ \vdots\\ \frac{1}{t}\sum_{j=0}^{t-1}[\zeta_{Nj}+\lambda_{Nj}]-\bar{\zeta}_N\\ \frac{1}{t}\sum_{j=0}^{t-1}[\tilde{\zeta}_j+\tilde{\lambda}_j]-\sum_{k=1}^N\kappa_k\bar{\zeta}_k\end{bmatrix}. \tag{A-3}$$

Consider Ψ_t^{-1}. Label the $N\times N$ upper-left block of Ψ_t as Ψ_A, the $N\times 1$ upper right block as Ψ_B, the $1\times N$ lower left block as Ψ_C, and the 1×1 lower right block as Ψ_D. From familiar results for block matrix inversion,

$$\Psi_t^{-1} = \begin{bmatrix}\Psi_A^{-1}+\Psi_A^{-1}\Psi_B(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1} & -\Psi_A^{-1}\Psi_B(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\\ -(\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1} & (\Psi_D-\Psi_C\Psi_A^{-1}\Psi_B)^{-1}\end{bmatrix}. \tag{A-4}$$

Element (i,i) of Ψ_A is $\tau_0^2+\sigma^2/t+\omega^2/t$, element (i,j) of Ψ_A is $\Gamma\tau_0^2$ for $i\neq j$, the ith element of Ψ_B and Ψ_C is $\kappa_i(\tau_0^2+\sigma^2/t)+(1-\kappa_i)\Gamma\tau_0^2$, and Ψ_D equals $(\tau_0^2+\sigma^2/t)\sum_{k=1}^N\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^N\sum_{j=k+1}^N\kappa_k\kappa_j+\tilde{\omega}^2/t$.

Conjecture that each diagonal of Ψ_A^{-1} is

$$\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}$$

and each off-diagonal of Ψ_A^{-1} is

$$\frac{-\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}.$$

Under the conjecture for Ψ_A^{-1}, each diagonal element in $\Psi_A\Psi_A^{-1}$ is

$$\begin{aligned}&[\tau_0^2+\sigma^2/t+\omega^2/t]\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\ &-\sum_{k=1}^{N-1}\Gamma\tau_0^2\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2-(N-1)[\Gamma\tau_0^2]^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2}\\ =&1,\end{aligned}$$

and each off-diagonal element in $\Psi_A \Psi_A^{-1}$ is

$$\begin{aligned}
&\Gamma\tau_0^2 \frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\
&-[\tau_0^2+\sigma^2/t+\omega^2/t]\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\
&-\sum_{k=1}^{N-2}\Gamma\tau_0^2\frac{\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2-(N-1)[\Gamma\tau_0^2]^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2}\\
=&0.
\end{aligned}$$

We have shown that $\Psi_A \Psi_A^{-1}$ is the identity matrix under the conjecture for Ψ_A^{-1} and thus have confirmed that the conjecture is correct. Observe that the denominator of each element simplifies to

$$\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right].$$

With Ψ_A^{-1} in hand, we now calculate Ψ_t^{-1} from (A-4). Element i of $\Psi_C \Psi_A^{-1}$ and also of $\Psi_A^{-1}\Psi_B$ is

$$\begin{aligned}
&\frac{\tau_0^2+\sigma^2/t+\omega^2/t+(N-2)\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}[\kappa_i(\tau_0^2+\sigma^2/t)+(1-\kappa_i)\Gamma\tau_0^2]\\
&+\sum_{k=1,\neq i}^{N}\frac{-\Gamma\tau_0^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\
&\qquad[\kappa_k(\tau_0^2+\sigma^2/t)+(1-\kappa_k)\Gamma\tau_0^2]\\
=&\frac{(1-\Gamma)\tau_0^2+\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_i+\frac{\Gamma\tau_0^2\omega^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]}.
\end{aligned}$$

We then have:

$$\begin{aligned}
\Psi_C\Psi_A^{-1}\Psi_B=&\sum_{i=1}^{N}\frac{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_i+\Gamma\tau_0^2\omega^2/t}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\
&\qquad\left[\kappa_i((1-\Gamma)\tau_0^2+\sigma^2/t)+\Gamma\tau_0^2\right]\\
=&\frac{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]^2\sum_{i=1}^N\kappa_i^2}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\\
&+\frac{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]+\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\omega^2/t}{(\tau_0^2+\sigma^2/t+\omega^2/t)^2+(\tau_0^2+\sigma^2/t+\omega^2/t)(N-2)\Gamma\tau_0^2-(N-1)[\Gamma\tau_0^2]^2}\Gamma\tau_0^2.
\end{aligned}$$

And

$$
\begin{aligned}
&(\Psi_D - \Psi_C \Psi_A^{-1} \Psi_B)^{-1} \\
=&\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big] \\
&\Bigg\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big] \\
&\qquad \Big[(\tau_0^2 + \sigma^2/t)\sum_{k=1}^{N}\kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j + \tilde{\omega}^2/t\Big] \\
&\quad - \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]^2 \sum_{i=1}^{N}\kappa_i^2 \\
&\quad - \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Gamma\tau_0^2 - \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\Gamma\tau_0^2\omega^2/t\Bigg\}^{-1} \\
=&\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big] \\
&\Bigg\{\omega^2/t\Bigg(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\Bigg) \\
&\quad + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\Bigg\}^{-1},
\end{aligned}
$$

where the final line uses $\sum_{k=1}^{N}\kappa_k = 1$. Element i of $\Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}$ and also of $(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is

$$
\begin{aligned}
&\Bigg\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_i + \Gamma\tau_0^2\omega^2/t\Bigg\} \\
&\Bigg\{\omega^2/t\Bigg(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\Bigg) \\
&\quad + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\Bigg\}^{-1}.
\end{aligned}
$$

Element (i,i) of $\Psi_A^{-1} + \Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is

$$\frac{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + (N-1)\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$+ \frac{\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\}$$
$$\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left. + \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1},$$

and element (m,n) of $\Psi_A^{-1} + \Psi_A^{-1}\Psi_B(\Psi_D - \Psi_C\Psi_A^{-1}\Psi_B)^{-1}\Psi_C\Psi_A^{-1}$ is, for $m \neq n$,

$$\frac{-\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$+ \frac{\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_m + \Gamma\tau_0^2\omega^2/t\right\}}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]}$$
$$\left\{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\kappa_n + \Gamma\tau_0^2\omega^2/t\right\}$$
$$\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left. + \left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}.$$

The foregoing pieces define Ψ_t^{-1} from (A-4). $\Sigma_t\Psi_t^{-1}$ is $1 \times N+1$ and, from (A-3), determines how $\hat{\mu}_t$ uses the signals in s_t. Element $k \in \{1, ..., N\}$ of $\Sigma_t \Psi_t^{-1}$ is

$$
\begin{aligned}
&[\kappa_k \tau_0^2 + (1-\kappa_k)\Gamma\tau_0^2] \frac{(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + (N-1)\Gamma\tau_0^2}{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]} \\
&+ \frac{\left\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_k + \Gamma\tau_0^2\omega^2/t\right\}}{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]} \\
&\quad \sum_{i=1}^{N}[\kappa_i \tau_0^2 + (1-\kappa_i)\Gamma\tau_0^2]\left\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_i + \Gamma\tau_0^2\omega^2/t\right\} \\
&\quad \left\{\omega^2/t\left(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad \left. + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\right\}^{-1} \\
&- [(1-\kappa_k)(1-\Gamma)\tau_0^2 + (N-1)\Gamma\tau_0^2] \frac{\Gamma\tau_0^2}{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]} \\
&- \left[\tau_0^2 \sum_{k=1}^{N}\kappa_k^2 + 2\Gamma\tau_0^2 \sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j\right] \\
&\quad \left\{\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\kappa_k + \Gamma\tau_0^2\omega^2/t\right\} \\
&\quad \left\{\omega^2/t\left(\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2 + \Gamma\tau_0^2\omega^2/t\right)\right. \\
&\qquad \left. + \Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{(1-\Gamma)\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_k\\
&+\frac{\sigma^2/t+\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}\\
&-[\omega^2/t]\left(\frac{(1-\Gamma)\tau_0^2+\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\kappa_k+\frac{\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}\frac{\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}\right)\\
&\quad\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2\sum_{i=1}^{N}\kappa_i^2+\left[\sigma^2/t+\omega^2/t\right]\Gamma\tau_0^2\right\}\\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}. \quad \text{(A-5)}
\end{aligned}
$$

Element $N+1$ of $\Sigma_t\Psi_t^{-1}$ is

$$
\begin{aligned}
&-\sum_{k=1}^{N}[\kappa_k\tau_0^2+(1-\kappa_k)\Gamma\tau_0^2]\left\{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\kappa_k+\Gamma\tau_0^2\omega^2/t\right\}\\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}\\
&+\left[\tau_0^2\sum_{k=1}^{N}\kappa_k^2+2\Gamma\tau_0^2\sum_{k=1}^{N}\sum_{j=k+1}^{N}\kappa_k\kappa_j\right]\\
&\quad\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\\
&\quad\left\{\omega^2/t\left(\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.\\
&\qquad\left.+\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\tilde{\omega}^2/t\right\}^{-1}
\end{aligned}
$$

$$=[\omega^2/t]\left\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\Big[\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2\right\}$$
$$\left\{\omega^2/t\left(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left.+\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\tilde{\omega}^2/t\right\}^{-1}. \quad \text{(A-6)}$$

With these elements, we can determine $\hat{\mu}_t$ from (A-3).

Define

$$\hat{Z}_t \triangleq[\omega^2/t]\left\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\Big[\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2\right\}$$
$$\left\{[\omega^2/t]\left(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t\Big]\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2\omega^2/t\right)\right.$$
$$\left.+[\tilde{\omega}^2/t]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\right\}^{-1}. \quad \text{(A-7)}$$

Clearly $\hat{Z}\geq 0$. Observe that $\sum_{k=1}^{N}\kappa_k^2$ is minimized when each $\kappa_k=1/N$ and thus

$$\hat{Z}_t <[\omega^2/t]\left\{\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\Big[\sigma^2/t+\omega^2/t\Big]\Gamma\tau_0^2\right\}$$
$$\left\{[\omega^2/t]\left(\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big](1-\Gamma)\tau_0^2\sum_{k=1}^{N}\kappa_k^2+\Gamma\tau_0^2[\sigma^2/t+\omega^2/t]\right)\right.$$
$$\left.+[\tilde{\omega}^2/t]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t\Big]\Big[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\Big]\right\}^{-1}$$
$$<1.$$

Therefore $\hat{Z}_t\in[0,1)$. By inspection, $\hat{Z}_t\to 0$ as $\tilde{\omega}/\omega\to\infty$. The expression in the proposition follows from (A-3), (A-5), (A-6), and the definition (A-7).

D Proof of Corollary 1

From Proposition 1, $\hat{Z}_t \to 0$ as $\omega^2 \to 0$ if $\tilde{\omega}^2 > 0$. In that case, the expression in part (i) follows from (3), $\lambda_{kj} = 0$, and the definition of $\tilde{\zeta}_j$. Now consider the case in which $\tilde{\omega}^2 = 0$ and $\omega^2 \to 0$. From (A-7),

$$\hat{Z}_t = \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + \left[\sigma^2/t + \omega^2/t\right]\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2 + \Gamma\tau_0^2\omega^2/t}$$

if $\tilde{\omega}^2 = 0$, which goes to

$$\frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 \sum_{k=1}^N \kappa_k^2 + [\sigma^2/t]\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2}$$

as $\omega^2 \to 0$. Substituting into (3), using $\lambda_{kj} = \tilde{\lambda}_j = 0$, and applying the definition of $\tilde{\zeta}_j$ yields the expression in part (i). We have established the first part of the corollary.

From (A-7),

$$\lim_{\Gamma,\tilde{\omega}^2 \to 0} \hat{Z}_t = \frac{\left[\tau_0^2 + \sigma^2/t + \omega^2/t\right]\tau_0^2 \sum_{k=1}^N \kappa_k^2}{\left[\tau_0^2 + \sigma^2/t + \omega^2/t\right]\left[\tau_0^2 + \sigma^2/t\right]\sum_{k=1}^N \kappa_k^2} = \frac{\tau_0^2}{\tau_0^2 + \sigma^2/t}.$$

Substituting into (3) and setting $\tilde{\lambda}_{kj} = 0$ yields the expression in part (ii) of the corollary.

From (A-7),

$$\begin{aligned}\lim_{\tilde{\omega}^2 \to 0, \kappa_i \to 1/N\,\forall i} \hat{Z}_t =& \frac{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right](1-\Gamma)\tau_0^2 + \left[\sigma^2/t + \omega^2/t\right]N\Gamma\tau_0^2}{\left[(1-\Gamma)\tau_0^2 + \sigma^2/t + \omega^2/t + N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2 + \sigma^2/t\right] + N\Gamma\tau_0^2\omega^2/t} \\ =& \frac{(1-\Gamma)\tau_0^2 + N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2 + \sigma^2/t + N\Gamma\tau_0^2}.\end{aligned}$$

Using this,

$$\lim_{\tilde{\omega}^2\to 0,\kappa_i\to 1/N\,\forall i} \frac{(1-\hat{Z}_t)(1-\Gamma)\tau_0^2 - \hat{Z}_t\sigma^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t}$$

$$= - \frac{N\Gamma\tau_0^2\sigma^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]+N\Gamma\tau_0^2\omega^2/t},$$

$$\lim_{\tilde{\omega}^2\to 0,\kappa_i\to 1/N\,\forall i} \frac{\sigma^2/t+(1-\hat{Z}_t)\omega^2/t}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t} \frac{N\Gamma\tau_0^2}{(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2}$$

$$= \frac{N\Gamma\tau_0^2\sigma^2/t}{\left[(1-\Gamma)\tau_0^2+\sigma^2/t+\omega^2/t+N\Gamma\tau_0^2\right]\left[(1-\Gamma)\tau_0^2+\sigma^2/t\right]+N\Gamma\tau_0^2\omega^2/t}.$$

Substituting into (3) and setting $\tilde{\lambda}_{kj}=0$ yields the expression in part (iii) of the corollary.

E Proof of Proposition 2

I first solve for market equilibrium conditional on the choice of tax and then consider how the regulator would design the tax to maximize welfare.

Let p_t be the price of consumption, p_{it} be the time t price of intermediate i, and p_t^R be the price paid for removal. Final-good firms solve:

$$\max_{R_t,\{Y_{it}\}_{i=1}^N} \left\{ p_t\left(1-c_t(R_t)\right)Y_t - \sum_{i=1}^N p_{it}Y_{it} + p_t^R R_t \right\}.$$

The first-order conditions are

$$\begin{aligned} p_{it} =& p_t\kappa_i \frac{C_t}{Y_{it}}, \\ p_t^R =& p_t \frac{c_t'(R_t)}{1-c_t(R_t)}\, C_t. \end{aligned} \tag{A-8}$$

The latter condition becomes a $\leq$ when $R_t=0$.

The representative intermediate-good producer in sector i solves

$$\max_{L_{it},e_{it},R_{it}\geq 0} \left\{ E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] L_{it}Y^{it}(e_{it}) - w_{it}L_{it} - \nu_t \max\{e_{it}-R_{it},0\} - p_t^R R_{it} \right\}.$$

The $\max\{e_{it}-R_{it},0\}$ reflects that, unless the regulator could disciriminate subsidies by sectors, the regulator cannot pay firms in sector i for negative emissions or else it would violate its revenue constraint if emissions in other sectors net out to zero or less. A maximum clearly has $e_{it}-R_{it}\geq 0$.

The first-order condition for L_{it} is

$$w_{it} = E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] Y^{it}(e_{it}).$$

If $e_{it} - R_{it} > 0$, then the first-order condition for e_{it} is

$$\nu_t = E_{it}\left[p_{it}\exp[-\zeta_{it}T_t]\right] L_{it} Y^{it\prime}(e_{it}),$$

and the first-order condition for a solution with $R_{it} > 0$ is

$$\nu_t = p_t^R.$$

Substitute for p_{it} and p_t^R in the first-order conditions:

$$w_{it} = p_t \kappa_i \frac{C_t}{L_{it}}, \tag{A-9}$$

$$\nu_t = p_t \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_t, \tag{A-10}$$

$$\nu_t = p_t \frac{c_t'(R_t)}{1 - c_t(R_t)} C_t. \tag{A-11}$$

At a corner solution with $e_{it} = 0$, the second condition's equality would become a $\geq$, and at a corner solution with $R_t = 0$, the third condition's equality would become a $\leq$.

Household maximization implies $p_t = u'(C_t)$. Therefore, using $p_0 = 1$,

$$p_t = \frac{u'(C_t)}{u'(C_0)} = \frac{C_0}{C_t}.$$

Households' budget constraint $\sum_{i=1}^N w_{it} L_{it} = p_t C_t$ and the first-order condition imply $1 = \sum_{i=1}^N \kappa_i$, which does hold. Substitute $p_t C_t = C_0$ in (A-9) through (A-11):

$$w_{it} = \kappa_i \frac{1}{L_{it}} C_0, \tag{A-12}$$

$$\nu_t = \frac{\kappa_i Y^{it\prime}(e_{it})}{Y^{it}(e_{it})} C_0, \tag{A-13}$$

$$\nu_t = \frac{c_t'(R_t)}{1 - c_t(R_t)} C_0. \tag{A-14}$$

The wage must be equal in sectors with nonzero employment, so $w_{it} = w_t$ for some $w_t > 0$. Equation (A-12) becomes:

$$L_{it} = \kappa_i \frac{1}{w_t} C_0.$$

From the budget constraint, $w_t = C_0$ and thus $L_{it} = \kappa_i$. Therefore equilibrium L_{it} is independent of e_{it}, ν_t, and T_t.

Conjecture that ν_t/C_0 is independent of T_t (we will confirm this conjecture when studying the regulator's problem). Then from (A-13) and (A-14), the time t market equilibrium is also independent of T_t. It is also independent of the random shocks ϵ_{it} and the unknown damage parameters ζ_{it} and thus is not stochastic.

As ν_t increases, e_{it} strictly decreases while $e_{it} > 0$. Aggregating over all i, $\sum_{i=1}^N e_{it}$ strictly decreases in ν_t and R_t weakly increases in ν_t. There exists some $\bar{\nu}_t > 0$ such that $\sum_{i=1}^N e_{it} - R_t = 0$ if $\nu_t = \bar{\nu}_t$ and $\sum_{i=1}^N e_{it} - R_t > 0$ if $\nu_t < \bar{\nu}_t$.

The regulator solves the following Bellman equation:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = \max_{\nu_t} \tilde{E}_t \Bigg[u\bigg((1 - c_t(R_t^*)) \prod_{i=1}^N [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i} \bigg) + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \Bigg],$$

where $\tilde{E}_t$ denotes expectations over the regulator's information set at the beginning of period t and where a star represents market equilibrium. Market outcomes will be insensitive to ν_t for $\nu_t > \bar{\nu}_t$ and thus the regulator's objective will be constant in ν_t for $\nu_t > \bar{\nu}_t$. Maximized welfare is therefore equivalent for a regulator who solves the following problem in which ν_t is constrained to be less than or equal to $\bar{\nu}_t$:

$$\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = \max_{\nu_t \leq \bar{\nu}_t} \tilde{E}_t \Bigg[u\bigg((1 - c_t(R_t^*)) \prod_{i=1}^N [\exp[-\zeta_{it} T_t] L_{it}^* Y^{it}(e_{it}^*)]^{\kappa_i} \bigg) + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \Bigg].$$

At an interior solution, the regulator's first-order condition is

$$0 = \sum_{i=1}^N \frac{\kappa_i Y_e^{i\prime}(e_{it}^*)}{Y^{it}(e_{it}^*)} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \right] \alpha \sum_{i=1}^N \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{c_t'(R_t^*)}{1 - c_t(R_t^*)} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \right] \alpha \frac{\partial R_t^*}{\partial \nu_t}. \tag{A-15}$$

Substitute from (A-13) and (A-14):[41]

$$\begin{aligned}
0 =& \frac{\nu_t}{C_0^*} \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \right] \alpha \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} \\
&- \frac{\nu_t}{C_0^*} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \right] \alpha \frac{\partial R_t^*}{\partial \nu_t}.
\end{aligned}$$

Factor out common terms:

$$0 = \left(\frac{\nu_t}{C_0} + \frac{1}{1+r} \alpha \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \right] \right) \left(\sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{\partial R_t^*}{\partial \nu_t} \right).$$

Applying the Implicit Function Theorem to equations (A-13) and (A-14), we find:

$$\frac{\partial e_{it}^*}{\partial \nu_t} = \frac{-1}{\frac{-Y_{ee}^{it}}{Y^{it\prime}} + \frac{Y^{it\prime}}{Y^{it}}} \frac{1}{\nu_t} < 0 \quad \text{if } e_{it}^* > 0, \tag{A-16}$$

$$\frac{\partial R_t^*}{\partial \nu_t} = \frac{1}{\frac{c_t''(R_t^*)}{1-c_t(R_t^*)} C_0^* + \frac{[c_t'(R_t^*)]^2}{[1-c_t(R_t^*)]^2} C_0^*} > 0 \quad \text{if } R_{it}^* > 0. \tag{A-17}$$

Therefore, if some firm is at an interior solution for either emissions or removal:

$$0 = \frac{\nu_t}{C_0^*} + \frac{1}{1+r} \alpha \tilde{E}_t \left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}) \right]. \tag{A-18}$$

From the envelope theorem:

$$\tilde{W}_T(T_t, \tilde{\mu}_t, \tilde{\Omega}_t) = -\tilde{\zeta}_t - \tilde{\lambda}_t + \frac{1}{1+r} \tilde{E}_t [\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1})],$$

where we recognize that the market equilibrium is independent of temperature under the conjecture that ν_t/C_0 is independent of temperature. Recursively substitute into (A-18) and rearrange:

$$\nu_t = C_0^* \, \alpha \frac{1}{r} \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \tilde{\mu}_t \right].$$

We have confirmed the conjecture that ν_t/C_0^* is independent of temperature when $\sum_{i=1}^{N} e_{it} - R_t > 0$. We have established Proposition 2.

[41] At a corner with either $e_{it} = 0$ or $R_{it} = 0$, we would have $\partial e_{it}^*/\partial \nu_t = 0$ or $\partial R_t^*/\partial \nu_t = 0$, respectively, and so would end up at the same optimal tax as below as long as at least one is interior.

F Proof of Corollary 2

Let G_t be cumulative revenue collected, which is invested with rate of return r:

$$G_{t+1} = (1+r)\left\{G_t + \nu_t \left[\sum_{i=1}^{N} e_{it} - R_t\right]\right\},$$

with $G_0 \geq 0$. The constraint in time t is now

$$G_t + \nu_t \left[\sum_{i=1}^{N} e_{it} - R_t\right] \geq 0.$$

This constraint binds only if $\sum_{i=1}^{N} e_{it} - R_t \leq 0$. Denote the smallest ν_t at which the constraint binds as $\bar{\nu}_t$. By the implicit function theorem,

$$\frac{\mathrm{d}\bar{\nu}_t}{\mathrm{d}G_t} = -\frac{1}{\sum_{i=1}^{N} e^*_{it}(\bar{\nu}_t) - R^*_t(\bar{\nu}_t) + \sum_{i=1}^{N} \frac{\partial e^*_{it}}{\partial \nu_t}\Big|_{\bar{\nu}_t} - \frac{\partial R^*_t}{\partial \nu_t}\Big|_{\bar{\nu}_t}} > 0,$$

where a star indicates market equilibrium. The sign follows from (A-16), (A-17), and emissions being net negative when the constraint binds.

The regulator solves the following Bellman equation:

$$\begin{aligned}\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \max_{\nu_t} \tilde{E}_t \Bigg[& u\left((1 - c_t(R^*_t)) \prod_{i=1}^{N} [\exp[-\zeta_{it} T_t] L^*_{it} Y^{it}(e^*_{it})]^{\kappa_i}\right) \\ & + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\Bigg],\end{aligned}$$

where $\tilde{E}_t$ denotes expectations over the regulator's information set at the beginning of period t. Market outcomes will be insensitive to ν_t for $\nu_t > \bar{\nu}_t$, so the regulator's objective will be constant in ν_t for $\nu_t > \bar{\nu}_t$. Maximized welfare is therefore equivalent for a regulator who solves the following problem in which ν_t is constrained to be less than or equal to $\bar{\nu}_t$:

$$\begin{aligned}\tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \max_{\nu_t \leq \bar{\nu}_t} \tilde{E}_t \Bigg[& u\left((1 - c_t(R^*_t)) \prod_{i=1}^{N} [\exp[-\zeta_{it} T_t] L^*_{it} Y^{it}(e^*_{it})]^{\kappa_i}\right) \\ & + \frac{1}{1+r} \tilde{W}(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\Bigg].\end{aligned}$$

At an interior solution, the regulator's first-order condition is

$$\begin{aligned}0 =& \sum_{i=1}^{N} \frac{\kappa_i Y_e^{i\prime}(e_{it}^*)}{Y^{it}(e_{it}^*)} \frac{\partial e_{it}^*}{\partial \nu_t} + \frac{1}{1+r} \tilde{E}_t\left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\right] \alpha \sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} \\ &- \frac{c_t'(R_t^*)}{1 - c_t(R_t^*)} \frac{\partial R_t^*}{\partial \nu_t} - \frac{1}{1+r} \tilde{E}_t\left[\tilde{W}_T(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\right] \alpha \frac{\partial R_t^*}{\partial \nu_t} \\ &+ \tilde{E}_t\left[\tilde{W}_G(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\right] \underbrace{\left\{\sum_{i=1}^{N} e_{it}^* - R_t^* + \nu_t \left[\sum_{i=1}^{N} \frac{\partial e_{it}^*}{\partial \nu_t} - \frac{\partial R_t^*}{\partial \nu_t}\right]\right\}}_{\text{Marginal effect of emission tax on revenue}}. \qquad \text{(A-19)}\end{aligned}$$

From the envelope theorem,

$$\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \tilde{E}_t\left[\tilde{W}_G(T_{t+1}, \tilde{\mu}_{t+1}, \tilde{\Omega}_{t+1}, G_{t+1})\right]$$

if $\nu_t < \bar{\nu}_t$ and

$$\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) = \frac{\mathrm{d}\bar{\nu}_t}{\mathrm{d}G_t} \frac{\partial \tilde{W}(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t)}{\partial \bar{\nu}_t}$$

if $\nu_t = \bar{\nu}_t$. Advancing these to later timesteps, we find

$$\begin{aligned}\tilde{W}_G(T_t, \tilde{\mu}_t, \tilde{\Omega}_t, G_t) =& \sum_{s=1}^{\infty} Pr(\nu_{t+j} < \bar{\nu}_{t+j}\, \forall j \in \{1, ..., s-1\}) Pr(\nu_{t+s} = \bar{\nu}_{t+s}) \\ & \frac{\mathrm{d}\bar{\nu}_{t+s}}{\mathrm{d}G_{t+s}} \frac{\partial \tilde{W}(T_{t+s}, \tilde{\mu}_{t+s}, \tilde{\Omega}_{t+s}, G_{t+s})}{\partial \bar{\nu}_{t+s}}\end{aligned}$$

if $\nu_t < \bar{\nu}_t$. Because increasing $\bar{\nu}_{t+s}$ loosens a constraint, the final derivative is strictly positive. We saw above that the other derivative on the final line is strictly positive. And the probabilities on the first line are strictly positive because they depend on $\tilde{\mu}_{t+j}$ and $\tilde{\mu}_{t+s}$, which in turn depend on draws from normally distributed variables that have infinite support. Therefore $\tilde{W}_G > 0$.

Use ν_t^{NoLB} to denote the optimal ν_t implied by (A-15) and ν_t^{LB} to denote the optimal ν_t implied by (A-19). ν_t^{NoLB} is the tax described in Proposition 2 when emissions are strictly positive. If we evaluate (A-19) around ν_t^{NoLB}, then it reduces to its final line. Since we have established that $\tilde{W}_G > 0$, the sign of that final line of (A-19) matches the sign of the term in curly braces, which is the change in revenue due to a marginal change in the tax. If that change is positive, then this term increases the first-order condition and makes $\nu_t^{LB} > \nu_t^{NoLB}$ by concavity around a maximum. And if that change is negative, then this term decreases the first-order condition and makes $\nu_t^{LB} < \nu_t^{NoLB}$ by concavity around a maximum. We have established the first part of the corollary.

The curly braces in the final line of (A-19) are weakly (strictly) negative when net emissions are weakly (strictly) negative. Because net emissions weakly (strictly) decrease in ν_t when net emissions are weakly (strictly) negative, we have established the second part of the corollary in the case that the dynamic revenue constraint does not bind. And the second part of the corollary holds trivially when the dynamic revenue constraint does bind.

G Proof of Lemma 1

Conjecture that the value of the carbon share depends linearly on each $\hat{E}_t[d_{t+j}]$ and $\hat{E}_t[\Delta_{t+j}]$ for $j \geq 0$:

$$\hat{q}_t = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \hat{E}_t[d_{t+j}] + \sum_{j=0}^{\infty} \Psi_{t,t+j} \hat{E}_t[\Delta_{t+j}],$$

with $\{\Lambda_{t,t+j}\}_{j=0}^{\infty}$ and $\{\Psi_{t,t+j}\}_{j=0}^{\infty}$ sequences to be determined.

First consider a case in which $R_t = 0$. The value of a carbon share in period t is $\hat{E}_t[d_t] + \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}]$.

Next consider a case in which $R_t > 0$. The payoff of a shareholder who removes carbon in period t is

$$(1+r)D - \hat{E}_t[\Delta_t] - p_t^R,$$

and the payoff of a shareholder who does not remove carbon in period t is

$$\hat{E}_t[d_t] + \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}].$$

In a competitive equilibrium with abundant carbon shares, shareholders must be indifferent between the two options, implying that

$$p_t^R = (1+r)D - \frac{1}{1+r}\hat{E}_t[\hat{q}_{t+1}] - \hat{E}_t[d_t] - \hat{E}_t[\Delta_t]. \tag{A-20}$$

Equilibrium payoffs are then identical whether $R_t = 0$ or $R_t > 0$. By absence of arbitrage, the value of the carbon share is:

$$\hat{q}_t = (1+r)D - \hat{E}_t[\Delta_t] - p_t^R.$$

Substitute for $\hat{q}_{t+1}$ from the guess:

$$\hat{q}_t = \hat{E}_t[d_t] + \frac{1}{1+r}\sum_{j=1}^{\infty} \Lambda_{t+1,t+j}\hat{E}_t[d_{t+j}] + \frac{1}{1+r}\sum_{j=1}^{\infty} \Psi_{t+1,t+j}\hat{E}_t[\Delta_{t+j}].$$

Matching coefficients, $\Lambda_{t,t} = 1$ and $\Psi_{t,t} = 0$. Advancing the analysis by one timestep, we find $\Lambda_{t+1,t+1} = 1$ and $\Psi_{t+1,t+1} = 0$. Therefore $\Lambda_{t,t+1} = 1/(1+r)$ and $\Psi_{t,t+1} = 0$. The lemma follows from repeating these steps for subsequent periods, deriving $\Lambda_{t+j,t+j}$ and $\Psi_{t+j,t+j}$, eventually $\Lambda_{t+1,t+j}$ and $\Psi_{t+1,t+j}$, and finally $\Lambda_{t,t+j}$ and $\Psi_{t,t+j}$.

H Proof of Proposition 4

The cost of emitting in period t is $D - (\hat{q}_t - \hat{E}[d_t])$. From (8),

$$D - (\hat{q}_t - \hat{E}[d_t]) = D - \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[d_{t+j}],$$

from which (7) implies

$$\begin{aligned} D - (\hat{q}_t - \hat{E}[d_t]) =& D - \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[r\, D - \Delta_{t+j}] \\ =& \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \hat{E}_t[\Delta_{t+j}]. \end{aligned}$$

Using (6) and the properties of truncated normal distributions,

$$D - (\hat{q}_t - \hat{E}[d_t]) = C_0\, \alpha \left[\frac{1}{r} \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \frac{1}{r} \hat{\mu}_t - \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \chi_{t,t+j} \right], \tag{A-21}$$

where

$$\chi_{t,t+j} \triangleq \frac{\phi\left(\frac{\bar{\mu}-\hat{\mu}_t}{\Sigma_{t,t+j}}\right)}{\Phi\left(\frac{\bar{\mu}-\hat{\mu}_t}{\Sigma_{t,t+j}}\right)} \Sigma_{t,t+j} \geq 0$$

is the adjustment to the mean of a normal distribution (for time j random variables, using the time t information set) for the upper truncation point (from the deposit's definition in (5)), $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and cumulative density function for the standard normal distribution, and $\Sigma_{t,t+j} \triangleq \left(\widehat{Var}_t(\hat{\mu}_{t+j})\right)^{1/2}$ is independent of $\bar{\mu}$. Setting $\nu_t = D - \hat{q}_t$ in (A-13), time t emissions are as in (1) as $\chi_{t,t+j} \to 0$ for all $j > 0$.

Assumption 1 ensures that some shares are outstanding. Applying the foregoing analysis to equation (A-20), we find that, if some shares are exercised,

$$p_t^R = \frac{1}{1+r} \sum_{j=1}^{\infty} \frac{1}{(1+r)^{j-1}} \hat{E}_t[\Delta_{t+j}].$$

Using in equation (A-8), we find that

$$\begin{aligned} \frac{c_t'(R_t)}{1 - c_t(R_t)} =& \frac{1}{1+r} \sum_{j=1}^{\infty} \frac{1}{(1+r)^{j-1}} \hat{E}_t[\Delta_{t+j}] \\ =& C_0\, \alpha \left[\frac{1}{r} \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \frac{1}{r} \hat{\mu}_t - \frac{1}{1+r} \sum_{j=1}^{\infty} \frac{1}{(1+r)^{j-1}} \chi_{t,t+j} \right], \end{aligned} \tag{A-22}$$

with $\chi_{t,t+j}$ as above. Time t removal of carbon emitted in all previous periods is as in (2) as $\chi_{t,t+j} \to 0$ for all $s \in \{1, ..., t\}$ and all $j > 0$.

$\chi_{t,t+j}$ decreases in $\bar{\mu}$ and goes to 0 as $\bar{\mu}$ goes to ∞ (i.e., the mean of a truncated-normal distribution increases in the upper bound and approaches the mean of the untruncated normal distribution as the truncation point goes to infinity). By the foregoing analysis, time t emissions and removal as as in (1) and (2) as $\bar{\mu} \to \infty$. So $\breve{L}_t \to 0$ as $\bar{\mu} \to \infty$. We have proved the proposition.

I Preliminaries for Proofs of Propositions 5 and 6

Let $\breve{\mu}_t$ and $\breve{\Omega}_t$ indicate the indicate the mean and variance for $\sum_{k=1}^{N} \kappa_k \zeta_k$ formed after observing $\tilde{\zeta}_s + \tilde{\lambda}_s$ and q_s for all $s < t$, with the corresponding variance of each ζ_i labeled τ_t^2 (note that this variance will be indepndent of i). I show below how to define τ_t^2 from Bayesian updating for $t > 0$.

Demand for carbon shares and market-clearing price

Conjecture that $\breve{q}_t$ is a linear function of the $\zeta_{it} + \lambda_{it}$ and that q_{t+1} is a linear function of $\tilde{\zeta}_t + \tilde{\lambda}_t$ and $\breve{q}_t$. In this case, $\breve{q}_t$ and q_{t+1} are normally distributed, and by standard properties of normal random variables and $D \to \infty$, the time t maximization problem for traders of type i is equivalent to:

$$\begin{aligned}
&\max_{X_{it}} - \exp\Bigg\{ - A(1+r)(w_{it} - X_{it}\breve{q}_t) - A(y_{it} + X_{it})\breve{E}_t[q_{t+1} + (1+r)d_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \\
&\qquad + \frac{1}{2}A^2(y_{it} + X_{it})^2 \breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] \Bigg\} \\
=&\max_{X_{it}} - \exp\Bigg\{ - A(1+r)(w_{it} - X_{it}\breve{q}_t) + \frac{1}{2}A^2(y_{it} + X_{it})^2 \breve{Var}_t\left[q_{t+1} - (1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] \\
&\qquad - A(y_{it} + X_{it})\left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left[rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t]\right]\right) \Bigg\},
\end{aligned}$$

where $\breve{E}_t$ and $\breve{Var}_t$ indicate the expectation and variance at the common time t beginning-of-period information set. The first-order condition for a maximum is

$$X_{it} = \frac{h_{it}}{A}\left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r)\left(rD - C_0\alpha\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t]\right) - (1+r)\breve{q}_t\right) - y_{it}, \tag{A-23}$$

where

$$h_{it} \triangleq \left(\breve{Var}_t \left[q_{t+1} - (1+r) C_0 \alpha [\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] \right)^{-1}. \tag{A-24}$$

The h_{it} are deterministic by standard properties of normal-normal updating. Aggregate net demand for carbon shares is

$$X_t = \sum_{i=1}^{N} \frac{h_{it}}{A} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) - (1+r) \breve{q}_t \right) - \sum_{i=1}^{N} y_{it}.$$

Market-clearing requires

$$X_t = 0.$$

Rearranging, the equilibrium price is:

$$\begin{aligned} \breve{q}_t^* = & \frac{1}{(1+r) \sum_{i=1}^{N} h_{it}} \Bigg[\sum_{i=1}^{N} h_{it} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, q_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) \right) \\ & - A \sum_{i=1}^{N} y_{it} \Bigg]. \end{aligned}$$

Define

$$\breve{h}_{it} \triangleq \frac{h_{it}}{\sum_{i=1}^{N} h_{it}}. \tag{A-25}$$

We have:

$$\begin{aligned} \breve{q}_t^* = & \frac{1}{1+r} \sum_{i=1}^{N} \breve{h}_{it} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) \right) \\ & - \frac{1}{(1+r) \sum_{k=1}^{N} h_{kt}} A \sum_{i=1}^{N} y_{it}. \end{aligned}$$

Observe that $\sum_{i=1}^{N} y_{it} = M_t - M_0$. Therefore

$$\begin{aligned} \breve{q}_t^* = & \frac{1}{1+r} \sum_{i=1}^{N} \breve{h}_{it} \left(\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] + (1+r) \left(rD - C_0 \alpha \breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \breve{q}_t] \right) \right) \\ & - A \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}}. \end{aligned} \tag{A-26}$$

Analyzing q_{t+1}

q_{t+1} is the equilibrium price determined by time $t+1$ agents who have common beliefs, so that Lemma 1 applies, modulo the information set. As $D \to \infty$,

$$q_{t+1} = \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_{t+1} \right]$$
$$= \frac{1+r}{r} \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_{t+1} \right]. \quad \text{(A-27)}$$

Define $\tilde{q}_t$ as the signal of $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k$ extracted from $\breve{q}_t$, which implies

$$\breve{E}_t[\tilde{q}_t] = \breve{\mu}_t. \quad \text{(A-28)}$$

Conjecture that

$$\breve{\mu}_{t+1} = a'_t \breve{\mu}_t + b'_t \left(\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) + B'_t \tilde{q}_t, \quad \text{(A-29)}$$

where a'_t, b'_t, and B'_t are constants to be determined. Taking the expectation of each side under the information set at the beginning of time t and using (A-28), we find that $1 = a'_t + b'_t + B'_t$. Under the conjecture,

$$\breve{E}_t[\breve{\mu}_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] = a'_t \breve{\mu}_t + b'_t \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] + B'_t \tilde{q}_t.$$

Using this with (A-27),

$$\breve{E}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t] = \frac{1+r}{r} \Bigg[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \left(a'_t \breve{\mu}_t + b'_t \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] + B'_t \tilde{q}_t \right) \Bigg].$$

From (A-26),

$$\breve{q}^*_t = \frac{1+r}{r} \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right] - \frac{1}{r} C_0 \alpha \left(a'_t \breve{\mu}_t + B'_t \tilde{q}_t \right) - \frac{b'_t + r}{r} C_0 \alpha \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right]$$
$$- A \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}}. \quad \text{(A-30)}$$

Deriving $\tilde{q}_t$

From (A-30),

$$\begin{aligned}&\sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] \\ =& \frac{1}{C_0 \alpha (b'_t + r)} \left[(1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha (a'_t \breve{\mu}_t + B'_t \tilde{q}_t) - r \breve{q}^*_t - rA \, \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right].\end{aligned} \tag{A-31}$$

If we set,

$$\tilde{q}_t = \frac{1}{C_0 \alpha (B'_t + b'_t + r)} \left[(1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 a'_t \breve{\mu}_t - r \breve{q}_t - rA \, \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right], \tag{A-32}$$

then from (A-31) and the definition $\breve{q}_t \triangleq \breve{q}^*_t + \theta_t$,

$$\tilde{q}_t = \sum_{i=1}^{N} \breve{h}_{it} \breve{E}_t \left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t \right] - \frac{r}{C_0 \alpha (B'_t + b'_t + r)} \theta_t. \tag{A-33}$$

Recalling that $\breve{h}_{it} \in (0,1)$ and $\sum_{i=1}^{N} \breve{h}_{it} = 1$, this $\tilde{q}_t$ satisfies the earlier definition of $\tilde{q}_t$ as the signal of $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k$ extracted from $\breve{q}_t$.

J Proof of Proposition 5

Section I contains preliminaries. A rational expectations equilibrium is fully revealing if and only if $\breve{\mu}_t = \hat{\mu}_{t+1}$ and, from (A-33), $\tilde{q}_t = \hat{\mu}_{t+1}$. From (A-32) and taking $\Theta^2 \to 0$, that $\tilde{q}_t$ clears the carbon share market if and only if

$$r \breve{q}^*_t = (1+r) \left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha a'_t \hat{\mu}_{t+1} - C_0 \alpha (B'_t + b'_t + r) \hat{\mu}_{t+1} - rA \, \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}}.$$

Using $a'_t + b'_t + B'_t = 1$,

$$\breve{q}^*_t = \frac{1+r}{r} \left[\left(rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \right) - C_0 \alpha \hat{\mu}_{t+1} - rA \, \frac{M_t - M_0}{(1+r) \sum_{k=1}^{N} h_{kt}} \right].$$

As $\Theta^2 \to 0$, $\breve{q}_t \to \breve{q}_t^*$, in which case traders can back out $\hat{\mu}_{t+1}$ from the observed price $\breve{q}_t$, so $\breve{\mu}_t = \hat{\mu}_{t+1}$. In addition, the conjectured form of $\breve{\mu}_{t+1}$ in (A-29) holds, with $B_t' = 1$, and τ_t^2 can be defined from the posterior variance of informationally efficient beliefs. We have described a rational expectations equilibrium that is fully revealing.

Observe that $\breve{q}_t$ is identical to $\hat{q}_t$ from Lemma 1. Assumption 1 and Proposition 4 then imply that $\breve{L} \to 0$ as $D \to \infty$.

K Proof of Proposition 6

Section I contains preliminaries.

Expected q_{t+1} as a function of time t information

The combination of normal random variables and an affine information structure implies that the posterior mean is a linear function of the prior and the signals:

$$\breve{E}_t\left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] = a_{it}\breve{\mu}_t + b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) + B_{it}\tilde{q}_t \tag{A-34}$$

for yet-to-be-determined coefficients a_{it}, b_{it}, and B_{it}. Substituting into (A-33), we find:

$$\tilde{q}_t = \sum_{i=1}^{N} \breve{h}_{it}\left[a_{it}\breve{\mu}_t + b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) + B_{it}\tilde{q}_t\right] - \frac{r}{C_0\alpha(B_t' + b_t' + r)}\theta_t.$$

Solving for $\tilde{q}_t$ yields:

$$\tilde{q}_t = \frac{1}{1 - \sum_{i=1}^{N} \breve{h}_{it}B_{it}}\left(\sum_{i=1}^{N} \breve{h}_{it}a_{it}\breve{\mu}_t + \sum_{i=1}^{N} \breve{h}_{it}b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(B_t' + b_t' + r)}\theta_t\right).$$

Taking the expectation under the information set at the beginning of time t and setting it equal to (A-28), we find:

$$\breve{\mu}_t = \frac{\breve{\mu}_t}{1 - \sum_{i=1}^{N} \breve{h}_{it}B_{it}} \sum_{i=1}^{N} \breve{h}_{it}(a_{it} + b_{it}).$$

This holds if and only if $\sum_{i=1}^{N} \breve{h}_{it}B_{it} = 1 - \sum_{i=1}^{N} \breve{h}_{it}(a_{it} + b_{it})$. Define

$$\chi_t \triangleq 1 - \sum_{i=1}^{N} \breve{h}_{it}B_{it}. \tag{A-35}$$

Then:

$$\tilde{q}_t = \frac{1}{\chi_t}\left(\sum_{i=1}^{N} \breve{h}_{it}a_{it}\breve{\mu}_t + \sum_{i=1}^{N} \breve{h}_{it}b_{it}(\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) - \frac{r}{C_0\alpha(B_t' + b_t' + r)}\theta_t\right). \tag{A-36}$$

Deriving $\breve{E}_t[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k | \zeta_{it} + \lambda_{it}, \breve{q}_t]$

In time t but prior to observing $\zeta_{it} + \lambda_{it}$ and $\breve{q}_t$, the 3×1 random vector

$$\breve{S}_{it} = \begin{bmatrix} \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k \bar{\zeta}_k \\ \zeta_{it} + \lambda_{it} - \bar{\zeta}_i \\ \tilde{q}_t \end{bmatrix} \tag{A-37}$$

is jointly normal with unconditional mean

$$\breve{E}_{it}[\breve{S}_{it}] = \begin{bmatrix} \breve{\mu}_t \\ \breve{\mu}_t \\ \breve{\mu}_t \end{bmatrix}$$

and covariance matrix

$$Cov_{it}(\breve{S}_{it}) = \begin{bmatrix} \sum_{j=1}^N \kappa_j^2[\tau_t^2+\sigma^2]+2\Gamma\tau_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \kappa_j\kappa_k+\tilde{\omega}^2 & \kappa_i\tau_t^2+(1-\kappa_i)\Gamma\tau_t^2+\kappa_i\sigma^2 & \sum_{k=1}^N(\kappa_k\tau_t^2+(1-\kappa_k)\Gamma\tau_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ \kappa_i\tau_t^2+(1-\kappa_i)\Gamma\tau_t^2+\kappa_i\sigma^2 & \tau_t^2+\sigma^2+\omega^2 & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ \sum_{k=1}^N(\kappa_k\tau_t^2+(1-\kappa_k)\Gamma\tau_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{matrix} \sum_{k=1}^N\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2(\tau_t^2+\sigma^2+\omega^2) \\ +2\Gamma\tau_t^2\sum_{j=1}^N\sum_{k=j+1}^N\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ +\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2 \end{matrix} \end{bmatrix}. \tag{A-38}$$

From the projection theorem,

$$\breve{E}_t\left[\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k\bar{\zeta}_k | \zeta_{it}+\lambda_{it}, \breve{q}_t\right]$$

$$=\breve{\mu}_t + \begin{bmatrix} \kappa_i\tau_t^2+(1-\kappa_i)\Gamma\tau_t^2+\kappa_i\sigma^2 \\ \sum_{k=1}^N(\kappa_k\tau_t^2+(1-\kappa_k)\Gamma\tau_t^2+\kappa_k\sigma^2)\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \end{bmatrix}^\intercal \begin{bmatrix} \tau_t^2+\sigma^2+\omega^2 & (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ (\tau_t^2+\sigma^2+\omega^2)\frac{\breve{h}_{it}b_{it}}{\chi_t}+\Gamma\tau_t^2\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} & \begin{matrix} \sum_{k=1}^N\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2(\tau_t^2+\sigma^2+\omega^2) \\ +2\Gamma\tau_t^2\sum_{j=1}^N\sum_{k=j+1}^N\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\ +\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2 \end{matrix} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \zeta_{it}+\lambda_{it}-\bar{\zeta}_i-\breve{\mu}_t \\ \tilde{q}_t-\breve{\mu}_t \end{bmatrix}.$$

Working through the matrix algebra and matching coefficients to (A-34), we find:

$$\begin{aligned}
b_{it} =&\frac{1}{det_{it}}\Bigg\{ \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \\
&\left[\sum_{k=1}^{N}\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2 + 2\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\sum_{j=1}^{N}\sum_{k=j+1}^{N}\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \right.\\
&\left. + \frac{1}{\tau_t^2+\sigma^2+\omega^2}\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2\right] \\
&- \left(\sum_{k=1}^{N}\left(\kappa_k \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_k)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right) \\
&\left(\frac{\breve{h}_{it}b_{it}}{\chi_t} + \frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)\Bigg\},
\end{aligned} \tag{A-39}$$

$$\begin{aligned}
B_{it} =&\frac{1}{det_{it}}\Bigg\{\sum_{k\neq i}\Bigg(\left[\kappa_k - \kappa_i\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right]\frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} \\
&+ \left[(1-\kappa_k) - (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right]\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\Bigg)\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\Bigg\},
\end{aligned} \tag{A-40}$$

and

$$a_{it} = 1 - b_{it} - B_{it}, \tag{A-41}$$

where

$$\begin{aligned}
det_{it} \triangleq& \sum_{k\neq i}\left(\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2 + 2\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\frac{\breve{h}_{jt}b_{jt}}{\chi_t}\frac{\breve{h}_{kt}b_{kt}}{\chi_t} \\
&+ \frac{1}{\tau_t^2+\sigma^2+\omega^2}\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\frac{\Theta}{\chi_t}\right)^2 - \left(\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)^2\left(\sum_{k\neq i}\frac{\breve{h}_{kt}b_{kt}}{\chi_t}\right)^2.
\end{aligned}$$

(A-41) implies that $\frac{1}{N}\sum_{i=1}^{N} B_{it} = 1 - \sum_{i=1}^{N}(a_{it}+b_{it})$, as required above.

Simplifying, (A-39) becomes:

$$
\begin{aligned}
b_{it} =& \Bigg\{ \sum_{k\neq i} \breve{h}_{kt}^2 b_{kt}^2 + 2\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \sum_{j=1,\neq i}^{N} \sum_{k=j+1,\neq i}^{N} \breve{h}_{jt} b_{jt} \breve{h}_{kt} b_{kt} + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2 \frac{\Theta^2}{\tau_t^2+\sigma^2+\omega^2} \\
&\quad - \left(\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)^2 \left(\sum_{k\neq i} \breve{h}_{kt} b_{kt}\right)^2 \Bigg\}^{-1} \\
&\Bigg\{ \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \sum_{k\neq i} \breve{h}_{kt}^2 b_{kt}^2 \\
&\quad + \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2 \frac{\Theta^2}{\tau_t^2+\sigma^2+\omega^2} \\
&\quad + \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) 2\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \sum_{j=1,\neq i}^{N} \sum_{k=j+1,\neq i}^{N} \breve{h}_{jt} b_{jt} \breve{h}_{kt} b_{kt} \\
&\quad + \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \breve{h}_{it} b_{it} \sum_{k\neq i} \breve{h}_{kt} b_{kt} \\
&\quad - \breve{h}_{it} b_{it} \sum_{k\neq i} \left(\kappa_k \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_k)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \breve{h}_{kt} b_{kt} \\
&\quad - \frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2} \left(\sum_{k\neq i} \left(\kappa_k \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_k)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \breve{h}_{kt} b_{kt}\right) \sum_{k\neq i} \breve{h}_{kt} b_{kt} \Bigg\}.
\end{aligned}
$$

Solve for b_{it}:

$$
\begin{aligned}
b_{it} = &\left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)\\
&\Bigg\{ \left(\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right)\left(\tau_t^2+\sigma^2+\omega^2+\Gamma\tau_t^2\right)\sum_{k\neq i}\breve{h}_{kt}^2 b_{kt}^2\\
&\quad + 2\Gamma\tau_t^2\left[\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right]\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\breve{h}_{jt}b_{jt}\breve{h}_{kt}b_{kt}\\
&\quad + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2(\tau_t^2+\sigma^2+\omega^2)\\
&\quad - \Bigg[\sum_{k\neq i}\Bigg(\left[\kappa_k(\tau_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\tau_t^2\right](\tau_t^2+\sigma^2)\\
&\qquad\qquad + \left[(1-\kappa_k)(\tau_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\tau_t^2\right]\Gamma\tau_t^2\Bigg)\breve{h}_{kt}b_{kt}\Bigg]\\
&\qquad\qquad \sum_{k\neq i}\frac{\Gamma\tau_t^2}{\kappa_i(\tau_t^2+\sigma^2)+(1-\kappa_i)\Gamma\tau_t^2}\,\breve{h}_{kt}b_{kt}\Bigg\}\\
&\Bigg\{ \left(\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right)\left(\tau_t^2+\sigma^2+\omega^2+\Gamma\tau_t^2\right)\sum_{k\neq i}\breve{h}_{kt}^2 b_{kt}^2\\
&\quad + 2\Gamma\tau_t^2\left[\tau_t^2+\sigma^2+\omega^2-\Gamma\tau_t^2\right]\sum_{j=1,\neq i}^{N}\sum_{k=j+1,\neq i}^{N}\breve{h}_{jt}b_{jt}\breve{h}_{kt}b_{kt}\\
&\quad + \left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2(\tau_t^2+\sigma^2+\omega^2)\\
&\quad + \breve{h}_{it}\sum_{k\neq i}\Bigg(\left[\kappa_k(\tau_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\tau_t^2\right](\tau_t^2+\sigma^2)\\
&\qquad\qquad + \left[(1-\kappa_k)(\tau_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\tau_t^2\right]\Gamma\tau_t^2\Bigg)\breve{h}_{kt}b_{kt}\Bigg\}^{-1}.
\end{aligned}
\tag{A-42}
$$

Observe that

$$
\begin{aligned}
&\left[\kappa_k(\tau_t^2+\sigma^2+\omega^2)-\kappa_i\Gamma\tau_t^2\right](\tau_t^2+\sigma^2)+\left[(1-\kappa_k)(\tau_t^2+\sigma^2+\omega^2)-(1-\kappa_i)\Gamma\tau_t^2\right]\Gamma\tau_t^2\\
=&\left[(1-\Gamma)\tau_t^2+\sigma^2+\omega^2\right]\left[\kappa_k(\tau_t^2+\sigma^2)+(1-\kappa_i)\Gamma\tau_t^2\right]
\end{aligned}
$$

decreases in κ_i and is strictly positive as $\kappa_i \to 1$, implying that it is strictly positive for all relevant κ_i. Because that expression is positive,

$$b_{it} < \kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2},$$

and because this last inequality holds for arbitrary i, inspection of (A-42) shows that $b_{it} > 0$. Therefore the set of functions defined by (A-42) for each $i \in \{1, ..., N\}$ maps a vector from

$$\bigtimes_{i=1}^{N} \left[0, \kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right]$$

into itself. By Brouwer's fixed-point theorem, there exists a fixed point in that space. By inspection, the fixed point does not have any b_{kt} on the boundary. Therefore, for each $i \in \{1, ..., N\}$,

$$b_{it} = \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right) \breve{Z}_{it} \tag{A-43}$$

for some $\breve{Z}_{it} \in (0, 1)$. Observe that $\breve{Z}_{it}$ and b_{it} are deterministic because each $\breve{h}_{kt}$ is deterministic.

Deriving $\breve{\mu}_{t+1}$

Substituting (A-43) into (A-36) and using $\sum_{i=1}^{N} \breve{h}_{it} a_{it} = \chi_t - \sum_{i=1}^{N} \breve{h}_{it} b_{it}$ from (A-35) and (A-41),

$$\begin{aligned}\tilde{q}_t = & \frac{\chi_t - \sum_{i=1}^{N} \breve{h}_{it} b_{it}}{\chi_t} \breve{\mu}_t + \sum_{i=1}^{N} \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right) \frac{\breve{h}_{it} \breve{Z}_{it}}{\chi_t} (\zeta_{it} + \lambda_{it} - \bar{\zeta}_i) \\ & - \frac{r}{C_0 \alpha (B_t' + b_t' + r)} \frac{\theta_t}{\chi_t}.\end{aligned} \tag{A-44}$$

Consider jointly updating about

$$\begin{bmatrix} \sum_{i=1}^{N} \kappa_i \zeta_i \\ \sum_{i=1}^{N} \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1 - \kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2}\right) \frac{\breve{h}_{it} \breve{Z}_{it}}{\chi_t} \zeta_i \end{bmatrix}. \tag{A-45}$$

Define the time t signal of this vector as

$$\breve{s}_t \triangleq \begin{bmatrix} \tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k \\ \tilde{q}_t - \frac{\chi_t - \sum_{i=1}^{N} \breve{h}_{it} b_{it}}{\chi_t} \breve{\mu}_t \end{bmatrix}. \tag{A-46}$$

Let $\Upsilon_{\mu,t}$ indicate the 2×2 precision matrix of the time t prior and $\Upsilon_{s,t}$ indicate the 2×2 conditional precision matrix of the time t signal. Apply the conventional normal-normal updating formula, recursively substitute, and recognize that the time 0 mean is a zero vector:

$$E\left[\left[\frac{\sum_{i=1}^N \kappa_i \zeta_i}{\sum_{i=1}^N \left(\kappa_i \frac{\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2} + (1-\kappa_i)\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right) \frac{\breve{h}_{it}\breve{Z}_{it}}{\chi_t}\zeta_i}\right] \middle| \{\tilde{\zeta}_j + \tilde{\lambda}_j, \tilde{q}_j\}_{j=0}^t\right] = \sum_{k=0}^t \left[\prod_{j=k+1}^t \pi_j\right](I-\pi_k)\breve{s}_k,$$

where $\pi_j \triangleq (\Upsilon_{\mu,j} + \Upsilon_{s,j})^{-1}\Upsilon_{\mu,j}$ and I is the 2×2 identity matrix. Define $\tilde{\pi}_k$ as element $(1,1)$ of $\left[\prod_{j=k+1}^t \pi_j\right][I-\pi_k]$ and $\breve{\pi}_k$ as element $(1,2)$ of $\left[\prod_{j=k+1}^t \pi_j\right][I-\pi_k]$. Using (A-46) and the definition of $\breve{\mu}_{t+1}$,

$$\breve{\mu}_{t+1} = \sum_{k=0}^t \left[\tilde{\pi}_k\left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^N \kappa_j\bar{\zeta}_j\right) + \breve{\pi}_k\left(\tilde{q}_k - \frac{\chi_k - \sum_{i=1}^N \breve{h}_{ik}b_{ik}}{\chi_k}\breve{\mu}_k\right)\right].$$

Substituting for $\breve{\mu}_t$, we have confirmed the conjecture in (A-29) that $\breve{\mu}_{t+1}$ is a linear function of $\breve{\mu}_t$, $\tilde{\zeta}_t + \tilde{\lambda}_t - \sum_{k=1}^N \kappa_k\bar{\zeta}_k$, and $\tilde{q}_t$. Matching coefficients yields a'_t, b'_t, and B'_t.

Using (A-44),

$$\begin{aligned}\breve{\mu}_{t+1} = \sum_{k=0}^t \Bigg[&\tilde{\pi}_k\left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^N \kappa_j\bar{\zeta}_j\right)\\ &+ \breve{\pi}_k \sum_{j=1}^N \left(\kappa_j \frac{\tau_k^2+\sigma^2}{\tau_k^2+\sigma^2+\omega^2} + (1-\kappa_j)\frac{\Gamma\tau_k^2}{\tau_k^2+\sigma^2+\omega^2}\right)\frac{\breve{h}_{jk}\breve{Z}_{jk}}{\chi_k}(\zeta_{jk}+\lambda_{jk}-\bar{\zeta}_j)\\ &- \breve{\pi}_k \frac{r}{C_0\alpha(B'_k + b'_k + r)}\frac{\theta_k}{\chi_k}\Bigg].\end{aligned}$$

Define

$$\breve{\kappa}_{is} \triangleq \breve{h}_{is}\breve{Z}_{is}. \tag{A-47}$$

$\breve{\kappa}_{is} \in (0,1)$, because both components are. Also define

$$\breve{\chi}_s \triangleq B'_s + b'_s < 1.$$

Then:

$$\begin{aligned}\breve{\mu}_{t+1} =& \sum_{k=0}^{t} \tilde{\pi}_k \left(\tilde{\zeta}_k + \tilde{\lambda}_k - \sum_{j=1}^{N} \kappa_j \bar{\zeta}_j \right) \\ &+ \sum_{k=0}^{t} \frac{\breve{\pi}_k}{\chi_k} \left[\frac{(1-\Gamma)\tau_k^2 + \sigma^2}{\tau_k^2 + \sigma^2 + \omega^2} \sum_{i=1}^{N} \breve{\kappa}_{ik} \kappa_i \left(\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i \right) \right. \\ &\qquad \left. + \frac{\Gamma \tau_k^2}{\tau_k^2 + \sigma^2 + \omega^2} \sum_{i=1}^{N} \breve{\kappa}_{ik} (\zeta_{ik} + \lambda_{ik} - \bar{\zeta}_i) \right] \\ &- \sum_{k=0}^{t} \frac{\breve{\pi}_k}{\chi_k} \frac{r}{C_0 \alpha (\breve{\chi}_k + r)} \theta_k. \end{aligned} \tag{A-48}$$

Letting $\tilde{\Upsilon}_t$ indicate element $(1,1)$ of $(\Upsilon_{\mu,t} + \Upsilon_{s,t})^{-1}$, the conventional normal-normal updating formula for the precision and the definition of τ_t^2 together imply:

$$\tau_{t+1}^2 = \frac{\tilde{\Upsilon}_t}{(1-\Gamma)\sum_{k=1}^{N} \kappa_k^2 + \Gamma}.$$

Finally, the following lemma establishes properties of the $\tilde{\pi}$ and the $\breve{\pi}$:

Lemma 2. *If Γ, ω^2, $\tilde{\omega}^2$, and Θ^2 are sufficiently small, then $\tilde{\pi}_k \in (0,1)$ and $\lim_{\Gamma,\tilde{\omega}^2 \to 0} \breve{\pi}_k$ is arbitarily close to zero.*

Proof. See Appendix M. □

Emissions and carbon removal

Adapting (A-21) and (A-22) to the current informational environment and applying the conditions of the proposition, time t firms equate both the marginal cost of emission reductions and the marginal cost of carbon removal to $D - (q_t - \breve{E}_t[d_t])$. Using (A-27),

$$\begin{aligned} D - (q_t - \breve{E}_t[d_t]) =& D - \left(\frac{1+r}{r} - 1 \right) \left[rD - C_0 \alpha \sum_{k=1}^{N} \kappa_k \bar{\zeta}_k - C_0 \alpha \breve{\mu}_t \right] \\ =& \frac{1}{r} C_0 \alpha \left[\sum_{k=1}^{N} \kappa_k \bar{\zeta}_k + \breve{\mu}_t \right]. \end{aligned}$$

The proposition follows from using (A-48) to define $\breve{\mu}_t$.

L Proof of Corollary 4

By inspection, equation (A-42) holds if $b_{it} = 0$ for all $i \in \{1, ..., N\}$ with $\Theta^2 = 0$, and because the denominator contains a term that is linear in the b_{kt} whereas the numerator contains only products of the b, equation (A-42) holds if each b_{it} is arbitrarily small with Θ^2 arbitrarily small. The first part of the corollary follows from these results, $\breve{h}_{kt} \in (0, 1)$, and the definition (A-47).

Now analyze h_{it}. From the definition (A-24),

$$1/h_{it} = \breve{Var}_t \left[q_{t+1} | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] + \breve{Var}_t \left[(1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right]. \tag{A-49}$$

From (A-29),

$$\breve{Var}_t[\mu_{t+1} | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = (b'_t)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t].$$

Using this in (A-27),

$$\breve{Var}_t[q_{t+1} | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = \left(\frac{1+r}{r} C_0 \alpha b'_t\right)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t],$$

$$\breve{Cov}_t \left[q_{t+1}, -(1+r)C_0\alpha[\tilde{\zeta}_t + \tilde{\lambda}_t] | \zeta_{it} + \lambda_{it}, \breve{q}_t\right] = [(1+r)C_0\alpha]^2 \frac{b'_t}{r} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t].$$

Substituting into (A-49),

$$1/h_{it} = \left[\frac{1+r}{r} C_0 \alpha\right]^2 (b'_t + r)^2 \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t]. \tag{A-50}$$

Apply the projection theorem to (A-37), via (A-38):

$$\begin{aligned}
&\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] \\
=&[\tau_t^2 + \sigma^2] \sum_{j=1}^N \kappa_j^2 + 2\Gamma\tau_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \kappa_j \kappa_k + \tilde{\omega}^2 \\
&- \begin{bmatrix} \kappa_i\tau_t^2 + (1-\kappa_i)\Gamma\tau_t^2 + \kappa_i\sigma^2 \\ \sum_{k=1}^N (\kappa_k\tau_t^2 + (1-\kappa_k)\Gamma\tau_t^2 + \kappa_k\sigma^2) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \end{bmatrix}^\intercal \begin{bmatrix} \tau_t^2 + \sigma^2 + \omega^2 & (\tau_t^2 + \sigma^2 + \omega^2) \frac{\breve{h}_{it} b_{it}}{\chi_t} + \Gamma\tau_t^2 \sum_{k \neq i} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \\ (\tau_t^2 + \sigma^2 + \omega^2) \frac{\breve{h}_{it} b_{it}}{\chi_t} + \Gamma\tau_t^2 \sum_{k \neq i} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} & \begin{matrix} \sum_{k=1}^N \left(\frac{\breve{h}_{kt} b_{kt}}{\chi_t}\right)^2 (\tau_t^2 + \sigma^2 + \omega^2) \\ + 2\Gamma\tau_t^2 \sum_{j=1}^N \sum_{k=j+1}^N \frac{\breve{h}_{jt} b_{jt}}{\chi_t} \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \\ + \left(\frac{r}{C_0\alpha(B'_t + b'_t + r)} \frac{\Theta}{\chi_t}\right)^2 \end{matrix} \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \kappa_i\tau_t^2 + (1-\kappa_i)\Gamma\tau_t^2 + \kappa_i\sigma^2 \\ \sum_{k=1}^N (\kappa_k\tau_t^2 + (1-\kappa_k)\Gamma\tau_t^2 + \kappa_k\sigma^2) \frac{\breve{h}_{kt} b_{kt}}{\chi_t} \end{bmatrix}
\end{aligned}$$

Using the b_{it} and B_{it} from (A-39) and (A-40), this becomes

$$\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\tau_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + \Gamma\tau_t^2 + \tilde{\omega}^2 - b_{it} \left[\kappa_i[(1-\Gamma)\tau_t^2 + \sigma^2] + \Gamma\tau_t^2\right] - B_{it} \left[\sum_{k=1}^{N} (\kappa_k[(1-\Gamma)\tau_t^2 + \sigma^2] + \Gamma\tau_t^2) \frac{\breve{h}_{kt} b_{kt}}{\chi_t}\right].$$

Substituting from (A-43),

$$\breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\tau_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + \Gamma\tau_t^2 + \tilde{\omega}^2 - \frac{\left[\kappa_i[(1-\Gamma)\tau_t^2 + \sigma^2] + \Gamma\tau_t^2\right]^2}{\tau_t^2 + \sigma^2 + \omega^2} \breve{Z}_{it} - B_{it} \left[\sum_{k=1}^{N} \frac{(\kappa_k[(1-\Gamma)\tau_t^2 + \sigma^2] + \Gamma\tau_t^2)^2}{\tau_t^2 + \sigma^2 + \omega^2} \frac{\breve{h}_{kt} \breve{Z}_{kt}}{\chi_t}\right].$$

From (A-42) and (A-43), $\lim_{\Theta^2 \to \infty} \breve{Z}_{it} = 1$. And from (A-40), $\lim_{\Theta^2 \to \infty} B_{it} = 0$. Therefore,

$$\lim_{\Theta^2 \to \infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = [(1-\Gamma)\tau_t^2 + \sigma^2] \sum_{j=1}^{N} \kappa_j^2 + \Gamma\tau_t^2 + \tilde{\omega}^2 - \frac{\left[\kappa_i[(1-\Gamma)\tau_t^2 + \sigma^2] + \Gamma\tau_t^2\right]^2}{\tau_t^2 + \sigma^2 + \omega^2}.$$

Simplifying, we find:

$$\lim_{\Theta^2 \to \infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t] = \tilde{\omega}^2 + [(1-\Gamma)\tau_t^2 + \sigma^2] \left(\sum_{j \neq i} \kappa_j^2 + \frac{\omega^2}{\tau_t^2 + \sigma^2 + \omega^2} \kappa_i^2 \right) + \frac{\Gamma\tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \left((1-\kappa_i)^2 [(1-\Gamma)\tau_t^2 + \sigma^2] + \omega^2 \right).$$

Now consider the second part of the corollary. If $\kappa_i = 1/N$ for all $i \in \{1, .., N\}$, then the foregoing implies that $\lim_{\Theta^2 \to \infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t | \zeta_{it} + \lambda_{it}, \tilde{q}_t]$ is independent of i. From that result, (A-50), and the definition (A-25), $\lim_{\Theta^2 \to \infty} \breve{h}_{kt} = 1/N$. And that result with $\lim_{\Theta^2 \to \infty} \breve{Z}_{it} = 1$ and the definition (A-47) implies $\lim_{\Theta^2 \to \infty} \breve{\kappa}_{kt} = 1/N$. Further, if $\kappa_i = 1/N$ for all $i \in \{1, .., N\}$, then b_{it} equals some constant b_t which, from (A-42), solves

$$\left(\tau_t^2 + \sigma^2 + \omega^2 + (N-1)\Gamma\tau_t^2\right) b_t^3 + \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)}\right)^2 \Theta^2 \frac{\tau_t^2 + \sigma^2 + \omega^2}{\tau_t^2 + \sigma^2 + \omega^2 - \Gamma\tau_t^2} \frac{N^2}{N-1} b_t$$
$$= \left(\frac{1}{N} \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + \frac{N-1}{N} \frac{\Gamma\tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right) \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r)}\right)^2 \Theta^2 \frac{\tau_t^2 + \sigma^2 + \omega^2}{\tau_t^2 + \sigma^2 + \omega^2 - \Gamma\tau_t^2} \frac{N^2}{N-1}.$$

Substitute from (A-43), writing $\breve{Z}_t$ for $\breve{Z}_{it}$:

$$\frac{1}{N^2}\left((1-\Gamma)\tau_t^2+\sigma^2+\omega^2+N\Gamma\tau_t^2\right)\left(\frac{(1-\Gamma)\tau_t^2+\sigma^2+N\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\right)^2\breve{Z}_t^3$$
$$=(1-\breve{Z}_t)\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)}\right)^2\Theta^2\frac{\tau_t^2+\sigma^2+\omega^2}{(1-\Gamma)\tau_t^2+\sigma^2+\omega^2}\frac{N^2}{N-1}.$$

Evaluating at $\breve{Z}_t = 0$ and $\breve{Z}_t = 1$ shows that there exists at least one root in the interval $(0, 1)$. For $\breve{Z}_t \in (0, 1)$, the left-hand side monotonically increases in $\breve{Z}_t$ and the right-hand side monotonically decreases in $\breve{Z}_t$. Therefore the root in the interval $\breve{Z}_t \in (0, 1)$ is unique. This root monotonically increases in Θ^2 for $\Theta^2 \in (0, \infty)$. Using $\breve{h}_{kt} = 1/N$ and the definition (A-47), we have established the second part of the corollary.

Finally, consider the third part of the corollary. Without loss of generality, order the sectors by increasing κ_i. Then $\lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \tilde{q}_t] \geq \lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{jt} + \lambda_{jt}, \tilde{q}_t]$ for all $j > i$ if, for all $\{n, n + k\}$ such that $\kappa_n < \kappa_{n+k}$,

$$\sum_{j\neq n}\kappa_j^2+\frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}\kappa_n^2+\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}(1-\kappa_n)^2$$
$$\geq\sum_{j\neq n+k}\kappa_j^2+\frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}\kappa_{n+k}^2+\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}(1-\kappa_{n+k})^2.$$

This sufficient condition is equivalent to

$$(\kappa_{n+k}^2-\kappa_n^2)+\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}\left[-2(\kappa_n-\kappa_{n+k})+\kappa_n^2-\kappa_{n+k}^2\right]\geq\frac{\omega^2}{\tau_t^2+\sigma^2+\omega^2}(\kappa_{n+k}^2-\kappa_n^2),$$

and thus is equivalent to

$$\frac{(1-\Gamma)\tau_t^2+\sigma^2}{\tau_t^2+\sigma^2+\omega^2}\frac{\kappa_n+\kappa_{n+k}}{2}\geq-\frac{\Gamma\tau_t^2}{\tau_t^2+\sigma^2+\omega^2}.$$

This condition clearly holds and thus so does the sufficient condition for $\lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{it} + \lambda_{it}, \tilde{q}_t] \geq \lim_{\Theta^2\to\infty} \breve{Var}_t[\tilde{\zeta}_t + \tilde{\lambda}_t|\zeta_{jt} + \lambda_{jt}, \tilde{q}_t]$ for all $j > i$. Therefore from (A-50), the sequence $\{h_{1t}, ..., h_{Nt}\}$ is monotone increasing as $\Theta^2 \to 0$ when sectors are ordered by increasing κ_i. And from the definition (A-25), the sequence $\{\breve{h}_{1t}, ..., \breve{h}_{Nt}\}$ is monotone increasing as $\Theta^2 \to 0$, with $\breve{h}_{1t} \leq 1/N$ and $\breve{h}_{Nt} \geq 1/N$. The latter inequalities are strict if any $\kappa_i \neq 1/N$, which in turn is equivalent to $\kappa_1 < 1/N$. The third part of the corollary follows from the foregoing, $\lim_{\Theta^2\to\infty} \breve{Z}_{it} = 1$, and the definition (A-47).

M Proof of Lemma 2

Define

$$\breve{w}_{it} \triangleq \left(\kappa_i \frac{\tau_t^2 + \sigma^2}{\tau_t^2 + \sigma^2 + \omega^2} + (1-\kappa_i) \frac{\Gamma \tau_t^2}{\tau_t^2 + \sigma^2 + \omega^2} \right) \frac{\breve{h}_{it} \breve{Z}_{it}}{\chi_t}.$$

By definitions of χ_t and $\breve{Z}_{it}$,

$$\breve{w}_{it} = \frac{\breve{h}_{it} b_{it}}{\sum_i \breve{h}_{it}[a_{it} + b_{it}]},$$

and hence $\breve{w}_{it} \in (0,1)$ with $\sum_{i=1}^N \breve{w}_{it} < 1$. Before updating for time t observations, the prior covariance matrix for the vector of unknown parameters in (A-45) is

$$V_{\mu,t} = \begin{bmatrix} (1-\Gamma)\tau_t^2 \sum_{i=1}^N \kappa_i^2 + \Gamma \tau_t^2 & (1-\Gamma)\tau_t^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} + \Gamma \tau_t^2 \sum_{i=1}^N \breve{w}_{it} \\ (1-\Gamma)\tau_t^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} + \Gamma \tau_t^2 \sum_{i=1}^N \breve{w}_{it} & (1-\Gamma)\tau_t^2 \sum_{i=1}^N \breve{w}_{it}^2 + \Gamma \tau_t^2 \left(\sum_{i=1}^N \breve{w}_{it} \right)^2 \end{bmatrix},$$

and the covariance matrix for the vector of signals in (A-46) is

$$V_{s,t} = \begin{bmatrix} \sigma^2 \sum_{i=1}^N \kappa_i^2 + \tilde{\omega}^2 & \sigma^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} \\ \sigma^2 \sum_{i=1}^N \kappa_i \breve{w}_{it} & [\sigma^2 + \omega^2] \sum_{i=1}^N \breve{w}_{it}^2 + \left(\frac{r}{C_0 \alpha (B_t' + b_t' + r) \chi_t} \right)^2 \Theta^2 \end{bmatrix}.$$

By definition, $\Upsilon_{\mu,j} = V_{\mu,j}^{-1}$ and $\Upsilon_{s,j} = V_{s,j}^{-1}$. Define $det_{\mu,j}$ and $det_{s,j}$ as the determinants of $V_{\mu,j}$ and $V_{s,j}$, respectively. These are weakly positive. Recall that $\pi_j \triangleq (\Upsilon_{\mu,j} + \Upsilon_{s,j})^{-1} \Upsilon_{\mu,j}$. Using superscripts to indicate elements, standard matrix inversion and matrix algebra yield:

$$\pi_j = \frac{1}{det_{\mu,j} + det_{s,j} + V_{\mu,j}^{(1,1)} V_{s,j}^{(2,2)} + V_{s,j}^{(1,1)} V_{\mu,j}^{(2,2)} - 2 V_{\mu,j}^{(1,2)} V_{s,j}^{(1,2)}} \begin{bmatrix} det_{s,j} + V_{s,j}^{(1,1)} V_{\mu,j}^{(2,2)} - V_{\mu,j}^{(1,2)} V_{s,j}^{(1,2)} & V_{\mu,j}^{(1,1)} V_{s,j}^{(1,2)} - V_{s,j}^{(1,1)} V_{\mu,j}^{(1,2)} \\ V_{\mu,j}^{(2,2)} V_{s,j}^{(1,2)} - V_{s,j}^{(2,2)} V_{\mu,j}^{(1,2)} & det_{s,j} + V_{\mu,j}^{(1,1)} V_{s,j}^{(2,2)} - V_{\mu,j}^{(1,2)} V_{s,j}^{(1,2)} \end{bmatrix}.$$

The leading fraction is the determinant of $V_{\mu,j} + V_{s,j}$ and thus is weakly positive. Substituting, we find that

$$\begin{aligned} V_{\mu,j}^{(1,1)} V_{s,j}^{(1,2)} - V_{s,j}^{(1,1)} V_{\mu,j}^{(1,2)} =& - \tilde{\omega}^2 \left[(1-\Gamma)\tau_j^2 \sum_{i=1}^N \kappa_i \breve{w}_{ij} + \Gamma \tau_j^2 \sum_{i=1}^N \breve{w}_{ij} \right] \\ & - \Gamma \tau_j^2 \sigma^2 \left(\left[\sum_{i=1}^N \kappa_i^2 \right] \left[\sum_{i=1}^N \breve{w}_{ij} \right] - \sum_{i=1}^N \kappa_i \breve{w}_{ij} \right), \end{aligned} \quad \text{(A-51)}$$

which goes to 0 as $\tilde{\omega}^2$ and Γ become small. We also find:

$$\begin{aligned} V_{\mu,j}^{(2,2)}V_{s,j}^{(1,2)} - V_{s,j}^{(2,2)}V_{\mu,j}^{(1,2)} =& \Gamma\tau^2\sigma^2\left(\sum_{i=1}^N \breve{w}_{ij}\right)\left[\left(\sum_{i=1}^N \breve{w}_{ij}\right)\sum_{i=1}^N \kappa_i\breve{w}_{ij} - \sum_{i=1}^N \breve{w}_{ij}^2\right] \\ &- \left[\omega^2\sum_{i=1}^N \breve{w}_{ij}^2 + \left(\frac{r}{C_0\alpha(B_j' + b_j' + r)\chi_j}\right)^2\Theta^2\right] \\ &\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{ij}\right]. \end{aligned}$$

This goes to 0 as Γ, ω^2, and Θ^2 become small. So π_j becomes approximately diagonal as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small.

Observe that:

$$\begin{aligned} & det_{s,j} + V_{s,j}^{(1,1)}V_{\mu,j}^{(2,2)} - V_{\mu,j}^{(1,2)}V_{s,j}^{(1,2)} \\ =& \left[\sigma^2\sum_{i=1}^N \kappa_i^2 + \tilde{\omega}^2\right]\left[[(1-\Gamma)\tau^2 + \sigma^2 + \omega^2]\sum_{i=1}^N \breve{w}_{ij}^2 + \Gamma\tau_j^2\left(\sum_{i=1}^N \breve{w}_{ij}\right)^2 + \left(\frac{r}{C_0\alpha(B_j' + b_j' + r)\chi_j}\right)^2\Theta^2\right] \\ & - \left[\sigma^2\sum_{i=1}^N \kappa_i\breve{w}_{ij}\right]\left[[(1-\Gamma)\tau_j^2 + \sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{ij}\right] \\ >& \left[\sigma^2\sum_{i=1}^N \kappa_i^2\right]\left[[(1-\Gamma)\tau^2 + \sigma^2]\sum_{i=1}^N \breve{w}_{ij}^2 + \Gamma\tau_j^2\left(\sum_{i=1}^N \breve{w}_{ij}\right)^2\right] \\ & - \left[\sigma^2\sum_{i=1}^N \kappa_i\breve{w}_{ij}\right]\left[[(1-\Gamma)\tau_j^2 + \sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{ij} + \Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{ij}\right]. \end{aligned}$$

As Γ becomes small, the final two lines go to

$$\begin{aligned} & \sigma^2[\tau_j^2 + \sigma^2]\left[\left(\sum_{i=1}^N \kappa_i^2\right)\left(\sum_{i=1}^N \breve{w}_{ij}^2\right) - \left(\sum_{i=1}^N \kappa_i\breve{w}_{ij}\right)^2\right] \\ =& [\tau_j^2 + \sigma^2]\lim_{\Theta^2,\omega^2,\tilde{\omega}^2\to 0} det_{s,j}. \end{aligned}$$

We also have:

$$
\begin{aligned}
& det_{s,j} + V_{\mu,j}^{(1,1)} V_{s,j}^{(2,2)} - V_{\mu,j}^{(1,2)} V_{s,j}^{(1,2)} \\
=& \left[[\sigma^2 + \omega^2] \sum_{i=1}^{N} \breve{w}_{ij}^2 + \left(\frac{r}{C_0 \alpha (B'_j + b'_j + r)\chi_j} \right)^2 \Theta^2 \right] \left[[(1-\Gamma)\tau_j^2 + \sigma^2] \sum_{i=1}^{N} \kappa_i^2 + \Gamma \tau_j^2 + \tilde{\omega}^2 \right] \\
& - \left[\sigma^2 \sum_{i=1}^{N} \kappa_i \breve{w}_{ij} \right] \left[[(1-\Gamma)\tau_j^2 + \sigma^2] \sum_{i=1}^{N} \kappa_i \breve{w}_{ij} + \Gamma \tau_j^2 \sum_{i=1}^{N} \breve{w}_{ij} \right] \\
>& \left[\sigma^2 \sum_{i=1}^{N} \breve{w}_{ij}^2 \right] \left[[(1-\Gamma)\tau_j^2 + \sigma^2] \sum_{i=1}^{N} \kappa_i^2 + \Gamma \tau_j^2 \right] \\
& - \left[\sigma^2 \sum_{i=1}^{N} \kappa_i \breve{w}_{ij} \right] \left[[(1-\Gamma)\tau_j^2 + \sigma^2] \sum_{i=1}^{N} \kappa_i \breve{w}_{ij} + \Gamma \tau_j^2 \sum_{i=1}^{N} \breve{w}_{ij} \right].
\end{aligned}
$$

As Γ becomes small, the final two lines go to

$$
\begin{aligned}
=& \left[\sigma^2 \sum_{i=1}^{N} \breve{w}_{ij}^2 \right] \left[[\tau_j^2 + \sigma^2] \sum_{i=1}^{N} \kappa_i^2 \right] - \left[\sigma^2 \sum_{i=1}^{N} \kappa_i \breve{w}_{ij} \right] \left[[\tau_j^2 + \sigma^2] \sum_{i=1}^{N} \kappa_i \breve{w}_{ij} \right] \\
=& [\tau_j^2 + \sigma^2] \lim_{\Theta^2, \omega^2, \tilde{\omega}^2 \to 0} det_{s,j}.
\end{aligned}
$$

Because $det_{s,j} \geq 0$ by properties of covariance matrices, $det_{s,j} + V_{s,j}^{(1,1)} V_{\mu,j}^{(2,2)} - V_{\mu,j}^{(1,2)} V_{s,j}^{(1,2)} > 0$ and $det_{s,j} + V_{\mu,j}^{(1,1)} V_{s,j}^{(2,2)} - V_{\mu,j}^{(1,2)} V_{s,j}^{(1,2)} > 0$. Therefore elements $(1,1)$ and $(2,2)$ of π_j are each between 0 and 1 for Γ sufficiently small.

These results imply that $\prod_{j=k+1}^{t} \pi_j$ is approximately diagonal (in that its off-diagonal elements are arbitarily small), with diagonal elements between 0 and 1, as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small.

To analyze $\tilde{\pi}_k$ and $\breve{\pi}_k$ under these same conditions, it remains to analyze the first row of $(\Upsilon_{\mu,k} + \Upsilon_{s,k})^{-1} \Upsilon_{s,k}$: the product of element $(1,1)$ of $(\Upsilon_{\mu,k} + \Upsilon_{s,k})^{-1} \Upsilon_{s,k}$ and element $(1,1)$ of $\prod_{j=k+1}^{t} \pi_j$ dominates $\tilde{\pi}_k$ as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small, and the product of element $(1,2)$ of $(\Upsilon_{\mu,k} + \Upsilon_{s,k})^{-1} \Upsilon_{s,k}$ and element $(1,1)$ of $\prod_{j=k+1}^{t} \pi_j$ dominates $\breve{\pi}_k$ as Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ become small.

Matrix algebra yields that element $(1,1)$ of $(\Upsilon_{\mu,k} + \Upsilon_{s,k})^{-1} \Upsilon_{s,k}$ is

$$
\frac{det_{\mu,k} + V_{\mu,k}^{(1,1)} V_{s,k}^{(2,2)} - V_{\mu,k}^{(1,2)} V_{s,k}^{(1,2)}}{det_{\mu,k} + V_{\mu,k}^{(1,1)} V_{s,k}^{(2,2)} - V_{\mu,k}^{(1,2)} V_{s,k}^{(1,2)} + det_{s,k} + V_{s,k}^{(1,1)} V_{\mu,k}^{(2,2)} - V_{\mu,k}^{(1,2)} V_{s,k}^{(1,2)}}.
$$

We already established that the denominator is strictly positive as Γ becomes small and that the final three terms in the denominator sum to a strictly positive number as Γ becomes small. The numerator is

$$\begin{aligned}
&\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i^2+\Gamma\tau_j^2\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2+\omega^2]\sum_{i=1}^N \breve{w}_{it}^2+\Gamma\tau_j^2\left(\sum_{i=1}^N \breve{w}_{it}\right)^2+\left(\frac{r}{C_0\alpha(B_t'+b_t'+r)\chi_t}\right)^2\Theta^2\right]\\
&-\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right]\\
>&\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i^2+\Gamma\tau_j^2\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^N \breve{w}_{it}^2+\Gamma\tau_j^2\left(\sum_{i=1}^N \breve{w}_{it}\right)^2\right]\\
&-\left[(1-\Gamma)\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right]\left[[(1-\Gamma)\tau_j^2+\sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{it}+\Gamma\tau_j^2\sum_{i=1}^N \breve{w}_{it}\right].
\end{aligned}$$

As Γ becomes small, the final two lines go to

$$\begin{aligned}
&\left[\tau_j^2\sum_{i=1}^N \kappa_i^2\right]\left[[\tau_j^2+\sigma^2]\sum_{i=1}^N \breve{w}_{it}^2\right]-\left[\tau_j^2\sum_{i=1}^N \kappa_i\breve{w}_{it}\right]\left[[\tau_j^2+\sigma^2]\sum_{i=1}^N \kappa_i\breve{w}_{it}\right]\\
=&\frac{\tau_j^2+\sigma^2}{\sigma^2}\frac{\tau_j^2}{\sigma^2}\lim_{\Theta^2,\omega^2,\tilde{\omega}^2\to 0} det_{s,j}.
\end{aligned} \tag{A-52}$$

Therefore the numerator is strictly positive, which means that element $(1,1)$ of $(\Upsilon_{\mu,k}+\Upsilon_{s,k})^{-1}\Upsilon_{s,k}$ is $\in(0,1)$ for Γ sufficiently small. Because element $(1,1)$ of $\prod_{j=k+1}^t \pi_j$ is $\in(0,1)$ and element $(1,2)$ of $\prod_{j=k+1}^t \pi_j$ is arbitarily small for Γ, ω^2, Θ^2, and $\tilde{\omega}^2$ small, we have established that $\tilde{\pi}_k\in(0,1)$ for $\Gamma,\omega^2,\tilde{\omega}^2,\Theta^2$ sufficiently small.

Element $(1,2)$ of $(\Upsilon_{\mu,j}+\Upsilon_{s,j})^{-1}\Upsilon_{s,j}$ is:

$$\frac{V_{s,k}^{(1,1)}V_{\mu,k}^{(1,2)}-V_{\mu,k}^{(1,1)}V_{s,k}^{(1,2)}}{det_{\mu,k}+V_{\mu,k}^{(1,1)}V_{s,k}^{(2,2)}-V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)}+det_{s,k}+V_{s,k}^{(1,1)}V_{\mu,k}^{(2,2)}-V_{\mu,k}^{(1,2)}V_{s,k}^{(1,2)}}.$$

From (A-51), the numerator is 0 if both Γ and $\tilde{\omega}^2$ are zero. Therefore as $\Gamma,\tilde{\omega}^2\to 0$ with ω^2 and Θ^2 not too large, $\breve{\pi}_k$ is equal to zero times element $(1,1)$ of $\prod_{j=k+1}^t \pi_j$ plus a term that scales with element $(1,2)$ of $\prod_{j=k+1}^t \pi_j$ and thus is arbitarily small.

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