# Dynamic Quantile Models of Rational Behavior*

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**Abstract**

This paper develops a dynamic model of rational behavior under uncertainty, in which the agent maximizes the stream of the future $\tau$-quantile utilities, for $\tau \in (0, 1)$. That is, the agent has a quantile utility preference instead of the standard expected utility. Quantile preferences have useful advantages, such as robustness and ability to capture heterogeneity. We provide an axiomatization of the recursive quantile preferences to motivate its use. Although quantiles do not have some of the helpful properties of expectations, such as linearity and the law of iterated expectations, we are able to establish all the standard results in dynamic models. Namely, we show that the quantile preferences are dynamically consistent, the corresponding dynamic problem yields a value function, via a fixed point argument, establish its concavity and differentiability and show that the principle of optimality holds. Additionally, we derive the corresponding Euler equation, which is well suited for using well-known quantile regression methods for estimating and testing the economic model. In this way, the parameters of the model can be interpreted as structural objects. Therefore, the proposed methods provide microeconomic foundations for quantile regression models. To illustrate the developments, we construct an asset-pricing model and estimate the discount factor and elasticity of intertemporal substitution parameters across the quantiles. The results provide evidence of heterogeneity in these parameters.

**Keywords:** Quantile utility, dynamic programing, quantile regression, asset pricing.

**JEL:** C22, C61, E20, G12

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# 1 Introduction

Modeling dynamic economic behavior has been a concern in economics for a long time (see, e.g., Samuelson (1958), Baumol (1959), Koopmans (1960), Brock and Mirman (1972)). These models are critical for learning about economic effects, incentives, and to design policy analysis. We contribute to this literature by developing a new dynamic model for an individual, who, when selecting among uncertain alternatives, chooses the one with the highest $\tau$-quantile of the stream of future utilities for a fixed $\tau \in (0,1)$, instead of the standard expected utility. This quantile preference model is tractable, simple to interpret, and substantially broadens the scope of economic applications, because it is robust to fat tails and allows to account for heterogeneity through the quantiles.[1]

Quantile preferences were first studied by Manski (1988) and axiomatized by Chambers (2009) and Rostek (2010). Manski (1988) develops the decision-theoretic attributes of quantile maximization and examines risk preferences of quantile maximizers. In the context of preferences over distributions, Chambers (2009) shows that monotonicity, ordinal covariance, and continuity characterize quantile preferences. Rostek (2010) axiomatizes the quantile preference in Savage (1954)'s framework, using a 'typical' consequence (scenario). Thus, quantile preferences are a useful alternative to the expected utility, and a plausible complement to the study of rational behavior under uncertainty.[2]

This paper initiates the use of quantile preferences in a dynamic economic setting by providing a comprehensive analysis of a dynamic rational quantile model. As a first step in the developments, and to motivate our model, we axiomatize the recursive quantile preferences. We build on the results in Bommier, Kochov, and Le Grand (2017) on monotone recursive preferences to derive the recursive quantile utility representation. The preferences induce an additively separable quantile utility model with standard discounting, that is, the recursive equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is a quantile function. In addition, we discuss the notion of risk attitude and elasticity of intertemporal substitution (EIS) in our model, and show that by using the recursive quantile preferences, it is possible to separate the notion of risk attitude from the intertemporal substitution.

We then introduce the dynamic programming for intertemporal decisions whereby the economic agent maximizes the present discounted value of the stream of future $\tau$-quantile utilities by choosing a decision variable in an feasible set. Our first main result establishes dynamic consistency of the quantile preferences, in the sense commonly adopted in decision

[1] Rostek (2010) discusses several advantages of the quantile preferences, such as robustness, ability to deal with categorical (instead of continuous) variables, and the flexibility of offering a family of preferences indexed by quantiles.

[2] Quantile preferences can be associated with Choquet expected utility (see, e.g., Chambers (2007); Bassett, Koenker, and Kordas (2004)). The method of Value-at-Risk, which is widespread in finance, also is an instance of quantiles (see, e.g., Engle and Manganelli (2004)).

theory. Second, we show that the optimization problem leads to a contraction, which therefore has a unique fixed-point. This fixed point is the value function of the problem and satisfies the Bellman equation. Third, we prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Fourth, we show that the principle of optimality holds. Fifth, using these results, we derive the corresponding Euler equation for the infinite horizon problem. To obtain our Euler equation, we offer a sufficient condition for exchanging derivative and quantile operators, which does not hold in general.

We note that the theoretical developments and derivations in this paper are of independent interest. The main results for the dynamic quantile model – dynamic consistency, value function, principle of optimality, and Euler equation – are parallel to those of the expected utility model. However, because quantiles do not share all of the convenient properties of expectations, such as linearity and the law of iterated expectations, the generalizations of the results from expected utility to quantile preference are not straightforward.

The derivation of the Euler equation is an important feature of this paper because it allows to connect economic theory with empirical applications. We show that the Euler equation has a conditional quantile representation and relates to quantile regression econometric methods, and hence, our methods provide microeconomic foundations for quantile regression. The Euler equation, which must be satisfied in equilibrium, implies a set of population orthogonality conditions that depend, in a nonlinear way, on variables observed by an econometrician and on unknown parameters characterizing the preferences. Thus, empirically, one can employ practical existing econometric methods, such as instrumental variables for nonlinear quantile regression, for estimating and testing the parameters of the model. In this fashion, these parameters can be interpreted as structural objects. In addition, varying the quantiles $\tau$ enables one to empirically estimate a set of parameters of interest as a function of the quantiles.[3]

Finally, we briefly illustrate the methods with a dynamic asset-pricing model, which is central to contemporary economics and finance.[4] We use a variation of Lucas (1978)'s model where the economic agent decides on how much to consume and save by maximizing a quantile utility function subject to a linear budget constraint. We solve the dynamic problem and obtain the Euler equation. Following a large body of literature, we specify an isoelastic utility function and estimate the implied discount factor and EIS parameters at different levels of risk attitude (quantiles). The empirical results document evidence that both parameters vary across quantiles. On the one hand, the discount factor is relatively larger for lower quantiles and smaller for upper quantiles; on the other hand, the EIS coefficient is relatively smaller for the lower quantiles and larger for the upper quantiles.

More broadly, this paper contributes to the literature by proposing methods that could be

[3] We note that the theoretical methods do not impose restrictions across quantiles, and thus the parameter estimates might (or might not) vary across quantiles.

[4] See, among others, Hansen and Singleton (1982), Campbell (2003), Cochrane (2005), and Ljungqvist and Sargent (2012), and references therein.

applied to any dynamic economic problem, substituting the standard expected utility preference by a dynamic quantile preference. This preference has the following advantages: it is robust to outliers and to monotonic transformations, and allows for a strong separation of beliefs and tastes, while still maintaining most of the useful characteristics of the standard model: dynamic consistency, probabilistic sophistication and monotonicity. Moreover, it also allows the separation between risk aversion and elasticity of intertemporal substitution, which the standard expected utility model is not able to deliver, while maintaining monotonicity, an important feature that the main method to obtain such separation (the Epstein-Zin-Weil preferences) fails to satisfy. Since dynamic economic models are now routinely used in many fields, such as macroeconomics, finance, international economics, public economics, industrial organization and labor economics, among others, our methods expand the scope of economic analysis and empirical applications, providing an alternative to expected utility models.

The remaining of the paper is organized as follows. Section 2 presents definitions and basic properties of quantiles and provides an axiomatization of quantile preferences both in the static and recursive cases. A reader that is not interested in the technical details of the decision theoretic axiomatization of the preferences can safely skip Subsections 2.2 and 2.3. Section 3 describes the dynamic economic model and presents the main theoretical results. Section 4 illustrates the empirical usefulness of the the new approach by applying it to an asset pricing model. Finally, Section 5 concludes. We relegate all proofs to the Appendix.

### 1.1 Review of the Literature

This paper has a broad scope and relates to a number of streams of literature in economic theory and econometrics.

First, the paper relates to the extensive literature on dynamic nonlinear rational expectations models. Many models of dynamic maximization that use expected utility have been proposed and discussed. These models have been workhorses in several economic fields. We refer the reader to more comprehensive works, such as Stokey, Lucas, and Prescott (1989) and Ljungqvist and Sargent (2012). Another related segment of the literature studies recursive utilities. We refer the reader to Epstein and Zin (1989), Marinacci and Montrucchio (2010), Bommier, Kochov, and Le Grand (2017), among others. We extend this literature by replacing expected utility and its variations with quantile utility.

Second, this paper is related to a few works on economic models using the quantile preferences, such as Manski (1988), Chambers (2007, 2009), Bhattacharya (2009), Rostek (2010) and Giovannetti (2013). We contribute to this line of research by taking the quantile maximization to a general dynamic optimization model and deriving its properties.

Third, the paper relates to an extensive literature on estimating Euler equations. Since the contributions of Hall (1978), Lucas (1978), Hansen and Singleton (1982), and Dunn and Singleton (1986) it has become standard in economics to estimate Euler equations based on conditional expectation models. There are large bodies of literature in micro and macroeconomics on this subject. We refer the reader to Attanasio and Low (2004) and Hall (2005), and the references therein, for a brief overview. The methods in this paper derive a Euler equation that has a conditional quantile function representation and estimate it using existing quantile regression (QR) econometric methods.

Finally, this paper relates to the QR literature, for which there is a large body of work in econometrics.[5] In a seminal paper Koenker and Bassett (1978) introduced QR methods for estimation of conditional quantile functions. These models have provided a valuable tool in economics and statistics applications to capture heterogeneous effects, and for robust inference when the presence of outliers is an issue (see, e.g., Koenker (2005)). QR has been largely used in program evaluation studies (Chernozhukov and Hansen (2005) and Firpo (2007)), identification of nonseparable models (Chesher (2003) and Imbens and Newey (2009)), nonparametric identification and estimation of nonadditive random functions (Matzkin (2003)), and testing models with multiple equilibria (Echenique and Komunjer (2009)). This paper contributes to the effort of providing microeconomic foundations for QR by developing a dynamic optimization decision model that generates a conditional quantile restriction (Euler equation).

## 2 Quantile Preferences

This section formally defines quantiles, provides an axiomatization of the static and dynamic quantile preferences and discusses the notions of risk and intertemporal substitution attitudes for these preferences. The main objective is to provide foundations and motivations for the recursive equation (equation (5) below), that is characterized by the sum of the current period utility function and the discounted value of the quantile certainty equivalent. The recursive equation is the central element in the definition of the dynamic quantile preferences, which is completed in Section 3.

The justification for the recursive equation (5) can be divided in two parts: the risk and/or uncertainty attitude, which has already been discussed in the literature about the static quantile preference, and the intertemporal attitude that is relevant for the dynamic setting.

For the static quantile preference, Rostek (2010) discusses different motivations among which are its robustness, invariance with respect to ordinal transformations of the consequences and the ability to model a family of preferences indexed by the quantiles, where the agent is governed by his/her attitude toward downside risk. The separation of tastes and believes, which is a desirable property of preferences as discussed by Ghirardato, Maccheroni, and Marinacci (2005), can be added as motivation. Quantile preferences allow for a very strong

[5] This paper is also related to an econometrics literature on identification, estimation, and inference of general conditional moment restriction models. We refer the reader to, among others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), Chen, Chernozhukov, Lee, and Newey (2014), and Chen and Liao (2015).

version of the separation between tastes and beliefs. Namely, the utility attributed to the tastes does not interfere with the beliefs part of the preference, which is robust to any monotonic transformation (and not only affine ones). These (mainly normative) motivations support our choice of quantiles as our certainty equivalent operators.

Regarding the intertemporal attitude, the ordinality that characterizes the static quantile preferences may seem at odds with the additive separability that characterizes the intertemporal aspect of (5). However, building on Bommier, Kochov, and Le Grand (2017), we show that this additive separability is a consequence of a widely accepted monotonicity property. This monotonicity property, which is well aligned with the ordinality property of quantiles, justifies our intertemporal attitude and additive separability. Therefore, taken together, one can say that ordinality and monotonicity justify our preferences.

The above aspects are formalized in the remaining of this section. Subsection 2.1 defines quantiles and establishes well-known basic results that are useful later. Subsection 2.2 offers a alternative axiomatization of the static quantile preferences. Subsection 2.3 provides an axiomatization of the recursive quantile preferences. Subsection 2.4 defines the notion of risk and elasticity of intertemporal substitution associated with the quantile preferences.

## 2.1 Preliminaries

Given a random variable (r.v.) $X$, let $F_X$ (or simply $F$) denote its cumulative distribution function (c.d.f.), that is, $F_X(\alpha) \equiv \Pr[X \leqslant \alpha]$. The quantile function $Q : [0,1] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the generalized inverse of $F$:

$$Q(\tau) \equiv \begin{cases} \inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant \tau\}, & \text{if } \tau \in (0,1] \\ \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}, & \text{if } \tau = 0. \end{cases}$$

The definition is special for $\tau = 0$ so that the quantile assumes a value in the support of $X$.[6] It is clear that if $F$ is invertible, that is, if $F$ is strictly increasing, its generalized inverse coincide with the inverse, that is, $Q(\tau) = F^{-1}(\tau)$. Usually, it will be important to highlight the random variable to which the quantile refers. In this case we will denote $Q(\tau)$ by $Q_\tau[X]$. For convenience, throughout the paper we will focus on $\tau \in (0,1)$, unless explicitly stated.

In Lemma A.1 in the appendix, we develop some useful properties of quantiles, such as the fact that it is left-continuous and $F(Q(\tau)) \geqslant \tau$. Another well-known and useful property of quantiles is "invariance" with respect to monotonic transformations, that is, if $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, then

$$Q_\tau[g(X)] = g\left(Q_\tau[X]\right). \tag{1}$$

[6] Indeed, $\inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant 0\} = -\infty$, no matter what is the distribution.

For $\tau \in (0,1]$, the conditional quantile of $X$ with respect to $Z$ is defined as:

$$Q_\tau[X|z] \equiv \inf\{\alpha \in \mathbb{R} : \Pr([X \leqslant \alpha] | Z = z) \geqslant \tau\}.$$

Lemma A.2, in the appendix, generalizes (1) to conditional quantiles. More precisely, Lemma A.2 proves that if $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ is non-decreasing and left-continuous in $Z \in \mathcal{Z}$, then,

$$Q_\tau[g(\theta,\cdot)|Z = z] = g(\theta, Q_\tau[X|Z = z]).$$

This property is repeatedly used in the rest of the paper.

## 2.2 Static Quantile Preferences

This subsection provides an axiomatization of the static quantile preferences. Manski (1988) was the first to study such preferences, which was recently axiomatized by Chambers (2009) and Rostek (2010). Rostek (2010) axiomatized the quantile preferences in the context of Savage (1954)'s subjective framework. Rostek (2010) modifies Savage's axioms to show that they are equivalent to the existence of a $\tau \in (0,1)$, probability measure and a quantile utility function.[7,8] In contrast, Chambers (2009) departs from a framework where the utility function and the probability distributions are already fixed (risk setting, in contrast with Rostek's uncertainty setting). He shows that the preference satisfies monotonicity, ordinal covariance, and continuity if and only if the preference is a quantile preference.[9]

We want to work with the uncertainty setting, but for a technical reason, we cannot appeal directly to Rostek (2010)'s results. The reason is that she works with a necessarily infinite state space (because of her event continuity axiom). This is not suitable for our extension to the dynamic case, which is based on Bommier, Kochov, and Le Grand (2017): they work with a finite state space. Therefore, we build on Chambers (2007) and provide an alternative set of axioms for the one-period preference. In doing so, we contribute to the literature by introducing a new axiom, which we call ordinality - axiom Q4 below, that is central to specifying the quantile preference.

### 2.2.1 Axiomatization of Static Quantile Preference

Let $S$ be a finite set and let $\Sigma = 2^S$ denote the set of all its subsets. The following definitions and notation will be useful below. Any $E \in \Sigma$ is called an *event*. The topological space $X$ is

[7]If $\tau \in \{0,1\}$, the statement is more complex; see her paper for details.

[8] Rostek (2010) also shows that the quantiles preferences are probabilistic sophisticated for $\tau \in (0,1)$, by using a variation of the original concept of probabilistic sophistication introduced by Machina and Schmeidler (1992).

[9]Since the upper semicontinuity property is a technical condition and first-order stochastic dominance is a mild property, also satisfied by expected utility, the really important property is invariance with respect to monotonic transformations. We have stated this property in equation (1). Thus, this property could be considered the essence of the quantile preference considered here.

called the set of *consequences*. Any function $f : S \to X$ is called an *act* and $\mathcal{F}$ denotes the set of all acts. Endow $\mathcal{F}$ with the product topology of $X^S$. An event $E$ is *null* if for all $f, g \in \mathcal{F}$, if $(f(s) = g(s), \forall s \notin E)$ implies $f \sim g$. We denote by $xEy$ the act $f \in \mathcal{F}$ defined by $f(s) = x$ if $s \in E$ and $f(s) = y$ if $s \notin E$. As usual, $X$ is seen a subset of $\mathcal{F}$. Also as usual, we write $f \succ g$ if $(f \succcurlyeq g) \wedge \neg(g \succcurlyeq f)$, $f \sim g$ if $(f \succcurlyeq g) \wedge (g \succcurlyeq f)$, $f \preccurlyeq g$ if $g \succcurlyeq f$ and analogously for $\prec$.

Our first three axioms are standard.

**Axiom Q1 (Weak order).** The binary relation $\succcurlyeq$ is complete and transitive on $\mathcal{F}$.

**Axiom Q2 (Continuity).** For any $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} : g \succcurlyeq f\}$ and $\{g \in \mathcal{F} : f \succcurlyeq g\}$ are closed.

**Axiom Q3 (Monotonicity).** For any $f, g \in \mathcal{F}$, $(f(s) \succcurlyeq g(s), \forall s \in S) \Rightarrow f \succcurlyeq g$.

The next axioms are the ones that allow to characterize quantile preferences. We need the following definition: we say that a function $\varphi : X \to X$ as *increasing* if $x \succcurlyeq y \Rightarrow \varphi(x) \succcurlyeq \varphi(y)$.

**Axiom Q4 (Ordinality).** For any increasing $\varphi : X \to X$, we have

$$f \succcurlyeq g \Rightarrow \varphi(f) \succcurlyeq \varphi(g). \tag{2}$$

Axiom Q4 requires that the preference is ordinal, that is, the only importance for the ranking of any act is the order of the consequences, not its cardinal value. Since $\varphi$ is not restricted to be "strictly increasing", the implication is required in just one direction. Of course, if $\varphi$ is "strictly increasing" with inverse $\varphi^{-1}$, then we could apply Q4 to $\varphi^{-1}$ to obtain $f \succcurlyeq g \Leftrightarrow \varphi(f) \succcurlyeq \varphi(g)$.[10]

Ordinality may seem unusual as an axiom for increasing functions $\varphi : X \to X$. However, the ordinality is similar to other axiom previously used in the literature. To see this, let $\mathcal{C}$ denote the set of functions $\varphi : X \to X$ for which there exists $y \in X$ such that $\varphi(x) = \alpha x + (1-\alpha)y, \forall x \in X$. Then, the weak certainty independence axiom used by Maccheroni, Marinacci, and Rustichini (2006a) is just the requirement that (2) holds for all $\varphi \in \mathcal{C}$. Moreover, it seems possible to rephrase ordinality in terms of comonotonicity (as used by Schmeidler (1989)), since $f$ and $\varphi(f)$ are comonotonic for an increasing $\varphi$. The above form seems simpler, however.

It should also be noted that ordinality allows full separation of beliefs and tastes, a characteristic that is desirable but not obtained in other models. Indeed, Ghirardato, Maccheroni, and Marinacci (2005) observe that economists often operate under an implicit assumption that the tastes of a decision maker are quite stable, while his beliefs change with the availability of

[10]This equivalence holds for a quantile preference because we focus on simple acts. In the general case, it is necessary to require extra conditions, as left-continuity; see Lemma A.2 in the appendix.

new information. They attempt to offer a result with this separation, but they are able to obtain only a partial separation, with their certainty independence axiom. Ordinality guarantees the desired full separation of beliefs and tastes.

Ordinality is the central axiom of our axiomatization of static quantile preferences. It is also our main contribution to this axiomatization, since our proof follows closely that of Chambers (2007).[11]

The following axiom is taken directly from Chambers (2007) and its role is to allow to obtain a probability that defines the preference as a quantile preference. Axioms Q1-Q4 alone are not sufficient for this conclusion.

**Axiom Q5 (Betting Consistency).** Let $\{A_1, ..., A_n\} \subset \Sigma$ and $\{B_1, ..., B_n\} \subset \Sigma$ be such that $\sum_{i=1}^n 1_{A_i} \geqslant \sum_{i=1}^n 1_{B_i}$. Then, there exists $\overline{x}, \underline{x} \in X$ and $i \in \{1, ..., m\}$ such that $\overline{x}A_i\underline{x} \succcurlyeq \overline{x}B_i\underline{x}$.

We refer the reader to Chambers (2007) for a discussion and justification of this axiom. Our axiomatization of quantile preferences is provided through the following:

**Proposition 2.1.** *The preference $\succcurlyeq$ satisfies axioms Q1-Q5 if and only if there exists utility function $u : X \to [0,1]$, probability $p : \Sigma \to [0,1]$ and $\tau \in (0,1)$ such that $U : \mathcal{F} \to [0,1]$ represents $\succcurlyeq$, where*

$$U(f) \equiv \inf\{\alpha : p(\{s \in S : u(f(s)) \geqslant \alpha\}) \leqslant \tau\}. \tag{3}$$

### 2.3 Recursive Quantile Preferences

Now we provide an axiomatization of the recursive quantile preferences. To do so, we will apply the results in Bommier, Kochov, and Le Grand (2017). We restrict the analysis to the particular case of stationary IID Markov.[12] The extension of the results to a general Markov case is left for future work. The notion of stationary IID relies on the following: (i) restriction of the analysis to cases where the passing of time has no impact on the structure of the domain of choice; and (ii) introduction of a set of assumptions implying that a decision maker who uses, at all dates, the same history independent preference relation is time-consistent. The setup and notation are taken verbatim from Bommier, Kochov, and Le Grand (2017).

#### 2.3.1 Setup and Notation

Let S be a finite set representing the states of the world to be realized in each period. We note that for the axiomatization of the dynamic preferences we work with a finite space $S$, which is required by Bommier, Kochov, and Le Grand (2017). A finite space $S$ is also used in

[11] Ordinality is simpler than the alternative proposed by Chambers (2007, footnote 2, p. 422).

[12] The notion of IID was first introduced in Epstein and Schneider (2003a) in the context of ambiguity for the case of max-min expected utility representation.

Maccheroni, Marinacci, and Rustichini (2006b). We assume that $S$ has at least three elements and let $\Sigma := 2^S$ be the associated algebra of events. The full state space is $\Omega \equiv S^\infty$, with a state $\omega \in \Omega$ specifying a complete history $(s_1, s_2, \dots)$. In each period $t > 0$, the individual knows the partial history $(s_1, \dots, s_t)$. Such knowledge can be represented by a filtration $\mathcal{G} = (\mathcal{G}_t)_t$ on $\Omega$ where $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and, for every $t > 0$, $\mathcal{G}_t := \Sigma^t \times \{\emptyset, S\}^\infty$. We again let $C = [\underline{c}, \bar{c}]$ be the set of all possible consumption levels. A consumption plan, or an act, is a $C$-valued, $\mathcal{G}$-adapted stochastic process, that is, a sequence $h = (h_0, h_1, \dots)$ such that $h_t : \Omega \to C$ is $\mathcal{G}_t$-measurable for every $t$. The set of all consumption plans is denoted by $\mathcal{H}$ and endowed with the topology of pointwise convergence.

We consider a binary relation $\succcurlyeq$ on $\mathcal{H}$ and introduce a set of axioms. Some notation is needed first. Given an act $h \in \mathcal{H}$ and state $\omega \in \Omega$, let $h(\omega) \in C^\infty$ be the deterministic consumption stream induced by $h$ in state $\omega \in \Omega$, that is, $h(\omega) = (h_0, h_1(\omega), \dots)$. Moreover, for any act $h \in \mathcal{H}$ and any $s \in S$, we define the conditional act $h^s \in \mathcal{H}$ by

$$\forall \omega = (s_1, s_2, \dots) \in \Omega : h^s(s_1, s_2, \dots) = h(s, s_2, \dots) = (h_0, h_1(s, s_2, \dots), \dots).$$

The act $h^s$ is obtained from $h$ when knowing that the first component of the state of the world is equal to $s \in S$. Notice that $h^s(s_1, s_2, \dots)$ is independent of $s_1$. Given $h = (h_0, h_1, h_2, \dots) \in \mathcal{H}$, let $h^1$ be defined by

$$\forall \omega = (s_1, s_2, \dots) \in \Omega : h^1(s_1, s_2, \dots) = (h_1(s_1, s_2, \dots), h_2(s_1, s_2, \dots) \dots).$$

The set of those $h^1$ can be identified with $\mathcal{H}^S$.

Last, for any $c \in C$ and $h \in \mathcal{H}$, we define the concatenated act $(c, h) \in \mathcal{H}$ by

$$(c, h) : \omega = (s_1, s_2, \dots) \in \Omega \mapsto (c, h)(\omega) = (c, h(s_2, \dots)) \in C^\infty.$$

The notions of conditional, continuation, and concatenated acts are related to each other. In particular, the conditional act is the concatenation of first-period consumption and the continuation act. Formally, for $h = (h_0, h_1, \dots) \in \mathcal{H}$ and $s \in S$, we have $h^s = (h_0, h^{s,1})$. Moreover, any concatenated act $(c, h)$ has continuation $h$. In mathematical terms, for any $c \in C$, $h \in \mathcal{H}$, and $s \in S$, we have $(c, h)^{s,1} = h$.

### 2.3.2 Axioms

We borrow the axioms below from Bommier, Kochov, and Le Grand (2017). See their paper for a discussion and motivation of these axioms.

**Axiom D1 (Weak Order).** The binary relation $\succcurlyeq$ is complete and transitive on $\mathcal{H}$.

**Axiom D2 (Continuity).** For all $h \in \mathcal{H}$, the sets $\{\hat{h} \in \mathcal{H} \mid \hat{h} \succcurlyeq h\}$ and $\{\hat{h} \in \mathcal{H} \mid h \succcurlyeq \hat{h}\}$ are closed in $\mathcal{H}$.

**Axiom D3 (Recursivity).** For all acts $h = (h_0, h_1, \dots)$ and $\hat{h} = (\hat{h}_0, \hat{h}_1, \dots)$ in $\mathcal{H}$ with $h_0 = \hat{h}_0$,

$$\left(\forall s \in S, h^s \succcurlyeq \hat{h}^s\right) \implies h \succcurlyeq \hat{h}.$$

If, in addition, one of the former rankings is strict, then the latter ranking is strict.

**Axiom D4 (History Independence).** For all acts $h = (h_0, h_1, \dots)$ and $\hat{h} = (h_0, \hat{h}_1, \dots)$ in $\mathcal{H}$, and $\hat{h}_0 \in C$,

$$(h_0, h_1, \dots) \succcurlyeq (h_0, \hat{h}_1, \dots) \iff (\hat{h}_0, h_1, \dots) \succcurlyeq (\hat{h}_0, \hat{h}_1, \dots).$$

**Axiom D5 (Stationarity).** For all $c \in C$ and $h, \hat{h} \in \mathcal{H}$, we have $(c, h) \succcurlyeq (c, \hat{h}) \iff h \succcurlyeq \hat{h}$.

**Axiom D6 (Monotonicity for Deterministic Prospects).** For all $c^\infty$, $\hat{c}^\infty \in C^\infty$, if $c^\infty \geq \hat{c}^\infty$, then $c^\infty \succcurlyeq \hat{c}^\infty$. The latter ranking is strict whenever $c^\infty \gneq \hat{c}^\infty$.

**Axiom D7 (Monotonicity).** For any $h$ and $\hat{h}$ in $\mathcal{H}$,

$$\left(h(\omega) \succcurlyeq \hat{h}(\omega) \text{ for all } \omega \in \Omega\right) \implies h \succcurlyeq \hat{h}.$$

### 2.3.3 Characterization of Monotone Preferences

Let $B_0(\Sigma)$ be the set of simple $\Sigma$-measurable functions from $S$ into $\mathbb{R}_+$. A function $I : B_0(\Sigma) \to \mathbb{R}_+$ is a *certainty equivalent* if it is continuous, strictly increasing and satisfies $I(x) = x$ for any $x \in \mathbb{R}_+$, where we see real numbers as constant functions in $B_0(\Sigma)$. A certainty equivalent $I : B_0(\Sigma) \to \mathbb{R}_+$ is *translation-invariant* if $I(x + f) = x + I(f)$ for any $x \in \mathbb{R}_+$ and $f \in B_0(\Sigma)$ and it is *scale-invariant* if for all $\lambda \in \mathbb{R}_+$ and $f \in B_0(\Sigma)$, $I(\lambda f) = \lambda I(f)$. Given a function $V : \mathcal{H} \to \mathbb{R}$ and an act $h \in \mathcal{H}$, we let $V \circ h^1$ denote the function $s \mapsto V(h^{s,1})$. If $V$ is a utility function, then $V \circ h^1$ is the state-contingent profile of continuation utilities induced by the act $h$ in period 1. A *time aggregator* is a function $W : C \times [0,1] \to [0,1]$.

We say that $\succcurlyeq$ admits a *recursive representation* $(V, W, I)$ if

$$h \succcurlyeq h' \iff V(h) \geq V(h'), \tag{4}$$

where $V : \mathcal{H} \to [0,1]$, $W$ is a time aggregator and $I$ is a certainty equivalent satisfying

$$V(h) = W(h_0, I(V \circ h^1)).$$

Bommier, Kochov, and Le Grand (2017)'s Proposition 4 is the following:

**Proposition 2.2** (Bommier-Kochov-Le Grand)**.** *A binary relation $\succcurlyeq$ on $\mathcal{H}$ satisfies axioms D1-D7 if and only if it admits a recursive representation $(V, W, I)$ such that either:*

1. $I$ *is translation-invariant and* $W(c, x) = u(c) + \beta x$*, where* $\beta \in (0, 1)$ *and* $u : C \to [0, 1]$ *is a continuous, strictly increasing function such that* $u(\underline{c}) = 0$ *and* $u(\bar{c}) = 1 - \beta$*, or*

2. $I$ *is translation-invariant and scale-invariant, and* $W(c, x) = u(c) + b(c)x$*, where* $u, b : C \to [0, 1]$ *are continuous functions such that* $b(C) \subset (0, 1)$*, the functions* $u$ *and* $u + b$ *are strictly increasing, and* $u(\underline{c}) = 0$ *and* $u(\bar{c}) = 1 - b(\bar{c})$*.*

### 2.3.4 Axiomatization of Recursive Quantile Preferences

Let $\Omega = S^\infty$, $C$ and $\mathcal{H}$ be as in Subsection 2.3.1. Consider the following axioms:

**Axiom A1 (Time Separability for Deterministic Prospects).** For all $c^\infty, \hat{c}^\infty \in C^\infty$, $c, c', d, d' \in C$, we have:

$$(c, d, c^\infty) \succcurlyeq (c', d', c^\infty) \Leftrightarrow (c, d, \hat{c}^\infty) \succcurlyeq (c', d', \hat{c}^\infty).$$

**Axiom A2 (Ordinality).** For any $c_0 \in C$, functions $f, g : S \to C$, $h = (h_0, h_1, h_2, ...) \in \mathcal{H}$, and increasing $\varphi : C \to C$,

$$(c_0, f(\cdot), h_2(\cdot), h_3(\cdot), ...) \succcurlyeq (c_0, g(\cdot), h_2(\cdot), h_3(\cdot), ...)$$
$$\Rightarrow (c_0, \varphi(f(\cdot)), h_2(\cdot), h_3(\cdot), ...) \succcurlyeq (c_0, \varphi(g(\cdot)), h_2(\cdot), h_3(\cdot), ...).$$

**Axiom A3 (Betting Consistency).** Let $\{A_1, ..., A_n\} \subset 2^S$ and $\{B_1, ..., B_n\} \subset 2^S$ be such that $\sum_{i=1}^n 1_{A_i} \geqslant \sum_{i=1}^n 1_{B_i}$. Then, there exists $c, c' \in C$ and $i \in \{1, ..., m\}$ such that $\mathbf{c}A_i\mathbf{c}' \succcurlyeq \mathbf{c}A_i\mathbf{c}'$, where $\mathbf{c} = (c, c, ...)$ and $\mathbf{c}' = (c', c', ...)$.

Axiom A1 is just Koopmans (1972)'s Postulate 3". Note that this is required only for deterministic prospects. A version of A1 has been required by Epstein and Schneider (2003b), Maccheroni, Marinacci, and Rustichini (2006b), Kochov (2015), among others. This axiom allows us to show that in Proposition 2.2, $b(c) = \beta \in (0, 1)$, that is, the discounting factor is a constant and does not depend on the consumption.

Axioms A2 and A3 are adaptations of axioms Q4 and Q5, respectively, which allow us to show that the certainty equivalent $I[\cdot]$ in Proposition 2.2 is in fact a quantile. We are now able to state and prove our axiomatization result:

**Proposition 2.3.** *A binary relation* $\succcurlyeq$ *on* $\mathcal{H}$ *satisfies axioms D1-D7 and A1-A3 if and only if there exist a utility function* $u : X \to \mathbb{R}$, *a probability* $p : \Sigma \to [0,1]$ *on* $\Sigma = 2^S$ *and* $\beta, \tau \in (0,1)$ *such that* $\succcurlyeq$ *admits a recursive representation* $(V, W, I)$, *where* $W(c,x) = u(c) + \beta x$ *and* $I : B_0(\Sigma) \to \mathbb{R}_+$ *is given by* $I(b) = Q^p_\tau[b]$, *for any* $b \in B_0(\Sigma)$. *That is,* $V$ *satisfying the recursive equation*

$$V(h) = u(h_0) + \beta Q^p_\tau[V(h^1)], \tag{5}$$

*represents* $\succcurlyeq$, *where* $h = (h_0, h_1, \ldots) \in \mathcal{H}$ *and* $h^1 = (h_1, h_2, \ldots) \in \mathcal{H}^S$.

It is important to discuss some aspects of Proposition 2.3 and the representation it delivers. Focusing only on the representation itself and ignoring the axioms that imply it, one could be tempted to describe it as combining cardinal and ordinal properties. With respect to the time component, the representation seems to have a "cardinal" flavor, in the sense that $W$ is given by an additively separable cardinal expression: $W(c,x) = u(c) + \beta x$. On the other hand, with respect to uncertainty, the certainty equivalent $I[\cdot] = Q^p_\tau[\cdot]$ is purely ordinal. An inspection of the axioms, however, reveal that these properties come, essentially, from monotonicity (D7) and ordinality (Q4). One could say that these axioms share the common principle of respecting the order on time, events and consequences and are, therefore, harmonious. Monotonicity with respect to deterministic prospects (D6) and Monotonicity (D7) indeed imply that $W(c,x) = u(c) + b(c)x$, according to Proposition 2.2. Ordinality and betting consistency, that come from A2 and A3, imply that the certainty equivalent is of the quantile form. Ordinality, which is the central axiom for this result, essentially requires that only the order of the consequences (not its cardinal index) matters for the rank of uncertain prospects. This also allows to separate beliefs and tastes, as commented above in Subsection 2.2.1, in the discussion after axiom Q4. It remains only to restrict $b(c)$ to be constant, which is obtained with the time separability of deterministic prospects (A1). In sum, the additive separability of the recursive function $W$ and the ordinality of the certainty equivalent $I[\cdot]$ are characteristics that combine well leading to our model.

## 2.4 Risk and Intertemporal Attitudes in the Quantile Model

In this section, we discuss risk and intertemporal attitudes in the recursive quantile model.

### 2.4.1 Risk Attitude in the Static Quantile Model

The risk attitude in the quantile model was studied by Manski (1988) and Rostek (2010). They show this model admits a notion of comparative risk attitude, related to $\tau$. Rostek (2010) argues that quantile maximizers are concerned with downside risk, which can be defined as follows, for the case of real valued consequences.

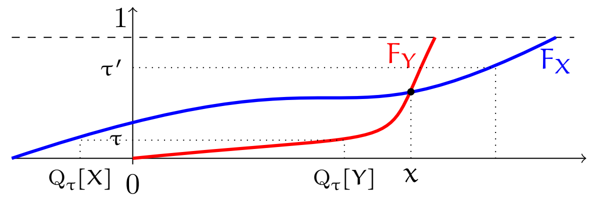



Figure 1: $X$ has more downside risk than $Y$.

**Definition 2.4.** 1. *We say that $F_Y$ crosses $F_X$ from below at $x$ if*

$$\text{(i) } F_Y(y) \leqslant F_X(y), \forall y < x \text{ and (ii) } F_Y(y) \geqslant F_X(y), \forall y > x. \tag{6}$$

*In this case, we say that $X$ involves more downside risk than $Y$ with respect to $x$.*

2. *The class of all r.v. with the single-crossing property is the set $\mathcal{SC} = \{(X,Y) : F_Y$ crosses $F_X$ from below at some $x\}$.*

3. *We say that $\succcurlyeq^1$ is more risk-averse than $\succcurlyeq^2$ if, for all pairs $(X,Y) \in \mathcal{SC}$,*

$$Y \succcurlyeq^2 X \Rightarrow Y \succcurlyeq^1 X. \tag{7}$$

Figure 1 illustrates (6), that is, that $F_Y$ crosses $F_X$ from below at $x$. Notice that $X$ is more widespread than $Y$, which justifies the notion that it is riskier or that it involves more downside risk than $Y$. The picture also helps to understand (7) and Proposition 2.5 below.

Intuitively, this comparative notion allows ranking the attractiveness of distributions by comparing the likelihood of losses with respect to outcome $x$. The following result establishes the connection between the risk attitude and quantiles; see Rostek (2010, Section 6.1) and Manski (1988, Section 5) for discussion.

**Proposition 2.5** (Rostek, 2010)**.** *In the Quantile Maximization model, $\tau < \tau'$ if and only if a $\tau$-maximizer is weakly more averse toward downside risk than a $\tau'$-maximizer.*

Proposition 2.5 shows that $\succcurlyeq^\tau$ is more risk-averse than $\succcurlyeq^{\tau'}$ if and only if $\tau < \tau'$. Thus, a decision maker that maximizes a lower quantile is more "risk-averse" than one who maximizes a higher quantile. In other words, the risk-attitude can be related to the quantile rather than to the concavity of the utility function. To understand this, fix $u(x) = x^\rho$ and remember that (1) implies $\varphi_\tau^\rho(X) \equiv Q_\tau[(X)^\rho] = (Q_\tau[X])^\rho$. Thus, $\varphi_\tau^\rho$ and $\varphi_\tau^{\rho'}$ represent the same preference, for any $\rho, \rho' > 0$, that is,

$$X \succcurlyeq Y \iff \varphi_\tau^\rho(X) \geqslant \varphi_\tau^\rho(Y) \iff Q_\tau[X] \geqslant Q_\tau[Y] \iff \varphi_\tau^{\rho'}(X) \geqslant \varphi_\tau^{\rho'}(Y). \tag{8}$$

Therefore, we can represent this preference just by $\varphi_\tau(X) \equiv Q_\tau[X]$. However, if $\varphi^\rho(X) \equiv$

$E[(X)^\rho]$ and $0 < \rho < \rho' < 1$, then $\varphi_\rho$ represents a more risk-averse preference (in the standard sense) than $\varphi_{\rho'}$.

### 2.4.2 Risk Attitude in the Dynamic Quantile Model

Now we discuss the notion of risk attitude in the dynamic quantile model. For this discussion, consider preferences $\succcurlyeq^i$, $i \in \{1,2\}$, satisfying axioms D1-D7 and A1, so that they are represented by $V^i$ satisfying the following recursive equation:

$$V^i(h) = u(h_0) + \beta I^i[V^i(h^1)].$$

As Epstein and Zin (1989), we adapt definition (7) for the dynamic case as follows: we say that $\succcurlyeq^1$ is more risk averse than $\succcurlyeq^2$ if, for all $c^\infty \in C^\infty$ and $h \in \mathcal{H}$,

$$c^\infty \succcurlyeq^2 h \implies c^\infty \succcurlyeq^1 h. \tag{9}$$

Observe that if $Y$ is a deterministic prospect, it crosses from below any other distribution, so that (7) holds. Since $c^\infty$ is a deterministic prospect, this justifies (9). We have the following:

**Lemma 2.6.** *$\succcurlyeq^1$ is more risk averse than $\succcurlyeq^2$ if and only if* $I^1[\cdot] \leqslant I^2[\cdot]$.

Specializing the above result to the quantile case, we conclude that $\succcurlyeq^1$ is more risk averse than $\succcurlyeq^2$ if and only if $I^1(\cdot) = Q_{\tau^1}[\cdot] \leq Q_{\tau^2}[\cdot] = I^2(\cdot)$ which is equivalent to $\tau^1 \leq \tau^2$. This is exactly the notion obtained in Proposition 2.5 above. Hence, as in Manski (1988) and Rostek (2010), the dynamic quantile model admits a notion of comparative risk attitude, where $\tau$ captures a measure of risk attitude, but agents are not characterized as risk-averse, risk-neutral, or risk-seeking. Moreover, this definition of risk allows for the risk attitudes to be disentangled from the degree of intertemporal substitutability, as we discuss next.

### 2.4.3 Timing of Resolution of Uncertainty and Intertemporal Substitution

Since Kreps and Porteus (1978) and Epstein and Zin (1989), it is well understood that a separation of risk and intertemporal attitudes is possible only if the timing of the resolution of uncertainty matters. More recently, Bommier, Kochov, and Le Grand (2017, Proposition 3) show that scale-invariant certainty equivalents generate what they call restricted indifference toward the timing of resolution of uncertainty. This is, in a sense, the weakest form of indifference toward the timing of the resolution of uncertainty that still accommodates the separation of risk and intertemporal substitution attitudes. Since the quantile certainty equivalent operator is scale-invariant, it belongs to this selected class and thus allows for this separation.

We illustrate how this separation can be achieved. Consider the utility index $u(c) = c^\rho$. If $\rho \in (0,1)$ this corresponds to the case of risk aversion in the expected utility model and if $\rho^1 < \rho^2$, 1 is more risk averse than 2, in the sense that he has a higher coefficient of relative risk aversion. However, in the static quantile preferences, any $\rho > 0$ leads to exactly the same choices, as discussed above; see (8). In other words, the parameter $\rho$ does not capture any aspect of the decision maker attitude towards risk.[13] If we have multiple periods, however, the parameter $\rho$ plays an important role. Indeed, consider equation (5) with the same utility index above, that is,

$$V(c_0, \tilde{c}_1) = c_0^{\rho} + \beta Q_{\tau}\left[\tilde{c}_1^{\rho}\right].$$

Applied to a deterministic prospect, that is, $\tilde{c}_1 = c_1$, this yields $c_0^{\rho} + \beta c_1^{\rho}$. It is easy to see that the elasticity of intertemporal substitution (EIS) in this case is simply $\frac{1}{1-\rho}$. Subsection 4.4 illustrates how to estimate the EIS with our dynamic quantile model using standard methods.

It is useful to compare our method with the most widely used method to separate risk aversion and the EIS, which is the following specification of Epstein and Zin (1989) and Weil (1990), with $\rho \neq 0$, $\alpha \neq 0$:

$$V^{EZ}(c_0, \tilde{c}_1) = \left(c_0^{\rho} + \beta\left(E\left[\tilde{c}_1^{\alpha}\right]\right)^{\frac{\rho}{\alpha}}\right)^{\frac{1}{\rho}}.$$

As observed by Bommier, Kochov, and Le Grand (2017), this model satisfies monotonicity if and only if $\rho = \alpha$, in which case the model collapses to the standard expected utility model, where the separation of risk aversion and EIS is not possible. In other words, for achieving its goal this popular Esptein-Zin-Weil preferences is necessarily non-monotonic. Bommier, Kochov, and Le Grand (2017, Lemmas 2 and 3) illustrate some of the problems that arise from this lack of monotonicity. In short, a Epstein-Zin-Weil decision maker may prefer to reduce lifetime utility in all states of the world, just out of his willingness to reduce risk. In contrast, the willingness to reduce risk by a decision maker with monotonic preferences will never lead him to reduce lifetime utility in all states of the world. This seems, therefore, a shortcoming of those preferences. Since dynamic quantile preferences are monotonic, they are immune to this criticism.

# 3 Economic Model and Theoretical Results

This section describes a dynamic economic model and develops a dynamic program theory for quantile preferences. We try to follow closely Stokey, Lucas, and Prescott (1989, chapter 9). Our first task is to define the dynamic quantile preference over plans from the recursive equation (5). This is necessary because (5) only describes the preference from one period to the next. We begin in Subsection 3.1 by defining a general dynamic environment, suitable for our analysis. Subsection 3.2 states and discusses the assumptions used for establishing the main results. Subsection 3.3 establishes the existence of recursive functions, necessary to extend

[13]As discussed in subsections 2.4.1 and 2.4.2, the attitude towards risk is captured by $\tau$ in both the static and dynamic cases.

the preferences from recursive equation (5) to the time contingent preferences. Subsection 3.4 shows that the preference is dynamically consistent. In Subsection 3.5 we establish the existence of the value function and its differentiability. Subsection 3.6 states and proves, in our context, the Bellman's Principle of Optimality, which allows to pass from plans to single period decisions and vice-versa, thus establishing that the value function corresponds to the original dynamic problem in a precise sense. Finally, Subsection 3.7 derives the Euler equation associated to this dynamic problem, which describes the agents behavior and is useful for the econometric part of the paper.

The main results in this section are generalizations to the quantile preferences' case of the corresponding ones in Stokey, Lucas, and Prescott (1989), which focus on expected utility. First, our results increase the scope of potential applications of economic models substantially by using quantile utility. Second, the generalizations are of independent interest. The demonstrations are not routine since quantiles do not possess several of the convenient properties of expectations, such as linearity and the law of iterated expectations.

## 3.1 Dynamic Model

In Subsection 2.3, we axiomatized the recursive equation (5). This equation describes the preference from one period to the next; it is now necessary to show how it determines the preferences over plans. To do this, we need some definitions and new notation.

### 3.1.1 States and Shocks

Let $\mathcal{X} \subset \mathbb{R}^p$ denote the state space, and $\mathcal{Z} \subseteq \mathbb{R}^k$ the range of the shocks (random variables) in the model. Let $x_t \in \mathcal{X}$ and $z_t \in \mathcal{Z}$ denote, respectively, the state and the shock in period $t$, both of which are known by the decision maker at the beginning of period $t$. We may omit the time indexes for simplicity, when it is convenient. Let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ ($t$-times, for $t \in \mathbb{N}$), $\mathcal{Z}^\infty = \mathcal{Z} \times \mathcal{Z} \times \cdots$ and $\mathbb{N}^0 \equiv \mathbb{N} \cup \{0\}$. Given $z \in \mathcal{Z}^\infty$, $z = (z_1, z_2, \ldots)$, we denote $(z_t, z_{t+1}, \ldots)$ by ${}_t z$ and $(z_t, z_{t+1}, \ldots, z_{t'})$ by ${}_t z_{t'}$. A similar notation can be used for $x \in \mathcal{X}^\infty$.

The random shocks will follow a time-invariant (stationary) Markov process. More precisely, a probability density function (p.d.f.) $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ establishes the dependence between $Z_t$ and $Z_{t+1}$, such that the process is invariant with respect to $t$. For simplicity of notation, we will frequently represent $Z_t$ and $Z_{t+1}$ by $Z$ and $Z'$, respectively. We will assume that $f$ and $\mathcal{Z}$ satisfy standard assumptions, as explicitly stated below in Subsection 3.2.

For any topological space $\mathcal{S}$, we will denote by $\sigma(\mathcal{S})$ the Borel $\sigma$-algebra. For each $z \in \mathcal{Z}$ and $A \in \sigma(\mathcal{Z})$, define

$$K(z, A) \equiv \int_A f(z'|z) dz',$$

where $f(z'|z) = \frac{f(z,z')}{\int_{\mathcal{Z}} f(z,z') dz'}$. Thus, $K$ is a probabilistic kernel, that is, (i) $z \mapsto K(z, A)$ is measurable for every $A \in \sigma(\mathcal{Z})$; and (ii) $A \mapsto K(z, A)$ is probability measure for every $z \in \mathcal{Z}$. In other words, $K$ represents a conditional probability, and we may emphasize this fact by writing $K(A|z)$ instead of $K(z,A)$. We will also abuse notation by denoting $K(z,\{\tilde{z}:\tilde{z}\leqslant z'\})$ simply by $K(z'|z)$.

#### 3.1.2 Plans

At the beginning of period $t$ , the decision maker knows the current state $x_t$ and learns the shock $z_t$ and decides (according to preferences defined below) the future state $x_{t+1}\in\Gamma(x_t,z_t)\subset\mathcal{X}$, where $\Gamma(x,z)$ is the constraint set.[14] From this, we can define plans as follows:

**Definition 3.1.** *A plan* $h$ *is a profile* $h=(h_t)_{t\in\mathbb{N}}$ *where, for each* $t\in\mathbb{N}$, $h_t$ *is a measurable function from* $\mathcal{X}\times\mathcal{Z}^t$ *to* $\mathcal{X}$.[15] *The set of plans is denoted by* $H$.[16]

The interpretation of the above definition is that a plan $h_t(x_t,z^t)$ represents the choice that the individual makes at time $t$ upon observing the current state $x_t$ and the sequence of previous shocks $z^t$. The following notation will simplify statements below.

**Definition 3.2.** *Given a plan* $h=(h_t)_{t\in\mathbb{N}}\in H$, $x\in\mathcal{X}$ *and realization* $z^\infty=(z_1,\ldots)\in Z^\infty$, *the sequence associated to* $(x,z^\infty)$ *is the sequence* $(x_t^h)_{t\in\mathbb{N}^0}\in\mathcal{X}^\infty$ *defined recursively by* $x_1^h=x$ *and* $x_t^h=h_{t-1}(x_{t-1}^h,z^{t-1})$, *for* $t\geqslant 2$. *Similarly, given* $h\in H$, $(x,z^t)\in\mathcal{X}\times Z^t$, *the* $t$*-sequence associated to* $(x,z^t)$ *is* $(x_l^h)_{l=1}^t\in\mathcal{X}^t$ *defined recursively as above.*

We may write $x_t^h(\cdot)$, $x_t^h\left(x,z^t\right)$ or $x_t^h\left(x,z^\infty\right)$ to emphasize that $x_t^h$ depends on the initial state $x$ and on the sequence of shocks $z^\infty$, up to time $t$.

**Definition 3.3.** *A plan* $h$ *is feasible from* $(x,z)\in\mathcal{X}\times\mathcal{Z}$ *if* $h_t(x_t^h,z^t)\in\Gamma\left(x_t^h,z_t\right)$ *for every* $t\in\mathbb{N}$ *and* $z^\infty\in\mathcal{Z}^\infty$ *such that* $x_1^h=x$ *and* $z_1=z$.

We denote by $H(x,z)$ the set of feasible plans from $(x,z)\in\mathcal{X}\times\mathcal{Z}$. Let $H$ denote the set of all feasible plans from some point, that is, $H\equiv\cup_{(x,z)\in\mathcal{X}\times\mathcal{Z}}H(x,z)$.

#### 3.1.3 Preferences

Let $\Omega_t$ represent all the information revealed up to time $t$.[17] We assume that in time $t$ with revealed information $\Omega_t$, the consumer/decision-maker has a preference $\succcurlyeq_{t,\Omega_t}$ over plans

[14] This model is very close to the one discussed in Stokey, Lucas, and Prescott (1989, Chapter 9). There are different, slightly more complicated dynamic models where the state is not chosen by the decision maker, but defined by the shock. The arguments in the current model can be extended to those models when preferences are expected utility, as Stokey, Lucas, and Prescott (1989, Chapter 9) discuss. In our setup, this extension may be more involved.

[15] In the expressions below, $h_0(z^0)$ should be understood as just $h_0\in\mathcal{X}$.

[16] The set $H$ is deeply related to the set $\mathcal{H}$ used in Subsection 2.3, but they are formally different. For this reason, we maintain slightly different notation.

[17] With the knowledge of a fixed $h$, $\Omega_t$ reduces to the initial state $x_1$ and the sequence of shocks $z^t$. More generally, we could take the sequence of states and shocks $(x^t,z^t)$.

$h, h' \in H(x,z)$, which is represented by a function $V_t : H \times \mathcal{X} \times Z^t \to \mathbb{R}$, that is,

$$h' \succcurlyeq_{t,x,\Omega_t} h \iff V_t(h', x, z^t) \geqslant V_t(h, x, z^t). \tag{10}$$

Notice that the preferences in (10) are time, information and state contingent.[18]

A special case of this model corresponds to the standard case of expected utility, that is,

$$V_t(h, x, z^t) = \mathrm{E}\left[\sum_{s \geqslant t} \beta^{s-t} u(x_s^h, x_{s+1}^h, Z_s) \middle| Z^t = z^t\right], \tag{11}$$

where $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is the current-period utility function. That is, $u(x, y, z)$ denotes the instantaneous utility obtained in the current period when $x \in \mathcal{X}$ denotes the current state, $y \in \mathcal{X}$, the choice in the current state, and $z \in \mathcal{Z}$, the current shock.

A first attempt to define the dynamic quantile preference would be to substitute the expectation operator $\mathrm{E}$ by the quantile operator $\mathrm{Q}_\tau$ in (11). Although this seems the natural way to adapt the standard definition, this would lead to dynamically inconsistent preferences, because the analog of the "law of iterated expectations" does not hold for quantiles.[19] Instead, we will need to take a different route. Note that that the functions $V_t$ defined by (11) satisfy the following recursive equation:

$$V_t(h, x, z^t) = u(x_t^h, x_{t+1}^h, z_t) + \beta \mathrm{E}\left[V_{t+1}(h, x, (z^t, Z_{t+1}))\middle| Z^t = z^t\right]. \tag{12}$$

We adapt equation (12) by replacing the expectation operator $\mathrm{E}$ with the quantile operator $\mathrm{Q}_\tau$, that is, we impose:

$$V_t(h, x, z^t) = u(x_t^h, x_{t+1}^h, z_t) + \beta \mathrm{Q}_\tau\left[V_{t+1}(h, x, (Z^t, z_{t+1}))\middle| Z^t = z^t\right]. \tag{13}$$

The recursive equation (13) is the building block of our dynamic quantile preferences and it leads to dynamically consistent preferences, as we show below. This is the reason why Section 2 provided a detailed justification of recursive equation (13) in the simpler form of (5).

In Subsection 3.3 below, we explicitly define a sequence of functions $V_t$ that satisfy (13) and will specify the preferences (10). Nevertheless, before we provide an additional formalization for the definition of the sequence of recursive functions, it is useful to build intuition on how the recursive equation (13) leads to an expression in quantiles that would be different from the expected utility case, developed from (12).

To see this, let us adopt $t = 1$ and substitute the expression of $V_{t+1} = V_2$ by the expression

[18] In contrast, the preferences in (4) considered in last section were stationary IID, not depending on the state.

[19] This fact is formally stated and proved in Proposition 3.7 in Subsection 3.4 below. See also Example 3.1 there.

in (12) and use superscript E to denote the expected utility case, we obtain:

$$V_1^E(h,x,z^t) \;=\; u(x_1^h,x_2^h,z_1)+\beta E\left[u(x_2^h,x_3^h,z_2)+\beta E\left[V_2^E(h,x,z^t)\Big|Z_2=z_2\right]\Bigg|Z_1=z\right].$$

Above, we could eliminate the expectation with respect to $Z_2 = z_2$ using of the law of iterated expectations. Since the same simplification is not possible in the quantile case, we will avoid it here. Moreover, we are able to put all the terms inside the expectations. That is, we can write:

$$\begin{aligned}
V_1^E(h,x,z^t) &= E\left[E\left[u(x_1^h,x_2^h,z_1)+\beta u(x_2^h,x_3^h,z_2)+\beta^2 V_2^E(h,x,z^t)\Big|Z_2=z_2\right]\Bigg|Z_1=z\right]\\
&= E\left[E\left[E\left[\sum_{t=1}^{3}\beta^{t-1}u(x_t^h,x_{t+1}^h,z_t)+\beta^3V_3^E(h,x,z^t)\Big|Z_3=z_3\right]\Bigg|Z_2=z_2\right]\Bigg|Z_1=z\right]\\
&= E\left[\cdots E\left[\sum_{t=1}^{n}\beta^{t-1}u(x_t^h,x_{t+1}^h,z_t)+\beta^nV_n^E(h,x,z^t)\Big|Z_n=z_n\right]\Bigg|\cdots\Bigg|Z_1=z\right],
\end{aligned}$$

where there are $n$ expectation operators E and corresponding conditionals $Z_t = z_t$ in the last line above. Following the same developments from (13), we obtain:

$$\begin{aligned}
V_1^{Q_\tau}(h,x,z^t) &= u(x_1^h,x_2^h,z_1)+\beta Q_\tau\left[u(x_2^h,x_3^h,z_2)+\beta Q_\tau\left[V_2^{Q_\tau}(h,x,z^t)\Big|Z_2=z_2\right]\Bigg|Z_1=z\right]\\
&= Q_\tau\left[Q_\tau\left[u(x_1^h,x_2^h,z_1)+\beta u(x_2^h,x_3^h,z_2)+\beta^2V_2^{Q_\tau}(h,x,z^t)\Big|Z_2=z_2\right]\Bigg|Z_1=z\right]\\
&= Q_\tau\left[Q_\tau\left[Q_\tau\left[\sum_{t=1}^{3}\beta^{t-1}u(x_t^h,x_{t+1}^h,z_t)+\beta^3V_3^{Q_\tau}(h,x,z^t)\Big|Z_3=z_3\right]\Bigg|Z_2=z_2\right]\Bigg|Z_1=z\right]\\
&= Q_\tau\left[\cdots Q_\tau\left[\sum_{t=1}^{n}\beta^{t-1}u(x_t^h,x_{t+1}^h,z_t)+\beta^nV_n^{Q_\tau}(h,x,z^t)\Big|Z_n=z_n\right]\Bigg|\cdots\Bigg|Z_1=z\right], \qquad (14)
\end{aligned}$$

where the operator $Q_\tau[\cdot]$ and corresponding conditionals $Z_t = z_t$ appear $n$ times in the last line above. In order to simplify the above equation, we will introduce the following notation:

$$Q_\tau^n[\cdot] \equiv Q_\tau\left[\cdots\left[Q_\tau\left[\,\cdot\,\Big|Z_n=z_n\right]\Big|\cdots\right]\Bigg|Z_1=z\right], \qquad (15)$$

where the operator $Q_\tau$ and corresponding conditionals appear $n$ times. Therefore, by using the notation defined by (15), we are able to rewrite (14) as

$$V_1^{Q_\tau}(h,x,z^t) \;=\; Q_\tau^n\left[\sum_{t=1}^{n}\beta^{t-1}u(x_t^h,x_{t+1}^h,z_t)+\beta^nV_n^{Q_\tau}(h,x,z^t)\right].$$

Our next step is to take the limit as $n$ goes to $\infty$. The formalization of such limit will be made in Subsection 3.3 below, but one can now intuitively understand the following:

$$V_1^{Q_\tau}(h, x, z^t) \quad = \quad Q_\tau^\infty\left[\sum_{t=1}^{\infty}\beta^{t-1}u(x_t^h, x_{t+1}^h, z_t)\right], \tag{16}$$

as a notation for an (infinite) sequence of applications of $Q_\tau^n[\cdot|Z^t = z^t]$.

Note that if we introduce an analogous notation of (15), that is $E^\infty$ for a(n infinite) sequence of conditional expectations, because of the law of iterated expectations, we obtain

$$V_1^E(h, x, z^t) \quad = \quad E^\infty\left[\sum_{t=1}^{\infty}\beta^{t-1}u(x_t^h, x_{t+1}^h, z_t)\right] = E\left[\sum_{t=1}^{\infty}\beta^{t-1}u(x_t^h, x_{t+1}^h, z_t)\middle| Z_1 = z_1\right],$$

which is the expression in (11). Therefore, our expression (16) is the corresponding generalization of (11): the difference, that is, the fact that we can substitute $E^\infty$ by E but not $Q_\tau^\infty$ by $Q_\tau$, is explained by whether or not the law of iterated expectations hold. Indeed, as Proposition 3.7 below shows, this law does not hold for quantiles.

## 3.2 Assumptions

Now we state the assumptions used for establishing the main results. We organize the assumptions in two groups. The first group collects basic assumptions, which will be assumed throughout the paper, even if they are not explicitly stated. The second group of assumptions will be used only to obtain special, desirable properties of the value function.

**Assumption 1** (Basic)**.** *The following properties are maintained throughout the paper:*

*(i)* $\mathcal{Z} \subseteq \mathbb{R}^k$ *is convex;*

*(ii)* $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ *is continuous, symmetric and* $f(z, z') > 0$*, for all* $(z, z') \in \mathcal{Z} \times \mathcal{Z}$*;*[20]

*(iii)* $\mathcal{X} \subset \mathbb{R}^p$ *is convex;*

*(iv)* $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is continuous and bounded;*

*(v) The correspondence* $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ *is continuous, with nonempty, compact and convex values.*

Note that in the above assumption, property $(i)$ allows an unbounded multidimensional Markov process, requiring only that the support is convex. Property $(ii)$ imposes continuity of $f$, the pdf that establishes the dependence between $Z_t$ and $Z_{t+1}$ and requires it to be strictly positive in the support of the Markov process, $\mathcal{Z}$. The state space $\mathcal{X}$ is not required to be compact, but only convex by property $(iii)$. Property $(iv)$ is the standard continuity

[20] Symmetry guarantees stationarity since $\Pr([Z_1 \in A]) = \int_{\mathcal{Z}}\int_A f(z_1, z_2)dz_1 dz_2 = \int_A\int_{\mathcal{Z}} f(z_1, z_2)dz_1 dz_2 = \Pr([Z_2 \in A])$.

assumption. Property $(v)$ and the continuity of $u$ required in property $(iv)$ guarantee that an optimal choice always exist.

For some results we will also require differentiability, concavity and monotonicity assumptions.

**Assumption 2** (Differentiability, Concavity and Monotonicity). *The following holds:*

*(i)* $\mathcal{Z} \subseteq \mathbb{R}$ *is an interval;*

*(ii) If* $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leqslant z'$*, then:*

$$\int_{\mathcal{Z}} h(\alpha) f(\alpha|z) d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha) f(\alpha|z') d\alpha; \tag{17}$$

*(iii)* $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is* $C^1$*, strictly concave in the first two variables and strictly increasing in the last variable;*

*(iv) For every* $x \in \mathcal{X}$ *and* $z \leqslant z'$*,* $\Gamma(x, z) \subseteq \Gamma(x, z')$*;*

*(v) For all* $z \in \mathcal{Z}$ *and all* $x, x' \in \mathcal{X}$*,* $y \in \Gamma(x, z)$ *and* $y' \in \Gamma(x', z)$ *imply*

$$\theta y + (1 - \theta) y' \in \Gamma[\theta x + (1 - \theta) x', z], \text{ for all } \theta \in [0, 1].$$

To work with monotonicity, we restrict the dimension of the Markov process to $k = 1$ in Assumption $2(i)$. Assumptions $2(ii) - 2(v)$ are standard conditions on dynamic models (see, e.g., Assumptions 9.8 – 9.15 in Stokey, Lucas, and Prescott (1989)). Assumption $2(ii)$ implies that whenever $z \leqslant z'$,

$$K(w|z') = \int_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}} f(\alpha|z') d\alpha \leqslant \int_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}} f(\alpha|z) d\alpha = K(w|z), \tag{18}$$

for all $w$.[21] In other words, $K(\cdot|z')$ first-order stochastically dominates $K(\cdot|z)$. Assumption $2(iii)$ allows us to establish the continuity and differentiability of the value function. Assumption $2(iv)$ only requires the monotonicity of the choice set. Assumption $2(v)$ implies that $\Gamma(x, z)$ is a convex set for each $(x, z) \in \mathcal{X} \times \mathcal{Z}$, and that there are no increasing returns.

It should be noted that monotonicity also is important for econometric reasons. Indeed, Matzkin (2003, Lemma 1, p. 1345) shows that two econometric models are observationally equivalent if and only if there are strictly increasing functions mapping one to another. Thus, in a sense, the quantile implied by a model is the essence of what can be identified by an econometrician.

[21] To obtain (18), it is enough to use $h(z) = -1_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}}(z)$ in (iii).

### 3.3 The Sequence of Recursive Functions

In this subsection, we define the sequence of functions $V_t$ that satisfy (13) and specify the preferences (10). For this, we need to define a transformation. Let $\mathcal{L}$ be the set of real-valued functions from $\mathcal{X} \times \mathcal{Z}$ to $\mathbb{R}$ and let $\mathcal{C} \subset \mathcal{L}$ denote the set of bounded continuous functions from $\mathcal{X} \times \mathcal{Z}$ to $\mathbb{R}$, endowed with the sup norm. It is well known that $\mathcal{C}$ is a Banach space. Let us fix $h \in H$ and $\tau \in (0,1)$, and define the transformation $\mathbb{T}^h : \mathcal{C} \to \mathcal{L}$ (the dependency on $\tau$ is omitted) by the following:

$$\mathbb{T}^h(V)(x,z) = u\left(x_1^h, x_2^h, z_1\right) + \beta Q_\tau[V(x_2^h, Z_2))|Z_1 = z],$$

where $\left(x_1^h, z_1\right) = (x,z)$ and $x_2^h = h(x,z)$. We show that the image of $\mathbb{T}^h$ is indeed in $\mathcal{C}$ continuous and that $\mathbb{T}^h$ is a contraction and, therefore, has a unique fixed point:

**Theorem 3.4.** $\mathbb{T}^h(V)$ *is continuous and bounded on* $\mathcal{X} \times \mathcal{Z}$*, that is,* $\mathbb{T}(\mathcal{C}) \subset \mathcal{C}$*. Moreover,* $\mathbb{T}^h$ *is a contraction and has a unique fixed point, denoted* $V^h \in \mathcal{C}$*.*

Now we can define $V_t$ as follows:

$$V_t(h, x, z^t) = V^h(x_t^h, z_t),$$

where $(x_l^h)_{l=1}^t$ is the associated $t$-sequence to $(x, z^t)$ (see Definition 3.2). From the fact that $V^h$ is the unique fixed point of $\mathbb{T}^h$, it is clear that (13) holds. This completes the definition of the preferences (10).

It is possible to write $V^h$ in a more explicit form. For this, let us define

$$\begin{aligned} V^{h,n}(x,z) &= u(x_1^h, x_2^h, z_1) + Q_\tau\Bigg[\beta u(x_2^h, x_3^h, z_2) + Q_\tau\Big[\beta^2 u(x_3^h, x_4^h, z_3) + \dots \\ &\qquad \dots + Q_\tau\Big[\beta^n u(x_{n+1}^h, x_{n+2}^h, z_n)\Big|Z_n = z_n\Big] \dots \Big|Z_1 = z\Bigg] \\ &= Q_\tau\left[\cdots\left[Q_\tau\left[\sum_{t=0}^{n} \beta^t u(x_{t+1}^h, x_{t+2}^h, z_t)\Big|Z_n = z_n\right]\cdots\right]\Big|Z_1 = z\right]\Bigg] \\ &= Q_\tau^n\left[\sum_{t=0}^{n} \beta^t u(x_{t+1}^h, x_{t+2}^h, z_t)\right], \end{aligned}$$

where the expression $Q_\tau^n[\cdot]$ in the last line is just a short notation for the conditional quantiles applied successively, as shown in the previous line; see (15). With this definition, we obtain:

**Proposition 3.5.** $V^h(x,z) = \lim_{n\to\infty} V^{h,n}(x,z)$.

Thus, the existence of the limit $\lim_{n\to\infty} V^{h,n}(x,z)$ allows us to define the notation $Q_\tau^\infty[\cdot]$, that is,

$$
\begin{aligned}
V^h(x,z) &= Q_\tau^\infty\left[\sum_{t=0}^{\infty}\beta^t u(x_t^h, x_{t+1}^h, z_t)\right] && (19)\\
&= u(x_1^h, x_2^h, z_1) + Q_\tau\Bigg[\beta u(x_2^h, x_3^h, z_2) + Q_\tau\Big[\beta^2 u(x_3^h, x_4^h, z_3) + \ldots \\
&\qquad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^h, x_{n+2}^h, z_{n+1}) + \cdots \Big| \cdots\Big] \cdots \Big| Z_2 = z_2 \Big| Z_1 = z\Bigg].
\end{aligned}
$$

We turn now to verify that this preference is dynamically consistent.

## 3.4 Dynamic Consistency

Our objective is to develop a dynamic theory for quantile preferences. Thus, the dynamic consistency of such preferences is of uttermost importance. In this subsection we formally define dynamic consistency and show that it is satisfied by the above defined preferences. The following definition comes from Maccheroni, Marinacci, and Rustichini (2006b); see also Epstein and Schneider (2003b).

**Definition 3.6** (Dynamic Consistency)**.** *The system of preferences* $\succcurlyeq_{t,\Omega_t}$ *is* dynamically consistent *if for every* $t$ *and* $\Omega_t$ *and for all plans* $h$ *and* $h'$, $h_{t'}(\cdot) = h'_{t'}(\cdot)$ *for all* $t' \leqslant t$ *and* $h' \succcurlyeq_{t+1,\Omega'_{t+1},x} h$ *for all* $\Omega'_{t+1}, x$, *implies* $h' \succcurlyeq_{t,\Omega_t,x} h$.

In principle, there is no reason to expect that quantile preferences would be dynamically consistent. For instance, the law of iterated expectations, which is important to the dynamic consistency of expected utility, does not have an analogous for quantile preferences, as the following result shows.

**Proposition 3.7.** *Let* $\Sigma_1 \supset \Sigma_0$ *be two* $\sigma$*-algebras on* $\Omega$, $\tau \in (0,1)$, *and consider random variables* $X : \Omega \to \mathbb{R}$ *and* $Y : \Omega \to \mathbb{R}$. *Then, in general,*

$$Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] \neq Q_\tau[X|\Sigma_0]. \tag{20}$$

*and it is possible that*

$$Q_\tau[X|\Sigma_1]_{(\omega)} \geqslant Q_\tau[Y|\Sigma_1]_{(\omega)}, \forall \omega \in \Omega, \text{ but } Q_\tau[X|\Sigma_0]_{(\omega)} < Q_\tau[Y|\Sigma_0]_{(\omega)}, \forall \omega \in \Omega. \tag{21}$$

Note that (21) suggests a potential negation of dynamic consistency for quantile preferences in general. Indeed, this failure would be fatal for dynamic consistency if we had chosen the preference to seek the maximization of $Q_\tau\left[\sum_{t=0}^{\infty}\beta^t u(x_t^h, x_{t+1}^h, z_t)|Z_t = z_t\right]$, because changing from one period to the other could imply a reversion of choices, which is exactly what (21) illustrates. However, because we have adopted as preference $Q_\tau^\infty\left[\sum_{t=0}^{\infty}\beta^t u(x_t^h, x_{t+1}^h, z_t)\right]$,which involves an infinite sequence of nested conditional quantiles, as explained in Subsection 3.1.3, where the notation $Q_\tau^\infty[\cdot]$ is also introduced. This is exactly what allows to obtain dynamic consistency. Indeed, in our framework, quantile preferences are dynamically consistent and amenable to the use of the standard techniques of dynamic programming, as the following result establishes.

**Theorem 3.8.** *The quantile preferences defined by (10) are dynamically consistent.*

This result is important, because it implies that no money-pump can be used against a decision maker with quantile preferences. Many preferences that departure from the expected utility framework do not satisfy dynamic consistency.

The reason why this result holds is that we imposed the recursive structure (13). This implies that in each period, the decision will be made taking in account a fixed succession of conditional quantiles. Thus, there is no reversal in choices. The following example allows to understand this point in more detail.

**Example 3.1.** *Let $\tau \in (0.5, 0.75)$ and consider the variables $X$ and $Y$ defined in the proof of Proposition 3.7 in the appendix to establish (21). Assume that both $X$ and $Y$ are paid in period 2 and let $\Sigma_t$ be the information $\sigma$-algebra known in period $t = 0, 1$; these are also defined in the appendix. Suppose now that the decision maker evaluates $X$ and $Y$ according to a preference defined from (11) by substituting the operator E by the operator $Q_\tau$, that is, by using $\tilde{V}_t(h, x, z_t) = Q_\tau\left[\sum_{s\geq t} \beta^{s-t} u(x_s^h(x, Z^s), x_{s+1}^h(x, Z^s), Z_s)|\Sigma_t\right]$, which particularized to our example gives:*

$$\tilde{V}_0(X) = Q_\tau[X|\Sigma_0] < Q_\tau[Y|\Sigma_0] = \tilde{V}_0(Y)$$

*as shown in the appendix. There we also show that $\forall \omega \in \Omega$,*

$$\tilde{V}_1(X, \omega) = Q_\tau[X|\Sigma_1]_{(\omega)} \geqslant Q_\tau[Y|\Sigma_1]_{(\omega)} = \tilde{V}_1(Y, \omega),$$

*which proves that this preference is not dynamically consistent: the initial preference for $Y$ is reversed at time 1 in all states of nature. Consider, however, our preference defined by $V_t(h, x, z_t) = Q_\tau^\infty\left[\sum_{s\geq t} \beta^{s-t} u(x_s^h(x, Z^s), x_{s+1}^h(x, Z^s), Z_s)\right]$ as in (19). Particularized to this example, we have:*

$$V_0(X) = Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] \geqslant Q_\tau[Q_\tau[Y|\Sigma_1]|\Sigma_0] = V_0(Y)$$

*and, $\forall \omega \in \Omega$,*

$$V_1(X, \omega) = Q_\tau[X|\Sigma_1]_{(\omega)} \geqslant Q_\tau[Y|\Sigma_1]_{(\omega)} = V_1(Y, \omega).$$

*In other words, $X$ is always preferred to $Y$ and there is no reversal because the sequence of events is fixed and taken into account.*

Our approach to establish dynamic consistency is similar to that taken by Epstein and

Schneider (2003b) for the maximin expected utility dynamic preferences, in the sense that the filtration of events where decisions are made is fixed. As discussed by Strzalecki (2013, p. 1048), this is one of the main approaches that have been used to obtain dynamic consistency for different preferences.

We also note that Epstein and Le Breton (1993) essentially prove that dynamic consistent preferences are "probabilistic sophisticated" in the sense of Machina and Schmeidler (1992). Probabilistic sophistication *roughly* means that the preference is "based" in a probability. Extending Machina-Schmeidler's definition, Rostek (2010) shows that the static quantiles preferences are probabilistic sophisticated for $\tau \in (0,1)$. Her observation is also valid for our dynamic quantile preference. However, we do not use these developments, since Theorem 3.8 offers a direct proof of dynamic consistency.

### 3.5 The Value Function

In this subsection we establish the existence of the value function associated to the dynamic programming problem for the quantile utility and some of its properties. This is accomplished through a contraction fixed point theorem.

The first step is to the define the contraction operator; this is similar to what we have defined in Subsection 3.3. For $\tau \in (0,1)$, define the transformation $\mathbb{M}^\tau : \mathcal{C} \to \mathcal{C}$ as

$$\mathbb{M}^\tau(v)(x,z) = \sup_{y \in \Gamma(x,z)} u(x,y,z) + \beta Q_\tau[v(y,w)|z]. \tag{22}$$

Note that this is similar to the usual dynamic program problem, in which the expectation operator $\mathsf{E}[\cdot]$ is in place of $Q_\tau[\cdot]$. The main objective is to show that the above transformation has a fixed point, which is the value function of the dynamic programming problem. The following result establishes the existence of the contraction $\mathbb{M}^\tau$ under the basic assumptions assumed throughout this paper.

**Theorem 3.9.** $\mathbb{M}^\tau$ *is a contraction and has a unique fixed point* $v^\tau \in \mathcal{C}$.

The unique fixed point of the problem will be the value function of the problem. Notice that the proof of this theorem is *not* just a routine application of the similar theorems from the expected utility case. In particular, the continuity of the function $(x,z) \mapsto Q_\tau[v(x,w)|z]$ is not immediate as in the standard case. Since $v$ is not assumed to be strictly increasing in the second argument, it can be constant at some level. Constant values may potentially lead to discontinuities in the c.d.f or quantile functions; see illustration in Subsection A.1 in the appendix. Thus, some careful arguments are needed for establishing this continuity.

The next step is to establish the differentiability and monotonicity of the value function.

**Theorem 3.10.** *If Assumption 2 holds, then* $v^\tau : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is differentiable in* $x$*, strictly* *increasing in* $z$*, and strictly concave in an interior point* $x$*. Moreover,*

$$\frac{\partial v^\tau}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x,y^*,z),$$

*where* $y^* \in \Gamma(x,z)$ *is the unique maximizer of* (22)*, assumed to be interior to* $\Gamma(x,z)$*.*

This theorem is the most important result in the paper since it delivers interesting and important properties of the value function. It establishes that the value function that one obtains from quantile functions possesses, essentially, the same basic properties of the value function of the corresponding expected utility problem. The second part of Theorem 3.10 is very important for the characterization of the problem. It is the extension of the standard envelope theorem for the quantile utility case. Notice that since the quantile function does not have some of the convenient properties of the expectation, we assumed that $z$ were unidimensional (see Assumption 2) in order to establish the conclusions of Theorem 3.10. However, this unidimensionality requirement does not seem overly restrictive in most practical applications. For example, it allows us to tackle the standard asset pricing model, as Section 4 shows.

**Remark 3.11.** *The result in Theorem 3.9 is related to that in Marinacci and Montrucchio (2010). They establish the existence and uniqueness of the value function in a more general setup. Nevertheless, we are able to provide sharper characterizations of the fixed point* $v^\tau$*. In particular, Theorem 3.9 establishes that* $v^\tau$ *is continuous and Theorem 3.10 that it is differentiable, concave, and increasing.*

### 3.6 The Principle of Optimality

This subsection establishes that the principle of optimality holds in our model (Theorem 3.15 below). That is, we show that optimizing period after period, as in the recursive problem (22), yields the same result as choosing the best plan for the whole horizon of the problem. In order to do that, we have to complete three tasks. First, we define the problem of choosing plans. Next, we revisit the recursive problem to establish a result that will be useful in the sequel. Finally, we show that choosing plans for the entire horizon and solving the problem step by step as in the recursive problem, lead to the same values.

Let us begin by establishing that the set of feasible plans departing from $(x,z) \in \mathcal{X} \times \mathcal{Z}$ at time $t$ is nonempty. More formally, let us define:

$$H_t(x,z) \equiv \{h \in H(x,z) : \exists (x,z^t) \in \mathcal{X} \times \mathcal{Z}^t, \text{ with } z_t = z, \text{ such that } x_t^h(x,z^t) = x\}.$$

Thus, $H_1(x,z)$ is just $H(x,z)$. We have the following result regarding the set of feasible plans:

**Lemma 3.12.** *For any* $x \in \mathcal{X}$ *and* $t \in \mathbb{N}$*,* $H_t(x,z) \neq \varnothing$*.*

This result allows us to define a supremum function as:

$$v_t^*(x,z) \equiv \sup_{h \in H_t(x,z)} V_t(h,x,z). \tag{23}$$

We first observe that $t$ plays no role in the above equation (23), that is, we prove the following:

**Lemma 3.13.** *For any $t \in \mathbb{N}$ and $(x,z) \in \mathcal{X} \times \mathcal{Z}$, $v_t^*(x,z) = v_1^*(x,z)$.*

Thus, we are able to drop the subscript $t$ from (23) and write $v^*(x,z)$ instead of $v_t^*(x,z)$.

The next step is to relate $v^*$ to $v^\tau$, the solution of the functional equation studied in the previous subsection, which was proved to exist in Theorem 3.9 and satisfies the Bellman equation:

$$v^\tau(x,z) \quad = \quad \sup_{y \in \Gamma(x,z)} \left\{ u(x,y,z) + \beta Q_\tau [v^\tau(y,w)|z] \right\}. \tag{24}$$

In the rest of this section we will denote $v^\tau$ simply by $v$.

To achieve this aim, we first establish important results relating $v$ in equation (24) to the policy function that solves the original problem. In particular, the next result allows us to define the policy function:

**Lemma 3.14.** *If $v$ is a bounded continuous function satisfying (24), then for each $(x,z) \in \mathcal{X} \times \mathcal{Z}$, the correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by*

$$\Upsilon(x,z) \equiv \{ y \in \Gamma(x,z) : v(x,z) = u(x,y,z) + \beta Q_\tau [v(y,w)|z] \}$$

*is nonempty, upper semi-continuous and has a measurable selection.*

Let $\psi : \mathcal{X} \times \mathcal{Z} \to \mathcal{X}$ be a measurable selection of $\Upsilon$. The policy function $\psi$ generates the plan $h^\psi$ defined by $h_t^\psi(z^t) = \psi(h_{t-1}(z^{t-1}), z_t)$ for all $z^t \in \mathcal{Z}^t$, $t \in \mathbb{N}$.

The next result provides sufficient conditions for a solution $v$ to the functional equation to the be supremum function, and for the plans generated by the associated policy function $\psi$ to attain the supremum.

**Theorem 3.15.** *Let $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be bounded and satisfy the functional equation (24) and $\psi$ be defined as above. Then, $v = v^*$ and the plan $h^\psi$ attains the supremum in (23).*

We highlight that this generalization is not straightforward. When working with expected utility, one can employ the law of iterated expectations. However, unfortunately a similar rule does not hold for quantiles, as we have already observed in Proposition 3.7.

### 3.7 Euler Equation

The final step is to characterize the solutions of the problem through the Euler equation. Let $v = v^\tau$ be the unique fixed point of $\mathbb{M}^\tau$, satisfying (24). By Theorem 3.10, $v$ is differentiable in its first coordinate, satisfying $v_{x_i}(x,z) = \frac{\partial v^\tau}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x,y^*,z) = u_{x_i}(x,y^*,z)$.

Given that we have shown the differentiability of value function, we are able to apply the standard technique to obtain the Euler equation, as formalized in the following theorem:

**Theorem 3.16.** *Let Assumption 2 hold and let $h = h^\psi$ be an optimal plan, as in Theorem 3.15. Assume that $x^h_{t+1} \in \mathrm{int}\Gamma(x^h_t, z_t)$, that is, the optima are interior, and $z_t \mapsto \frac{\partial u}{\partial x_i}\left(x^h_t, x^h_{t+1}, z_t\right)$ is strictly increasing. Then, the following first order condition (called Euler equation in this setting) necessarily holds for every $t \in \mathbb{N}$ and $i = 1, \ldots, p$:*

$$u_{y_i}\left(x^h_t, x^h_{t+1}, z_t\right) + \beta Q_\tau[u_{x_i}\left(x^h_{t+1}, x^h_{t+2}, z_{t+1}\right)|z_t] = 0. \tag{25}$$

In the expression above, $u_y$ represents the derivative of $u$ with respect to (some of the coordinates of) its second variable ($y$) and $u_x$ represents the derivative of $u$ with respect to (some of the coordinates of) its first variable ($x$).

Theorem 3.16 provides the Euler equation, that is the optimality conditions for the quantile dynamic programming problem. This result is the generalization the traditional expected utility to the quantile utility. The Euler equation in (25) is displayed as an implicit function, nevertheless for any particular application, and given utility function, one is able to solve an explicitly equation as a conditional quantile function.

The proof of Theorem relies on a result about the differentiability inside the quantile function. Indeed, for a general function $h$, we have $\frac{\partial}{\partial x_i} Q_\tau[h(x,Z)] \neq Q_\tau\left[\frac{\partial h}{\partial x_i}(x,Z)\right]$. However, we are able to establish this differentiability under some assumptions. We are not aware of this result in the theory of quantiles, and given its usefulness, we state it here:

**Proposition 3.17.** *Assume that $h : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is differentiable in $x$ and that $h$ and $d$ are increasing in $z$, where $d(z) \equiv h(x'_i, x_{-i}, z) - h(x_i, x_{-i}, z)$ for $x_i, x'_i$ satisfying $0 < x'_i - x_i < \epsilon$, for some small $\epsilon > 0$. Then,*

$$\frac{\partial Q_\tau}{\partial x_i}[h(x,Z)] = Q_\tau\left[\frac{\partial h}{\partial x_i}(x,Z)\right].$$

## 4 Application: Asset Pricing Model

This section illustrates the usefulness of the new quantile utility maximization methods through an empirical example. We apply the methodology to the standard intertemporal allocation of consumption model, which is central to contemporary economics and finance. It has been used extensively in the literature and has had remarkable success in providing empirical estimates for the study of the elasticity of intertemporal substitution (EIS) and discount-factor parameters.

We refer the readers to Campbell (2003), Cochrane (2005), and Ljungqvist and Sargent (2012), and the references therein, for a comprehensive overview.

There is a large empirical literature that attempts to estimate the EIS; among others, Hansen and Singleton (1983), Hall (1988), Campbell and Mankiw (1989), Epstein and Zin (1991), Blundell, Browning, and Meghir (1994), Attanasio and Browning (1995), Atkeson and Ogaki (1996), Campbell and Viceira (1999), Campbell (2003), and Yogo (2004). The majority of the literature relies on the traditional expected utility framework.

## 4.1 Economic Model

We employ a variation of Lucas (1978)'s endowment economy (see, also, Hansen and Singleton (1982)). The economic agent decides on the intertemporal consumption and savings (assets to hold) over an infinity horizon economy, subject to a linear budget constraint. The decision generates an intertemporal policy function, which is used to estimate the parameters of interest for a given utility function.

Let $c_t$ denote the amount of consumption good that the individual consumes in period $t$. At the beginning of period $t$, the consumer has $x_t$ units of the risky asset, which pays dividend $z_t$. The price of the consumption good is normalized to one, while the price of the risky asset in period $t$ is $p(z_t)$. Then, the consumer decides how many units of the risky asset $x_{t+1}$ to save for the next period, and its consumption $c_t$. Using the notation introduced in (19), we can write the consumer problem as[22]

$$\max_{(c_t)_{t=0}^{\infty}} Q_{\tau}^{\infty}\left[\sum_{t=0}^{\infty} \beta_{\tau}^{t} U(c_t)\right], \tag{26}$$

subjected to

$$\begin{aligned} c_t + p(z_t)x_{t+1} &\leqslant [z_t + p(z_t)] \cdot x_t, &(27)\\ c_t, x_{t+1} &\geqslant 0, &(28)\end{aligned}$$

where $\beta_{\tau} \in (0,1)$ is the discount factor for the quantile $\tau \in (0,1)$ of interest, and $U : \mathbb{R}_+ \to \mathbb{R}$ is the utility function. Note that $\beta_{\tau}$ is written with a subscript $\tau$, to emphasize the fact that we may have a different parameter for each $\tau \in (0,1)$. Because there is a single agent, the holdings must not exceed one unit. In fact, in equilibrium, we must have $x_{tk}^{*} = 1, \forall t, k$. Let $\bar{x} > 1$ and $\mathcal{X} = [0, \bar{x}]$.

From (27), we can determine the consumption entirely from the current and future states, that is, $c_t = z_t \cdot x_t + p(z_t) \cdot (x_t - x_{t+1})$. Following the notation of the previous sections, we denote $x_t$ by $x$, $x_{t+1}$ by $y$, and $z_t$ by $z$. Then, the above restrictions are captured by the

[22]Recall (19): $Q_{\tau}^{\infty}\left[\sum_{t=0}^{\infty} \beta_{\tau}^{t} u(c_t)\right] = u(c_0) + Q_{\tau}\Big[\beta_{\tau} U(c_1) + Q_{\tau}\big[\beta_{\tau}^{2} U(c_2) + Q_{\tau}[\beta_{\tau}^{3} U(c_3) + \cdots \mid \Omega_2]\big|\Omega_1\big]\Big|\Omega_0\Big]$.

feasible correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{X} = \mathcal{X}$ defined by:

$$\Gamma(x, z) \equiv \{y \in \mathcal{X} : p(z) \cdot y \leqslant (z + p(z)) \cdot x\}.$$

For each pricing function $p : \mathcal{Z} \to \mathbb{R}_+$, define the utility function as:

$$u(x, y, z) \equiv U[z \cdot x + p(z) \cdot (x - y)].$$

We assume the following:

**Assumption 3.** *(i)* $\mathcal{Z} \subseteq \mathbb{R}$ *is a bounded interval and* $\mathcal{X} = [0, \bar{x}]$*;*

*(ii)* $U : \mathbb{R}_+ \to \mathbb{R}$ *is given by* $U(c) = \frac{1}{1-\gamma} c^{1-\gamma}$*, for* $\gamma > 0$*;*

*(iii)* $z$ *follows a Markov process with pdf* $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$*, which is continuous, symmetric,* $f(z, w) > 0$*, for all* $(z, w) \in \mathcal{Z} \times \mathcal{Z}$ *and satisfies the property: if* $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leqslant z'$*, then:*

$$\int_{\mathcal{Z}} h(\alpha) f(\alpha|z) d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha) f(\alpha|z') d\alpha;$$

*(iv)* $z \mapsto z + p(z)$ *is* $C^1$ *and non-decreasing, with* $z\left(\ln(z + p(z))\right)' \geqslant \gamma$*.*

Assumptions 3$(i) - (ii)$ are standard in economic applications. Assumption 3$(i)$ specifies an isoelastic utility function (constant elasticity of substitution – CES). This is a standard assumption in a large body of the literature, as for example, among others, Hansen and Singleton (1982), Stock and Wright (2000), and Campbell (2003). Condition 3$(ii)$ states that the idiosyncratic shocks follow a Markov process. Assumption 3$(iii)$ means that a high value of the dividend today makes a high value tomorrow more likely. It implies Assumption 2(ii). Assumption 3$(iv)$, $z \mapsto z + p(z)$ is non-decreasing, is natural. It states that the price of the risky asset and its return are a non-decreasing function of the dividends. Note that it is natural to expect that the price is non-decreasing with the dividends, but Assumption 3$(iv)$ is even weaker than this, as it allows the price to decrease with the dividend; only $z + p(z)$ is required to be non-decreasing.[23]

Given Assumption 3, we can verify the assumptions for establishing the quantile utility model in the asset pricing model context. Thus, we have the following:

**Lemma 4.1.** *Assumption 3 implies Assumptions 1 and 2 and that* $z_t \mapsto \frac{\partial u}{\partial x_i}\left(x_t^h, x_{t+1}^h, z_t\right)$ *is strictly increasing.*

Therefore, Theorems 3.9 and 3.10 imply the existence of a value function $v$, which is strictly

[23] In our dataset, when regressing the returns on the dividends, we find a statistically positive correlation.

concave and differentiable in its first variable, satisfying

$$v(x,z) = \max_{y\in\Gamma(x,z)}\{U[z\cdot x + p(z)\cdot(x-y)] + \beta_\tau Q_\tau[v(y,z')|z]\},$$

where, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$. Note that

$$\frac{\partial u}{\partial x}(x,y,z) = U'[z\cdot x + p(z)\cdot(x-y)](z+p(z));$$
$$\frac{\partial u}{\partial y}(x,y,z) = U'[z\cdot x + p(z)\cdot(x-y)](-p(z)).$$

Because, in equilibrium, the holdings are $x_t = 1$ for all $t$, we can derive the Euler equation as in (25) for this particular problem to obtain:

$$-p(z_t)U'(c_t) + \beta_\tau Q_\tau[U'(c_{t+1})(z_{t+1}+p(z_{t+1}))|\Omega_t] = 0. \tag{29}$$

Let us define the return by: $1 + r_{t+1} \equiv \frac{z_{t+1}+p(z_{t+1})}{p(z_t)}$. Assumption 3(i) implies that the ratio of marginal utilities can be written as $\frac{U'(c_{t+1})}{U'(c_t)} = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$. Thus, the Euler equation in (29) simplifies to

$$Q_\tau\left[\beta_\tau(1+r_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma_\tau} - 1\middle|\Omega_t\right] = 0. \tag{30}$$

Now, for illustration purposes, we compare the quantile utility maximization results with the corresponding ones for the expected utility, which can be written as

$$\max_{(c_t)_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t U(c_t)\right],$$

subject to the same constraints in (27)-(28). This problem can be rewritten and the associated value function is:

$$v(x,z) = \max_{y\in\Gamma(x,z)}\{U[z\cdot x + p(z)\cdot(x-y)] + \beta E[v(y,z')|z]\}.$$

Finally, the corresponding Euler equation can be expressed as

$$-p(z_t)U'(c_t) + \beta E[U'(c_{t+1})(z_{t+1}+p(z_{t+1}))|\Omega_t] = 0,$$

and by rearranging the previous equation we obtain

$$E\left[\beta(1+r_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - 1\middle|\Omega_t\right] = 0. \tag{31}$$

When comparing the Euler equations in (30) and (31) one can notice similarities and differences. The expressions inside the conditional quantile in (30) and the conditional expectation in (31) are practically the same, except that, for the quantile model, the parameters $(\beta_\tau, \gamma_\tau)$ depend on the quantile $\tau$. That is, for each $\tau$, we may have (potentially) different parameters.

## 4.2 Estimation and Inference

The previous section derives the Euler equation for the quantile utility model. Now we describe how to estimate the vector of parameters of interest. The basic idea underlying our proposed estimation strategy is to use the theoretical economic model to generate a family of nonlinear conditional quantile functions, and apply the instrumental variables (IV) quantile regression (QR) for nonlinear models developed in de Castro, Galvao, and Kaplan (2017).[24]

For a given parametrized utility function, one is able to isolate the implicit quantile function defined by equation (25). In particular, (30) can be written as the following nonlinear conditional quantile model:

$$Q_\tau[m(c_t, r_t, \theta_{0\tau})|\Omega_t] = 0, \tag{32}$$

where $\tau \in (0,1)$ is a quantile of interest, $m(c_t, r_t, \theta_{0\tau}) \equiv \beta_\tau(1 + r_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma_\tau} - 1$, with $\theta_{0\tau} = (\beta_\tau, \gamma_\tau)^\top$, is a function known up to a finite dimensional vector of parameter of interest $\theta_{0\tau}$, $\Omega_t$ denotes the $\sigma$-field generated by $\{w_s, s \le t\}$ that contains the information up to time $t$. The vector $(c_t, r_t, w_t)$ are the observable variables, with consumption $c_t \in \mathcal{Y}$, the real return on the asset $r_t \in \mathcal{X}$, the full instrument vector $w_t \in \mathcal{W}$, and parameters $\theta_{0\tau} \in \mathcal{B} \subseteq \mathbb{R}^2$. The quantile model in (32) is valid for each given quantile $\tau$. We aim to estimate the parameters $\theta_{0\tau}$ that describe the Euler equation for specified quantiles of interest.

The model in (32) can be represented by non-smooth conditional moment restrictions as

$$E[\tau - 1\{m(c_t, r_t, \theta_{0\tau}) \le 0\}|w_t] = 0, \tag{33}$$

where $1\{\cdot\}$ is the indicator function. Since $E[1\{m(c_t, r_t, \theta_{0\tau}) \le 0\}|w_t] = F[m(c_t, r_t, \theta_{0\tau})|w_t]$, when $F(\cdot)$ is invertible, one is able to recover (32) from (33).

For a given quantile index $\tau \in (0,1)$, estimation of the parameter vector $\theta_{0\tau}$ uses the method of moments. Rewrite (33) as the following moment condition

$$E\left[w_t\left[1\{m(c_t, r_t, \theta_{0\tau}) \le 0\} - \tau\right]\right] = 0, \tag{34}$$

where instruments, $w_t$, are used to achieve a valid orthogonality condition, that is, the (con-

[24] It has been standard in the literature to estimate Euler equations derived from the expected utility models. It is an important exercise to learn about the structural parameters that characterize the economic problem of interest. After parametrizing the utility function, the restrictions imply a conditional average model and the parameters are commonly estimated by the generalized method of moments (GMM) of Hansen (1982). Estimation and inference of GMM have been discussed by, among many others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), and Chen and Liao (2015).

ditional) moment condition equals to zero.

We now present the smoothed IVQR (SIVQR) estimator. Let the population map $M : \mathcal{B} \times \mathcal{T} \mapsto \mathbb{R}^2$ be

$$M(\theta, \tau) \equiv \mathrm{E}\left[g_t^u(\theta, \tau)\right], \tag{35}$$

$$g_t^u(\theta, \tau) \equiv g^u(c_t, r_t, w_t, \theta, \tau) \equiv w_t \left[1\{m(c_t, r_t, \theta) \leq 0\} - \tau\right], \tag{36}$$

where superscript "$u$" denotes "unsmoothed." The population moment condition (34) is

$$M(\theta_{0\tau}, \tau) = 0. \tag{37}$$

The method of moments is constructed by replacing the population moments, the expectation $\mathrm{E}(\cdot)$, with their corresponding sample expectation $\widehat{\mathrm{E}}(\cdot)$, i.e., the sample average. Analogous to (35), using (36), the unsmoothed *sample* moment map is

$$\widehat{M}_T^u(\theta, \tau) \equiv \widehat{\mathrm{E}}\left[g^u(c, r, w, \theta, \tau)\right] \equiv \frac{1}{T} \sum_{t=1}^{T} g_t^u(\theta, \tau). \tag{38}$$

Even if population system (37) has a unique solution, the unsmoothed system $\widehat{M}_T^u(\theta, \tau) = 0$ may have zero or multiple solutions. Although this issue can be overcome theoretically, smoothing addresses it directly. The SIVQR estimator is the solution to the system of smoothed sample moments, shown in (40) below. With smoothing (no "$u$" superscript), the sample analogs of (35), (36), and (37) are

$$\begin{aligned} g_{Tt}(\theta, \tau) &\equiv g_T(c_t, r_t, w_t, \theta, \tau) \equiv w_t\left[\tilde{I}\left(-m(c_t, r_t, \theta)/h_T\right) - \tau\right], \\ \widehat{M}_T(\theta, \tau) &\equiv \frac{1}{T} \sum_{t=1}^{T} g_{Tt}(\theta, \tau), \end{aligned} \tag{39}$$

$$\widehat{M}_T(\hat{\theta}_\tau, \tau) = 0, \tag{40}$$

where $h_T$ is a bandwidth (sequence), $\tilde{I}(\cdot)$ is a smoothed version of the indicator function $1\{\cdot > 0\}$, and $I(\cdot)$ may stand for "indicator-like function" or "integral of kernel." An example function $\tilde{I}(\cdot)$ is the integral of a fourth-order polynomial kernel.

Finally, given a random sample $\{(c_t, r_t, w_t) : t = 1, \ldots T\}$, the parameters $\theta_{0\tau}$ can be estimated by (39) and (40). The objective function depends only on the available sample information, the known function $m(\cdot)$, and the unknown parameters. Solutions of the above problem are denoted by $\widehat{\theta}_\tau$, the SIVQR estimator. de Castro, Galvao, and Kaplan (2017) discuss and give conditions for identification of the parameters of interest, and consider estimation and inference with weakly dependent data. The parameter $\theta_{0\tau}$ is "locally identified" if there exists a neighborhood of $\theta_{0\tau}$ in which only $\theta_{0\tau}$ satisfies (34). This property holds if the partial derivative matrix of the right-hand side of (34) with respect to the $\theta$ argument is full rank.[25] In addition, de Castro, Galvao, and Kaplan (2017) establish the asymptotic properties, namely consistency and asymptotic normality, of the SIVQR estimator.

**Theorem 4.2** (de Castro, Galvao and Kaplan, 2017)**.** *Under standard regularity conditions, as $T \to \infty$, the estimator is consistent, i.e., $\widehat{\theta}_\tau \xrightarrow{p} \theta_{0\tau}$, and*

$$\sqrt{T}(\widehat{\theta}_\tau - \theta_{0\tau}) \xrightarrow{d} N(0, G^{-1}\Sigma_{M_\tau}[G^\top]^{-1}),$$

*where $\Sigma_{M_\tau} = \lim_{T\to\infty} \mathrm{Var}\left(T^{-1/2}\sum_{t=1}^{T} g_{Tt}(\theta_{0\tau}, \tau)\right)$, $G = \frac{\partial}{\partial\theta^\top}\mathrm{E}[w_t 1\{m(c_t, r_t, w_t, \theta) \le 0\}]\big|_{\theta=\theta_{0\tau}} = -\mathrm{E}[w_t d_t^\top f(0|w_t, d_t)]$, $d_t \equiv \nabla_\theta m(c_t, r_t, \theta_{0\tau})$, and $f_{m|w,d}(\cdot|w, d)$ is the conditional pdf of $m_t$ given $w_t = w$ and $d_t = d$.*

Given the result in Theorem 4.2, one is able to estimate the variance-covariance matrix and conduct practical inference. One is also able to apply the SIVQR methods and estimate $(\gamma_\tau, \beta_\tau)$ across different quantiles by simply varying $\tau$. One advantage of the quantile Euler equation is that it may be log-linearized with no approximation error, differently from the standard Euler equation. Thus, we use a model as $Q_{1-\tau}[\ln(c_{t+1}/c_t) - \gamma_\tau^{-1}\ln(\beta_\tau) - \gamma_\tau^{-1}\ln(1 + r_{t+1})|\Omega_t] = 0$. From $\ln(\beta_\tau)/\gamma_\tau$ and $1/\gamma_\tau$ we are able to recover the parameters of interest.

**Remark 4.3.** *In this paper, we are interested in estimating the conditional quantile functions to learn about the potential underlying heterogeneity among quantiles. Nevertheless, it is possible to see the quantile $\tau$ as a parameter to be estimated together with the parameters $\theta_{0\tau}$. Bera, Galvao, Montes-Rojas, and Park (2016) develop an approach that delivers estimates for the coefficients together with a representative quantile. In their framework, $\tau$ captures a measure of asymmetry of the conditional distribution of interest and is associated with the "most probable" quantile in the sense that it maximizes the entropy.*

### 4.3 Data

We use a data set that is common in the literature. We use monthly data from 1959:01 to 2015:11, which produces 683 observations. As is standard in the literature (see, e.g., Hansen and Singleton (1982)), two different measures of consumption were considered: nondurables, and nondurables plus services. The monthly, seasonally adjusted observations of aggregate nominal consumption (measured in billions of dollars unit) of nondurables and services were obtained from the Federal Reserve Economic Data. Real per capita consumption series were constructed by dividing each observation of these series by the corresponding observation of population, deflated by the corresponding CPI (base 1973:01).

Each measure of consumption was paired with four sets of stock returns from the Center for Research in Security Prices (CRSP) U.S. Stock database, which contains month-end prices

[25] Identification for general nonlinear semiparametric and nonparametric conditional moment restrictions models is presented in Chen, Chernozhukov, Lee, and Newey (2014).

for primary listings for the New York Stock Exchange (NYSE). We use the equally-weighted average of returns (EWR) (including and excluding dividends) on the NYSE. The nominal returns were converted to real returns by dividing by the deflator associated with the measure of consumptions. Instruments include lags of log real consumption growth, nominal interest rate, inflation, and a log dividend-price ratio for equities. We use two instruments (same excluded instruments used in Yogo (2004)): constant, and the linear projection of the real interest rate onto a constant and nominal interest rate, inflation, and log consumption growth. All instruments are lagged twice to avoid problems with time aggregation in consumption data. We use smoothing bandwidth of $h = 10^{-4}$. The bandwidth for the estimation of the variance-covariance matrix is usually $k = 1$.

### 4.4 Results

Before we present the estimation results, it is important to discuss the interpretation of the parameters of interest $(\beta_\tau, \gamma_\tau)$. We notice that as discussed in Subsection 2.4 it is possible to separate the risk attitude from the intertemporal substitutability in the quantile model. First, the present notion of risk preference differs in several respects from the one familiar in the expected utility literature. The quantile $\tau$ captures the risk attitude in the model. As discussed previously, given that the notion of risk attitude is comparative and captured by varying the quantile index, we estimate the model for several different quantiles. Thus, an important point in the application is to compare estimates across quantiles, that is, different measures of risk. Second, for a given quantile $\tau$, $\beta_\tau$ is the usual discount factor. Finally, from the discussion in Subsection 2.4 and equation (26), one can notice that the parameter $1/\gamma_\tau$ captures the standard measure of EIS implicit in the CES utility function. Thus, by employing the quantile maximization model, for each specific risk attitude $\tau$, we are able to estimate the associated discount factor and EIS.

Now we present the empirical results. Because the literature reports results for conditional mean models, for comparison purposes, we also include estimates of the standard conditional average regression IV (TSLS) version of the model.

The results for the estimates of the parameters at different quantiles are reported in Table 1 and Figure 2. We present estimates using consumption of nondurables and stock return with and without dividends. The panels on the left display the estimates for EWR excluding dividends (EWRwo), and the right panel including dividends (EWRw). The results for consumption nondurables plus services are qualitatively similar, for brevity, we omit them.

Table 1: SIVQR and TSLS estimates for Discount Factor and EIS

| | EWRwo | | EWRw | |
|---|---|---|---|---|
| $\tau$ | $\beta_\tau$ | $1/\gamma_\tau$ | $\beta_\tau$ | $1/\gamma_\tau$ |
| 0.1 | 1.156* | 0.105* | 1.146* | 0.112* |
| 0.2 | 1.061* | 0.165* | 1.059* | 0.159** |
| 0.3 | 1.033* | 0.190* | 1.029* | 0.193* |
| 0.4 | 1.006* | 0.380* | 1.003* | 0.368* |
| 0.5 | 0.991* | 0.314* | 0.989* | 0.325* |
| 0.6 | 0.979* | 0.543* | 0.974* | 0.195* |
| 0.7 | 0.968* | 0.836* | 0.963* | 0.527*** |
| 0.8 | 0.802*** | 0.159* | 0.832 | 0.170 |
| 0.9 | 0.806*** | 0.300* | 0.729*** | 0.160* |
| TSLS | 0.992* | 0.203 | 0.989* | 0.204 |

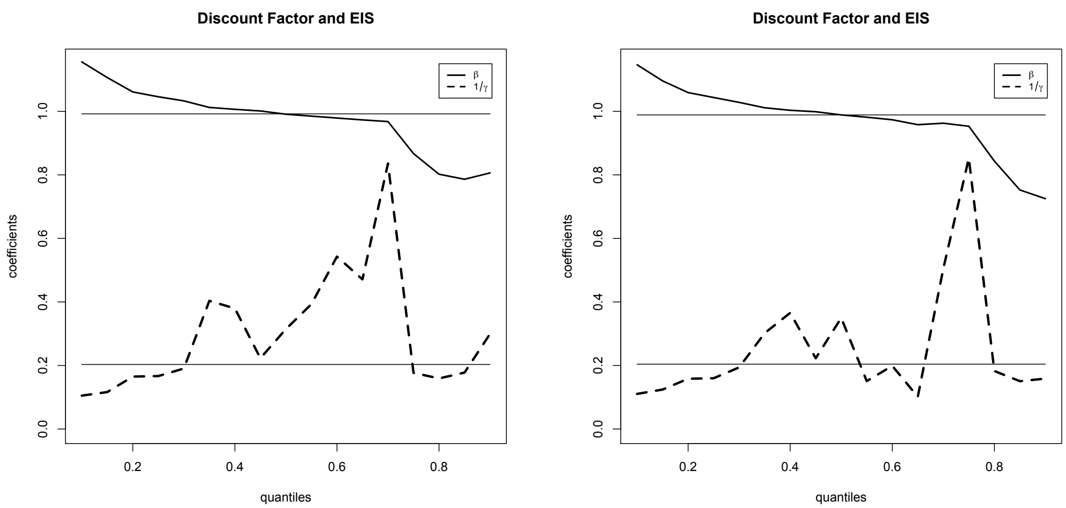
This table shows coefficients returned from applying SIVQR and TSLS methods to estimate the Euler equation. *, **, and *** represent statistical significance at the 1%, 5%, and 10% levels, respectively.

First we consider the estimates of the discount factor using EWRwo. The results show empirical evidence that estimates of the discount factor are decreasing across the quantiles. For the upper quantiles the estimates are close to 0.80. Table 1 shows that, for low quantiles, the discount factor estimates are larger than one. Epstein and Zin (1991, p. 282) also estimate a discount factor larger than one. Nevertheless, for the first four deciles we are not able to reject the null hypothesis of the discount factors being statistically equal to one. Overall, Table 1 shows evidence that the discount factor is smaller for upper quantiles (higher risk aversion). In the same way, the smaller risk aversion, the more patient. The results for the TSLS case are consistent with the literature and show a discount factor of 0.992.

Next we consider the estimates the EIS, $1/\gamma_\tau$. The left panels in Table 1 and Figure 2 present the results using EWRwo. The first interesting observation is that the results document evidence of heterogeneity in EIS across quantiles. In particular, the table shows that overall the EIS increases across quantiles, especially for $\tau \in (0.1, 0.7)$, such that EIS is relatively larger for upper quantiles. The smaller EIS, for low quantiles (less risk preferring), means less sensitivity to changes in intertemporal consumption. There is an existing literature exploring whether the EIS varies with the level of consumption (or wealth) which rejects the constant-EIS hypothesis (Blundell, Browning, and Meghir (1994); Attanasio and Browning (1995); Atkeson and Ogaki (1996)). In this paper we shed light on the discussion and allow the EIS to vary with the risk attitude, indexed by the quantile. We show that the EIS varies substantially in this case.

The right panels in Table 1 and Figure 2 display the estimates when considering EWRw. They serve as a robustness check. The results are qualitatively similar to those in the left panel and Figure 2. The coefficients of EIS present variation over the quantiles. The relative EIS increases across quantiles. The discount factor estimates also present heterogeneity, especially

Figure 2: Nondurables plus services and EWRwo and EWRw




for upper quantiles. The discount factor is smaller for larger quantiles (more risk-taking), which suggests less patient. On the contrary, for lower quantiles, the risk aversion is large, as is the discount factor, providing evidence that more risk-averse is related to more patient.

**Remark 4.4.** *It should be noted that our model does not control for income or wealth. Thus, the agents that correspond to low quantiles* do not *necessarily correspond to low income, but to low risk aversion. This observation is important to avoid confusion with the results in a branch of the literature that links discount factors with income and wealth (see, e.g., Hausman (1979), and Lawrance (1991)). Moreover, there is empirical evidence that documents small discount factors estimates. This literature estimates discount factors by using a quasi-hyperbolic discount function (see, e.g., Paserman (2008), Fang and Silverman (2009), and Laibson, Maxted, Repetto, and Tobacman (2015)). In contrast to these streams of literature, this paper abstracts from a relationship between discount rates and poverty and employs a simple model to estimate the discount factor. Our objective is to illustrate the potential empirical application of the quantile utility maximization model. We leave the connection with income and wealth and related extensions for future research.*

In all, the application illustrates that the new methods serve as an important alternative tool to study economic behavior. The methods allow one to estimate the discount factor and EIS at different levels of risk attitude (quantiles). Our empirical results document heterogeneity in both discount factor and EIS across quantiles.

# 5 Summary and Open Questions

This paper develops a dynamic model of rational behavior under uncertainty for an agent maximizing the quantile utility function indexed by a quantile $\tau \in (0, 1)$. More specifically, an agent maximizes the stream of future $\tau$-quantile utilities, where the quantile preferences induce the quantile utility function. We first axiomatize the recursive quantile preferences. We then show dynamic consistency of the preferences and that this dynamic problem yields a value function, using a fixed-point argument. We also obtain desirable properties of the value function. In addition, we derive the corresponding Euler equation.

Empirically, we show that one can employ existing general instrumental variables nonlinear quantile regression methods for estimating and testing the rational quantile models directly from stochastic Euler equations. An attractive feature of this method is that the parameters of the dynamic objective functions of economic agents can be interpreted as structural objects. Finally, to illustrate the methods, we construct an asset-pricing model and estimate the implied discount factor and elasticity of intertemporal substitution (EIS) parameters for different quantiles. The results provide evidence that both discount factor and EIS vary across quantiles.

Many issues remain to be investigated. First, although Subsection 2.4 discusses risk attitude for quantile preferences, much more needs to be studied and clarified. Our axiomatization could be further generalized to include the Markov case. Another interesting direction would be to generalize our model to the case where the future state is a randomly defined instead of directly chosen. The extension of the quantile maximization model from considering a single quantile to multiple quantiles simultaneously would be important. Extensions of the methods to general equilibrium models pose challenging new questions. In addition, aggregation of the quantile preferences is also a critical direction for future research. Applications to asset pricing and consumption models would appear to be a natural direction for further development of quantile utility maximization models.

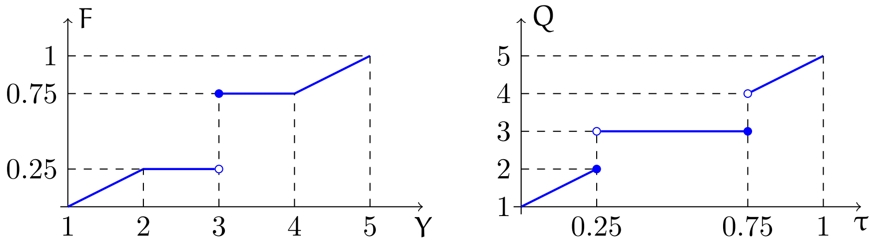



Figure 3: c.d.f. and quantile function of a random variable.

# A Appendix

## A.1 Properties of Quantiles

Figure 3 illustrates the c.d.f. $F$ of a random variable $X$, and its corresponding quantile function $Q(\tau) = \inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant \tau\}$, for $\tau > 0$.[26] In this case, $X$ assumes the value 3 with 50% probability and is uniform in $[1,2] \cup [4,5]$ with 50% probability. This picture is useful to inspire some of the properties that we state below. Note, for instance, the discontinuities and the values over which the quantile is constant.

The following lemma is an auxiliary result that will be helpful for the derivations below.

**Lemma A.1.** *The following statements are true:*

*(i)* $Q$ *is increasing, that is,* $\tau \leqslant \hat{\tau} \implies Q(\tau) \leqslant Q(\hat{\tau})$.

*(ii)* $\lim_{\tau \downarrow \hat{\tau}} Q(\tau) \geqslant Q(\hat{\tau})$.

*(iii)* $Q$ *is left-continuous, that is,* $\lim_{\tau \uparrow \hat{\tau}} Q(\tau) = Q(\hat{\tau})$.

*(iv)* $\Pr(\{z : z < Q(\tau)\}) \leqslant \tau \leqslant \Pr(\{z : z \leqslant Q(\tau)\}) = F(Q(\tau))$.

*(v)* *If* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous and strictly increasing function, then* $Q_\tau[g(X)] = g(Q_\tau[X])$.

*(vi)* *If* $g, h : \mathbb{R} \to \mathbb{R}$ *are such that* $g(\alpha) \leqslant h(\alpha), \forall \alpha$, *then* $Q_\tau[g(Z)] \leqslant Q_\tau[h(Z)]$.

*(vii)* $F$ *is continuous if and only if* $Q$ *is strictly increasing.*

*(viii)* $F$ *is strictly increasing if and only if* $Q$ *is continuous.*

*Proof.* (i) Let us first assume $\tau > 0$. If $\tau \leqslant \hat{\tau}$, then $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \tau\} \supseteq \{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \hat{\tau}\}$. This implies $Q_Z(\tau) \leqslant Q_Z(\hat{\tau})$. Next, if $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = -\infty$, there is nothing else to prove. If $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = x \in \mathbb{R}$, then $F_Z(x - \epsilon) = 0$ for any $\epsilon > 0$. Let $\hat{\tau} > 0$. Then, $y \in \{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \hat{\tau}\} \implies y > x - \epsilon$, which in turn implies $Q_Z(\hat{\tau}) \geqslant x - \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $Q_Z(\hat{\tau}) \geqslant x = Q_Z(0)$, which concludes the proof.

(ii) From (i), $\lim_{\tau \downarrow \hat{\tau}} Q_Z(\tau) \geqslant \inf_{\tau \geqslant \hat{\tau}} Q_z(\tau) \geqslant Q_z(\hat{\tau})$. Figure 3 illustrates (for example for $\hat{\tau} = 0.25$) that the inequality can be strict.

(iii) From (i), we know that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) \leqslant Q_z(\hat{\tau})$. For the other inequality, assume that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) + 2\epsilon < Q_z(\hat{\tau}) < \infty$, for some $\epsilon > 0$. This means that for each $k \in \mathbb{N}$, we can find $\alpha^k \in \{\alpha : F_Z(\alpha) \geqslant \hat{\tau} - \frac{1}{k}\}$ such

[26] For $\tau = 0$, $Q(0) = \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}$ is just the lower limit of the support of the variable.

that $Q_Z(\hat{\tau}-\frac{1}{k}) \leqslant \alpha^k \leqslant Q_Z(\hat{\tau}-\frac{1}{k})+\epsilon < Q_Z(\hat{\tau})-\epsilon$. We may assume that $\{\alpha^k\}$ is an increasing sequence bounded by $Q_z(\hat{\tau})$ and thus converges to some $\bar{\alpha} \in \mathbb{R}$. Then, $\lim_{\tau\uparrow\hat{\tau}} Q_Z(\tau) \leqslant \bar{\alpha} \leqslant Q_z(\hat{\tau})-\epsilon < Q_z(\hat{\tau})$. Since $F_Z(\alpha^k) \geqslant \hat{\tau}-\frac{1}{k}$ and $F_Z$ is upper semi-continuous, $F_Z(\bar{\alpha}) \geqslant \hat{\tau}$, which implies that $\bar{\alpha} \geqslant Q_Z(\hat{\tau})$, a contradiction. Now, assume that $Q_Z(\hat{\tau}) = \infty$. Since $\lim_{\alpha\to\infty} F_Z(\alpha) = 1$, the set $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geqslant \tau\}$ is non-empty for all $\tau < 1$, that is, $Q_Z(\tau) < \infty$ for all $\tau < 1$. Thus, $\hat{\tau} = 1$. If $\lim_{\tau\uparrow 1} Q_Z(\tau) = x \in \mathbb{R}$, then $F_Z(x+1) \geqslant 1-\epsilon$ for all $\epsilon > 0$, which implies that $F_Z(x+1) = 1$ and $Q_Z(1) \leqslant x+1$, a contradiction.

$(iv)$ As above, if $Q_Z(\tau) = \infty$, then $\tau = 1$, which implies $1 = \Pr(\{w : z < \infty\}) = \Pr(\{w : z \leqslant \infty\})$ and there is nothing to prove. Let $\bar{\alpha} = Q_Z(\tau) < \infty$. If $\alpha^k \downarrow \bar{\alpha}$ is such that $F_Z(\alpha^k) \geqslant \tau$, then $F_Z(\bar{\alpha}) \geqslant \tau$, by the well-known upper-semicontinuity of $F_Z$. That is, $\tau \leqslant F_Z(Q_Z(\tau))$. For the other inequality, let $\alpha^k \uparrow \bar{\alpha} = Q_Z(\tau)$. Since $\alpha^k < \bar{\alpha}$, then $\Pr[Z \leqslant \alpha^k] < \tau$, by the definition of $\bar{\alpha}$. Thus, $\Pr[Z < \alpha^k] \leqslant \Pr[Z \leqslant \alpha^k] < \tau$ and $\Pr[Z < \bar{\alpha}] \leqslant \sup_k \Pr[Z < \alpha^k] \leqslant \tau$.

$(v)$ The proof is direct as follows:

$$
\begin{aligned}
Q_\tau[g(Z)] &= \inf\{\alpha \in \mathbb{R} : \Pr[g(Z) \leqslant \alpha] \geqslant \tau\} \\
&= \inf\{\alpha \in \mathbb{R} : \Pr\left[Z \leqslant g^{-1}(\alpha)\right] \geqslant \tau\} \\
&= \inf\{\alpha \in \mathbb{R} : g^{-1}(\alpha) = \beta, \Pr[Z \leqslant \beta] \geqslant \tau\} \\
&= \inf\{g(\beta) : \Pr[Z \leqslant \beta] \geqslant \tau\} \\
&= g\left(\inf\{\beta : \Pr[Z \leqslant \beta] \geqslant \tau\}\right) \\
&= g\left(Q_\tau[Z]\right).
\end{aligned}
$$

$(vi)$ Since $g \leqslant h$, then for any $\alpha$, $\{z : g(z) \leqslant \alpha\} \supseteq \{z : h(z) \leqslant \alpha\}$, which implies $F_{g(Z)}(\alpha) = \Pr[g(Z) \leqslant \alpha] \geqslant \Pr[h(Z) \leqslant \alpha] = F_{h(Z)}(\alpha)$. If $\tau > 0$, $\{\alpha \in \mathbb{R} : \Pr[g(Z) \leqslant \alpha] \geqslant \tau\} \supseteq \{\alpha \in \mathbb{R} : \Pr[h(Z) \leqslant \alpha] \geqslant \hat{\tau}\}$. Taking infima, we obtain $Q_{g(Z)}(\tau) \leqslant Q_{h(Z)}(\tau)$. On the other hand, $\{\alpha \in \mathbb{R} : F_{h(Z)}(\alpha) = 0\} \subset \{\alpha \in \mathbb{R} : F_{g(Z)}(\alpha) = 0\}$ and taking the supremum in both sides we obtain the same conclusion.

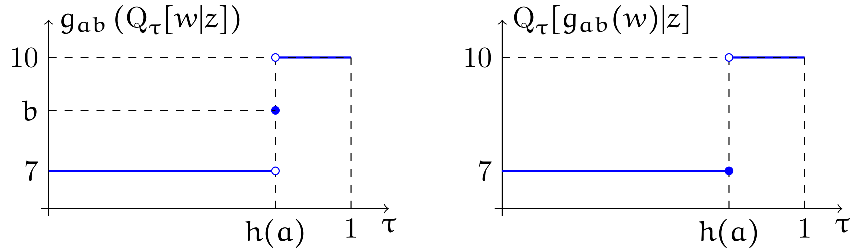
$(vii)$ Assume that $F_Z$ is discontinuous at $x_0$, that is, $\lim_{x\uparrow x_0} F_Z(x) = y_0 < y_1 = F_Z(x_0)$. If $y_0 < y_2 < y_3 < y_1$, then $Q_Z(y_2) = \inf\{\alpha : F_Z(\alpha) \geqslant y_2\} = \inf\{\alpha : F_Z(\alpha) \geqslant y_3\} = Q_Z(y_3)$, that is, $Q_Z$ is not strictly increasing. Conversely, assume that $Q_Z$ is not strictly increasing, that is, there exists $y_2 < y_3$ such that $Q_Z(y_2) = Q_Z(y_3) = x$. By definition, this means that $F_Z(x-\epsilon) < y_2 < y_3 \leqslant F_Z(x+\epsilon)$, for all $\epsilon > 0$. But this implies that $F_Z$ is not continuous at $x$.

$(viii)$ Suppose that $F_Z$ is not strictly increasing, that is, there exists $x_1 < x_2$ such that $F_Z(x_1) = F_Z(x_2) = y$. Then, $Q_Z(y-\epsilon) = \inf\{\alpha : F_Z(\alpha) \geqslant y-\epsilon\} \leqslant x_1 < x_2 \leqslant \inf\{\alpha : F_Z(\alpha) \geqslant y+\epsilon\} = Q_Z(y+\epsilon)$. Thus, $Q_Z$ cannot be continuous at $y$. Conversely, assume that $Q_Z$ is not continuous at $y_0$. Since $Q_Z$ is increasing by $(i)$ and left-continuous by $(iii)$, this means that $Q_Z(y_0) = x_0 < x_1 = \lim_{y\downarrow y_0} Q_Z(y)$. If $x_0 < x_2 < x_1$, then $F_Z(x_2) \leqslant y_0$, otherwise $\lim_{y\downarrow y_0} Q_Z(y) \leqslant x_2$. By $(iv)$, we have $y_0 \leqslant F_Z(Q_Z(y_0)) = F_Z(x_0) \leqslant F_Z(x_2) \leqslant y_0$ , that is, $F_Z$ is not strictly increasing between $x_0$ and $x_2$. □

Let $\Theta$ be a set (of parameters) and $g : \Theta \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be a measurable function. We denote by $Q_\tau[g(\theta,\cdot)|z]$ the quantile function associated with $g$, that is:

$$
Q_\tau[g(\theta,\cdot)|z] \equiv \inf\{\alpha \in \mathbb{R} : \Pr\left([g(\theta, W) \leqslant \alpha]|Z = z\right) \geqslant \tau\}. \tag{41}
$$

The following Lemma generalizes equation (1) to conditional quantiles.



(a): $g_{ab}(Q_\tau[w|z])$. (b): $Q_\tau[g_{ab}(w)|z]$.

Figure 4

**Lemma A.2.** *Let* $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ *be non-decreasing and left-continuous in* $\mathcal{Z}$. *Then,*

$$Q_\tau[g(\theta,\cdot)|z] = g(\theta, Q_\tau[w|z]). \tag{42}$$

It is useful to illustrate the above result with an example. Let us define the function $g_{ab} : [1,5] \to \mathbb{R}$ by:

$$g_{ab}(x) = \begin{cases} 7, & \text{if } x < a \\ b, & \text{if } x = a \\ 10, & \text{if } x > a \end{cases}$$

The function $g_{ab}$ thus defined is non-decreasing if $b \in [7,10]$ and it is left-continuous if $b = 7$.

Consider the r.v. $X$ whose c.d.f. $F$ and quantile function $Q$ are shown in Figure 3 above. Let $F_{ab}$ and $Q_{ab}$ denote respectively the c.d.f. and quantile functions associated to $g_{ab}(Z)$. Figure 4 shows $Q_\tau[g_{ab}(w)|z]$ and $g_{ab}(Q_\tau[w|z])$ for $a \in [1,5]$ and $b \in [7,10]$. The point of discontinuity is a function of $a$ ($h(a) \in [0,1]$).

**Proof of Lemma A.2:** For a contradiction, let us first assume that

$$Q_\tau[g(\theta,\cdot)|z] > g(\theta, Q_\tau[w|z]) \equiv \hat{\alpha}.$$

This means that $\hat{\alpha} \notin \{\alpha \in \mathbb{R} : \Pr(\{w : g(\theta,w) \leqslant \alpha\}|z) \geqslant \tau\}$, that is,

$$\Pr(\{w : g(\theta,w) \leqslant \hat{\alpha}\}|z) < \tau.$$

Since $\hat{\alpha} = g(\theta, Q_\tau[w|z])$ and $g$ is non-decreasing in $w$, $\{w : w \leqslant Q_\tau[w|z]\} \subset \{w : g(\theta,w) \leqslant \hat{\alpha}\}$. Thus, $\Pr(\{w : w \leqslant Q_\tau[w|z]\}|z) < \tau$, but this contradicts Lemma A.1(iv).

Conversely, assume that

$$Q_\tau[g(\theta,\cdot)|z] < g(\theta, Q_\tau[w|z]).$$

This means that there exists $\tilde{\alpha} < g(\theta, Q_\tau[w|z])$ such that

$$\Pr(\{w : g(\theta,w) \leqslant \tilde{\alpha}\}|z) \geqslant \tau.$$

Let $\tilde{w}$ be the supremum of the set $\{w : g(\theta,w) \leqslant \tilde{\alpha}\}$. Since $g$ is non-decreasing and left-continuous,

$g(\theta, \tilde{w}) \leqslant \tilde{\alpha}$. Moreover,

$$\Pr\left(\{w : w \leqslant \tilde{w}\}|z\right) = \Pr\left(\{w : g(\theta, w) \leqslant \tilde{\alpha}\}|z\right) \geqslant \tau.$$

Thus, $\tilde{w} \in \{\alpha \in \mathbb{R} : \Pr\left(\{w : w \leqslant \alpha\}|z\right) \geqslant \tau\}$, which implies that $\tilde{w} \geqslant Q_\tau[w|z]$. Thus, $\tilde{\alpha} \geqslant g(\theta, \tilde{w}) \geqslant g(\theta, Q_\tau[w|z]) > \tilde{\alpha}$, which is a contradiction. □

The following Corollary to the above Lemma will be useful.

**Corollary A.3.** *Let $T \in \mathbb{N} \cup \{\infty\}$, $h : \Theta \times \mathcal{Z}^T \times \mathcal{Z} \to \mathbb{R}$, $g : \Lambda \times \mathcal{Z}^T \times \mathcal{Z} \to \mathbb{R}$ be non-decreasing and left-continuous in $\mathcal{Z}$. Then,*

$$Q_\tau\left[h(\theta, z^T, Q_\tau[g(\lambda, z^T, z_{t+1})|z_t])|z_1\right] = Q_\tau\left[h\left(\theta, z^T, g(\lambda, z^T, Q_\tau[z_{t+1}|z_t])\right)|z_1\right].$$

*Proof.* Let $X$ denote the random variable $Q_\tau[g(\lambda, z^t, z_{t+1})|z_t])$ and similarly, let $Y$ denote $g(\lambda, z^t, Q_\tau[z_{t+1}|z_t])$. Then, by Lemma A.2, $X = Y$. Therefore, $h(\theta, z^t, X) = h(\theta, z^t, Y)$ and the result follows. □

The following result will be useful below.

**Proposition A.4.** *Given the random variables $X$ and $Y$, assume that there exists random variable $Z$ and continuous and increasing functions $h$ and $g$ such that $X = h(Z)$ and $Y = g(Z)$. Then $Q_\tau[X + Y] = Q_\tau[X] + Q_\tau[Y]$.*

*Proof.* Let $Z$, $h$ and $g$ be as in the definition. Define $\tilde{h}(Z) \equiv h(Z) + g(Z)$. This function is clearly continuous and increasing. Therefore,

$$\begin{aligned} Q_\tau[X + Y] &= Q_\tau[\tilde{h}(Z)] = \tilde{h}(Q_\tau[Z]) = h(Q_\tau[Z]) + g(Q_\tau[Z]) \\ &= Q_\tau[h(Z)] + Q_\tau[g(Z)] = Q_\tau[X] + Q_\tau[Y]. \end{aligned}$$

by applying Lemma A.2 twice. □

## A.2 Proofs of Section 2

The proof of Proposition 2.1 follows closely that of Chambers (2007). We need to introduce some notation. If $x \sim y$ for all $x, y \in X$, then Proposition 2.1 holds trivially. Indeed, we can define $u(x) = 1$ for all $x \in X$ and choose any $\tau \in (0, 1)$. We obtain $U(f) = 1$ for all $f \in \mathcal{F}$, which represents the preference from monotonicity. Therefore, from now on, we assume that there exists $\overline{x}$ and $\underline{x} \in X$, such that $\overline{x} \succ \underline{x}$. For the rest of the proof, let this $\overline{x}$ and $\underline{x}$ be fixed. We begin with the following auxiliary result:

**Lemma A.5.** *For any $x, y \in X$ and any $E \subset S$, we have $xEy \sim x$ or $xEy \sim y$.*

*Proof.* If $x \sim y$, there is nothing to prove, as this would imply by monotonicity, $xEy \sim x \sim y$. Thus, let us assume, without loss of generality, that $x \succ y$. By monotonicity, $xEy \succcurlyeq y$. Assume that $xEy \succ y$. Let $z \in X$ be such that $z \sim xEy$, which exists by continuity. Define $\varphi : X \to X$ by:

$$\varphi(w) = \begin{cases} x, & \text{if } w \succ y \\ y, & \text{if } w \preccurlyeq y \end{cases}$$

Clearly, $\varphi$ is increasing. Since $z \sim xEy \succ y$, $\varphi(z) = x$. Since $xEy \sim z$, by ordinality we have $xEy = \varphi(xEy) \sim \varphi(z) = x$. This concludes the proof. $\square$

A collection of sets $\mathcal{E} \subset \Sigma$ is a downset if $A \in \mathcal{E}$ and $B \subset A$ and $B \in \Sigma$ implies that $B \in \mathcal{E}$.

**Proposition A.6.** *If $\succcurlyeq$ satisfies axioms Q1-Q4, there exists $u : X \to \mathbb{R}$ and a unique downset $\mathcal{E} \subset \Sigma$ such that $\varnothing \in \mathcal{E}$ and $S \notin \mathcal{E}$ for which*

$$U(f) = \inf\{\alpha : \{s \in S : u(f(s)) \geqslant \alpha\} \in \mathcal{E}\}.$$

*Proof.* From continuity, there exists a continuous function $U : \mathcal{F} \to \mathbb{R}$ that represents $\succcurlyeq$. Since we see $X$ as a subset of $\mathcal{F}$, we can define $u : X \to \mathbb{R}$ by $u(x) = U(x)$. Without loss of generality, we may assume that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$. For notation simplicity, in this proof only we denote by $1_E$ the act that is equal to $\overline{x}$ if $s \in E$ and is equal to $\underline{x}$ if $s \notin E$. By Lemma A.5, we have $U(1_E) \in \{0,1\}$.

Let us define $\mathcal{E}$ as the set of those $E \in \Sigma$ such that $U(1_E) = 0$. It is easy to see that this defines a downset: if $B \subset E$, $B \in \Sigma$, $E \in \mathcal{E}$ then $1_E \succcurlyeq 1_B$ by monotonicity. Therefore, by Lemma A.5, $0 = U(1_E) \geqslant U(1_B) \in \{0,1\}$, which implies $U(1_B) = 0 \Rightarrow B \in \mathcal{E}$. Moreover, $S \notin \mathcal{E}$.

We will show that for all $E \in \Sigma$, $U(1_E) = \inf\{\alpha \in [0,1] : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\}$. Consider first the case $E \in \mathcal{E}$. For all $\alpha \in (0,1)$, $\{s \in S : u(1_E(s)) \geqslant \alpha\} = E$, so that $\{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}$. Therefore, $\inf\{\alpha : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\} \leqslant 0$. However, for all $\alpha < 0$, $\{s \in S : 1_E(s) \geqslant \alpha\} = S \notin \mathcal{E}$. Hence, we may conclude that $\inf\{\alpha : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\} = 0$, so that $U(1_E) = \inf\{\alpha : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\}$.

Suppose now that $E \notin \mathcal{E}$. Then for all $\alpha > 1$, $\{s \in S : u(1_E(s)) \geqslant \alpha\} = \varnothing$, so that $\inf\{\alpha : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\} \leqslant 1$. But for $\alpha \in (0,1)$, $\{s \in S : u(1_E(s)) \geqslant \alpha\} = E$, so that $\{s \in S : u(1_E(s)) \geqslant \alpha\} \notin \mathcal{E}$. Hence, $\inf\{\alpha : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\} = 1$. Therefore, $U(1_E) = \inf\{\alpha : \{s \in S : u(1_E(s)) \geqslant \alpha\} \in \mathcal{E}\}$. This also shows that $\mathcal{E}$ is the unique downset that can satisfy (3) for acts $1_E$. Moreover, for any act $xEy$, with $x \succ y$, we have $xEy = \varphi(1_E)$ for some increasing $\varphi$. Thus, Lemma A.5 and ordinality imply that

$$U(xEy) = u(x), \text{ if } E \notin \mathcal{E} \text{ and } U(xEy) = u(y) \text{ if } E \in \mathcal{E}. \tag{43}$$

Next, we extend the result to all functions in $\mathcal{F}$. Let $f \in \mathcal{F}$ be arbitrary, and set $\alpha^*(f) = \inf\{\alpha : \{s \in S : u(f(s)) \geqslant \alpha\} \in \mathcal{E}\}$. We want to conclude that $U(f) \leqslant \alpha^*(f)$. Let $\epsilon > 0$. Then $\{s \in S : u(f(s)) \geqslant \alpha^*(f) + \epsilon\} \in \mathcal{E}$ by definition of $\alpha^*(f)$. Let $g^\epsilon \in \mathcal{F}$ be defined by $xEy$, where $E = \{s \in S : u(f(s)) \geqslant \alpha^*(f) + \epsilon\}$, and $x, y \in X$ are any consequences such that $x \succcurlyeq f(s), \forall s \in E$ and $y \succcurlyeq f(s), \forall s \notin E$, so that $u(y) \leqslant \alpha^*(f) + \epsilon$. These $x$ and $y$ exist since $S$ is finite. Thus, $g^\epsilon(s) \succcurlyeq f(s), \forall s \in S$, which implies, by monotonicity, $U(f) \leqslant U(g^\epsilon)$. By Lemma A.5 and (43), $U(g^\epsilon) = u(y) \leqslant \alpha^*(f) + \epsilon$. Since $\epsilon$ is arbitrary, $U(f) \leqslant \alpha^*(f)$.

Now we wish to conclude that $U(f) \geqslant \alpha^*(f)$. Let $\epsilon > 0$. Then $\{s \in S : u(f(s)) \geqslant \alpha^*(f) - \epsilon\} \notin \mathcal{E}$ by definition of $\alpha^*(f)$. Let $h^\epsilon \in \mathcal{F}$ be the act $xEy$ where $E = \{s \in S : u(f(s)) \geqslant \alpha^*(f) - \epsilon\}$, and $x, y \in X$ are any consequences such that $f(s) \succcurlyeq x, \forall s \in E$, $f(s) \succcurlyeq y, \forall s \notin E$ and $u(x) \geqslant \alpha^*(f) - \epsilon$. This implies that $f(s) \succcurlyeq h^\epsilon(s), \forall s \in S$ and $U(f) \geqslant U(h^\epsilon)$. Moreover, $\{s \in S : u(f(s)) \geqslant \alpha^*(f) - \epsilon\} \notin \mathcal{E}$. By (43), $U(h^\epsilon) = u(x) \geqslant \alpha^*(f) - \epsilon$. Thus, $U(f) \geqslant \alpha^*(f) - \epsilon$. As $\epsilon$ is arbitrary, $U(f) \geqslant \alpha^*(f)$. Therefore $U(f) = \inf\{\alpha : \{s \in S : u(f(s)) \geqslant \alpha\} \in \mathcal{E}\}$. $\square$

**Proof of Sufficiency in Proposition 2.1:** This proof adapts the argument of Chambers (2007, Theorem 2) to exclude the case $\tau = 0$. We reproduce the whole argument here for completeness and readers' convenience. Let $U : \mathcal{F} \to \mathbb{R}$ and $\mathcal{E}$ be respectively the utility function and the downset shown to exist by Proposition A.6. Let $S = \{s_1, ..., s_{|S|}\}$ and for any $E \subset S$, let $\bar{u}^E$ denote the vector in $\{0,1\}^{|S|}$ such that $\bar{u}^E_i = 1$ if and only if $s_i \in E$.

We want to show that there exists a probability measure $p$, represented by a vector in $[0,1]^{|S|}$, and a number $\tau \in (0,1)$ such that $\mathcal{E} = \{E \in \Sigma : p(E) \leqslant \tau\} = \{E \in \Sigma : \bar{u}^E \cdot p \leqslant \tau\}$. For this, let us argue first that there exists $(\tau, p)$ solution of the following system of linear inequalities:

- for all $E \in \mathcal{E}$, $(1, -\bar{u}^E) \cdot (\tau, p) \geqslant 0$;
- for all $E \notin \mathcal{E}$, $(-1, \bar{u}^E) \cdot (\tau, p) > 0$;
- for all $s \in S$, $(0, 1_{\{s\}}) \cdot (\tau, p) \geqslant 0$,
- $(0, 1_S) \cdot (\tau, p) > 0$, and

For a contradiction, assume that this system of linear inequalities does not have a solution. Then there must exist (see, for example, Rockafellar (1970, Theorem 22.2)), nonnegative integers for each of the preceding constraints, so that

$$\sum_{E \in \mathcal{E}} n_E (1, -\bar{u}^E) + \sum_{E \notin \mathcal{E}} n_E (-1, \bar{u}^E) + \sum_{s \in S} n_s (0, 1_{\{s\}}) + m(0, 1_S) = 0.$$

Furthermore, one of the integers associated with one of the strict inequalities must be positive. Therefore, we may also conclude that at least one of the $n_E$ corresponding to an $E \in \mathcal{E}$ must be positive. Moreover, in order to equal zero, $\sum_{E \in \mathcal{E}} n_E = \sum_{E \notin \mathcal{E}} n_E$. Define $n \equiv \sum_{E \in \mathcal{E}} n_E$, and list out all of the sets $E \in \mathcal{E}$ a total of $n_E$ times each to form a sequence $\{A_1, \ldots, A_n\} \subset \mathcal{E}$. List out all of the sets $E \notin \mathcal{E}$ a total of $n_E$ times each to form a sequence $\{B_1, \ldots, B_n\} \subset \Sigma \backslash \mathcal{E}$. By definition of $\mathcal{E}$, for all $i = 1, \ldots, n$, $U(1_{A_i}) < U(1_{B_i})$. As the constraints sum to zero, and the weights $n_s$ and $m$ are nonnegative, $\sum_{i=1}^n 1_{A_i} \geqslant \sum_{i=1}^n 1_{B_i}$. This contradicts betting consistency.

Therefore, there exists $(\tau, p) \in \mathbb{R}_+^{|S|+1}$ the above system of linear inequalities. By dividing $\tau, p)$ by $\bar{u}^S \cdot p > 0$, we may assume that $\bar{u}^S \cdot p = 1$, that is, $p$ is a probability measure, satisfying $\mathcal{E} = \{E \in \Sigma : p(E) \leqslant \tau\}$.

Since $S \notin$ E, $p(S) = 1 > \tau$. If $\tau = 0$, let $S^0 \equiv \{s \in S : p(\{s\}) = 0\}$ and fix $\hat{s} \in S \setminus S^0$ such that $p(\{\hat{s}\}) = \min\{p(\{s\}) : s \in S \setminus S^0\}$. Let $n \equiv |S^0|$ and $\epsilon \equiv p(\{\hat{s}\})$. Define $\hat{\tau} = \frac{\epsilon}{2}$ and for each $s \in S$,

$$\hat{p}(\{s\}) = \begin{cases} p(\{s\}), & \text{if } s \notin S^0 \cup \{\hat{s}\} \\ \frac{2\epsilon}{3}, & \text{if } s = \hat{s} \\ \frac{\epsilon}{3n}, & \text{if } s \in S^0 \end{cases}$$

Then, $\hat{p}$ is a probability, $\hat{\tau} \in (0,1)$ and $\mathcal{E} = \{E \in \Sigma : p(E) \leqslant \tau\} = \{E \in \Sigma : \hat{p}(E) \leqslant \hat{\tau}\}$. □

**Proof of Necessity in Proposition 2.1:** Assume that $U(f) = Q^p_\tau[u(f)]$ represents $\succcurlyeq$, for some utility $u : X \to \mathbb{R}$, probability $p : \Sigma \to [0,1]$ and $\tau \in (0,1)$, where

$$Q^p_\tau[u(f)] \equiv \inf\{\alpha : p(\{s \in S : u(f(s)) \geqslant \alpha\}) \leqslant \tau\}.$$

Axioms Q1, Q2 and Q3 are easily seen to be satisfied. To see that the probabilistic quantiles satisfy betting consistency (Q5), let $U$ represent $\succcurlyeq$ be given by (3), for some $u : X \to \mathbb{R}$, probability $p : \Sigma \to [0,1]$ and $\tau \in (0,1)$. Let $\{A_1, \ldots, A_n\} \subset 2^S$ and $\{B_1, \ldots, B_n\} \subset 2^S$ for which $\sum_{i=1}^n 1_{A_i} \geqslant \sum_{i=1}^n 1_{B_i}$. Suppose, by means of contradiction, that for all $i \in \{1, \ldots, n\}$, $U(\bar{x} B_i \underline{x}) > U(\bar{x} A_i \underline{x})$. This is only possible if for all $i \in \{1, \ldots, n\}$, $U(\overline{x} B_i \underline{x}) = 1$ and $U(\overline{x} A_i \underline{x}) = 0$. Hence, for all $i \in \{1, \ldots, n\}$, $p(B_i) > \tau$ and $p(A_i) \leqslant \tau$. Let $E_p[\cdot]$ denote the expectation with respect to $p$. As $\sum_{i=1}^n 1_{A_i}(s) \geqslant \sum_{i=1}^n 1_{B_i}(s)$, for all $s \in S$, $E_p[\sum_{i=1}^n 1_{A_i}] = \sum_{i=1}^n p(A_i) \geqslant E_p[\sum_{i=1}^n 1_{B_i}] = \sum_{i=1}^n p(B_i)$. However, $\sum_{i=1}^n p(B_i) > n\tau$ and $n\tau \geqslant \sum_{i=1}^n p(A_i)$, a contradiction.

To see that Q4 (ordinality) is satisfied, fix $f, g \in \mathcal{F}$ and an increasing $\varphi : X \to X$. Let $f(S) = X^f = \{x_1^f, ..., x_n^f\}$ and $g(S) = X^g = \{x_1^g, ..., x_m^g\}$. Without loss of generality, we may assume $u(x_1^f) < \ldots < u(x_n^f)$ and similarly for $X^g$.

It is easy to see that $Q_\tau^p[u(f)] \in \{u(x_1^f), ..., u(x_i^f), ..., u(x_n^f)\}$. Let $i$ be such that $u(x_i^f) = Q_\tau^p[u(f)]$. Similarly, $u(x_j^g) = Q_\tau^p[u(g)]$. We assume that $f \succcurlyeq g$, that is, $u(x_j^g) \leqslant u(x_i^f)$.

Of course $Q_\tau^p[u(\varphi(f))] \in \{u(\varphi(x_1^f)), ..., u(\varphi(x_i^f)), ..., u(\varphi(x_n^f))\}$ and $u(\varphi(x_1^f)) \leqslant \ldots \leqslant u(\varphi(x_n^f))$. In fact,

$$Q_\tau^p[u(\varphi(f))] = u(\varphi(x_k^f)) \implies u(\varphi(x_k^f)) = \cdots = u(\varphi(x_i^f))$$

and

$$Q_\tau^p[u(\varphi(g))] = u(\varphi(x_\ell^g)) \implies u(\varphi(x_\ell^g)) = \cdots = u(\varphi(x_j^g)).$$

Note that $u\left(x_j^g\right) \leqslant u\left(x_i^f\right) \Rightarrow u\left(\varphi\left(x_j^g\right)\right) \leqslant u\left(\varphi\left(x_i^f\right)\right)$. This implies that $u(\varphi(x_\ell^g)) \leqslant u(\varphi(x_k^f))$. Thus, $\varphi(f) \succcurlyeq \varphi(g)$, that is, Q4 holds. □

**Proof of Proposition 2.3:** As usual, it is easy to verify that the axioms are satisfied if the preference has the representation, that is, satisfies the recursive equation (5). Conversely, from Proposition 2.2, we know that if $\succcurlyeq$ satisfies D1-D7, then $\succcurlyeq$ admits a recursive representation $(V, W, I)$ such that $W(c, x) = u(c) + b(c)x$, with $b(c) \in (0, 1)$. From Koopmans (1972) and A1, $b(c)$ is constant, that is, $b(c) = \beta \in (0, 1)$.

Let $\mathcal{F}$ denote, as before, the set of functions $f : S \to C$. Fix some $c^\infty = (c_0, c_1, c_2, ...) \in C^\infty$, define $\succcurlyeq^*$ on $\mathcal{F}$ by:

$$f \succcurlyeq^* g \iff (c_0, f(\cdot), c_2, ...) \succcurlyeq (c_0, g(\cdot), c_2, ...). \tag{44}$$

This preference is well defined and does not depend on $c^\infty \in C^\infty$. It is clear that $\succcurlyeq^*$ satisfies Q1-Q5. Therefore, by Proposition 2.1, there exists $p : \Sigma \to [0, 1]$ and $\tau \in (0, 1)$ such that

$$f \succcurlyeq^* g \iff Q_\tau^p[u(f)] \geqslant Q_\tau^p[u(g)] \tag{45}$$

where

$$Q_\tau^p[u(f)] \equiv \inf\{\alpha : p(\{s \in S : u(f(s)) \geqslant \alpha\}) \leqslant \tau\}.$$

Note that in (45), we can use the same $u : X \to \mathbb{R}$ provided by Proposition 2.2. By the definition of $\succcurlyeq^*$ and the recursive representation $(V, W, I)$, (45) is equivalent to

$$f \succcurlyeq^* g \iff I[u(f)] \geqslant I[u(g)].$$

Since $I(x) = x = Q_\tau^p[x]$ for any $x \in \mathbb{R}$, we can take $I[\cdot] = Q_\tau^p[\cdot]$ to represent the same preference. □

**Proof of Lemma 2.6:** Assume that there exists $f : S \to C$ such that $I^1[u(f)] > I^2[u(f)]$. Define

$h = (h_0, f, f, \ldots)$. Pick $c \in C$ such that $I^1[u(f)] > u(c) > I^2[u(f)]$. Let $c^\infty = (c, c, \ldots)$. Then,

$$V^1(h) = u(h_0) + \beta I^1[V^1(h^1)] > V^1(c^\infty) = \sum_{t=0}^{\infty} \beta^t u(c_t) = V^2(c^\infty) \geqslant V^2(h) = u(h_0) + \beta I^1[V^2(h^1)],$$

that is, $V^2(c^\infty) \geqslant V^2(h)$ but $V^1(h) > V^1(c^\infty)$, thus contradicting (9). Conversely, $I^1[\cdot] \leqslant I^2[\cdot]$ implies $V^1(h) \leqslant V^2(h)$ for any $h \in \mathcal{H}$. Thus, it cannot happen the negation of (9), that is, $V^1(h) > V^1(c^\infty) = V^2(c^\infty) \geqslant V^2(h)$. □

## A.3 Proofs of Section 3

**Proof of Theorem 3.4:** This is essentially the same proof of Theorem 3.9, presented in detail below. Thus, we omit it. □

**Proof of Proposition 3.5:** Let $L$ be a bound for $V^h$. Using repeated times the recursive property (13), we can write

$$\begin{aligned}
V^h(x,z) &= u(x_1^h, x_2^h, z_1) + Q_\tau\Bigg[\beta u(x_2^h, x_3^h, z_2) + Q_\tau\Big[\beta^2 u(x_3^h, x_4^h, z_3) + \ldots \\
&\qquad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^h, x_{n+2}^h, z_n) + \beta^{n+1} V^h(x_n^h, Z_n)\Big] \Big| Z_n = z_n\Big] \ldots \Big| Z_1 = z\Bigg] \\
&\leqslant u(x_1^h, x_2^h, z_1) + Q_\tau\Bigg[\beta u(x_2^h, x_3^h, z_2) + Q_\tau\Big[\beta^2 u(x_3^h, x_4^h, z_3) + \ldots \\
&\qquad \ldots + Q_\tau\Big[\beta^n u(x_{n+1}^h, x_{n+2}^h, z_n) + \beta^{n+1} L\Big] \Big| Z_n = z_n\Big] \ldots \Big| Z_1 = z\Bigg] \\
&= V^{h,n}(x,z) + \beta^{n+1} L,
\end{aligned}$$

where in the last line we have used the property of quantiles that $Q_\tau[X + \alpha] = \alpha + Q_\tau[X]$ for $\alpha \in \mathbb{R}$; see Lemma A.2. Repeating the same argument with the lower bound $-L$, we can write:

$$V^n(x,z) - \beta^{n+1} L \leqslant V^h(x,z) \leqslant V^{h,n}(x,z) + \beta^{n+1} L.$$

This concludes the proof. □

**Proof of Proposition 3.7:** Let $\Omega = \{1,2,3,4\}$ and $P(\{\omega\}) = 1/4$ for all $\omega \in \Omega$. Define $\Sigma_0 = \{\varnothing, \Omega\}$ and $\Sigma_1 = \{\varnothing, E_1, E_2, \Omega\}$, where $E_1 = \{1,2\}$ and $E_2 = \{3,4\}$. Let $X(\omega) = \omega$. Then for $\tau \in (0.5, 0.75)$,

$$Q_\tau[X|\Sigma_1]_\omega = \begin{cases} 2, & \text{if } \omega \in E_1 \\ 4, & \text{if } \omega \in E_2 \end{cases}$$

Therefore, $Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] = 4$ but $Q_\tau[X|\Sigma_0] = Q_\tau[X] = 3$, which establishes (20).

To see (21), consider $\Omega = [0,4]$, $\Sigma_0 = \{\varnothing, \Omega\}$ and let $\Sigma_1$ be generated by the partition $\{E_1, E_2\}$, where $E_1 = [1,2)$ and $E_2 = [2,4]$. Consider $P$ as the uniform distribution on $\Omega$. Let $X$ and $Y$ be two random variables with c.d.f. given respectively by $F_X(x) = \frac{1}{4}\left[x - \frac{1}{4}\sin(hx)\right]$ and $F_Y(x) = \frac{1}{4}\left[x + \frac{1}{4}\sin(hx)\right]$. The graphs of these two c.d.f.s are shown in Figure 5 below. Let $\tau \in (0.5, 0.75)$.

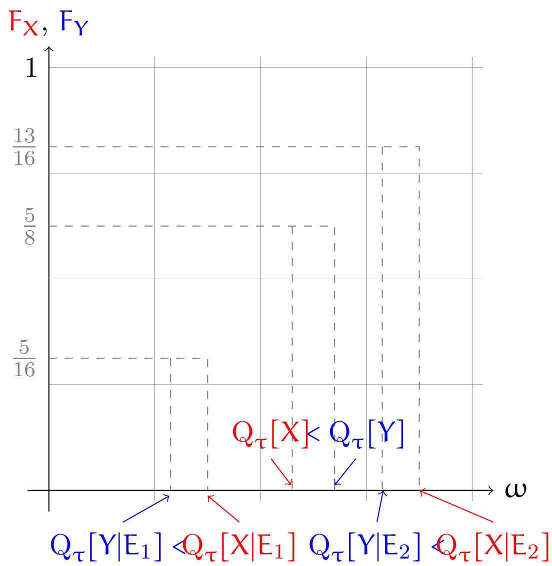



Figure 5: Graph of X and Y, with respective quantiles.

In the graph above, we plot the quantiles for $\tau = \frac{5}{8} \in (0.5, 0.75)$. We can easily see that $Q_\tau[X|\Sigma_1]_{(\omega)} \geqslant Q_\tau[Y|\Sigma_1]_{(\omega)}, \forall \omega \in \Omega$, but $Q_\tau[X] = Q_\tau[X|\Sigma_0] < Q_\tau[Y|\Sigma_0] = Q_\tau[Y]$, that is, (21) holds. □

**Proof of Theorem 3.8:** Assume that plans $h$ and $h'$ are such that $h_{t'}(\cdot) = h'_{t'}(\cdot)$ for all $t' \leqslant t$ and $h' \succcurlyeq_{t+1,\Omega'_{t+1},x} h$ for all $\Omega'_{t+1}, x$. From (10), this means that

$$V_{t+1}(h', x, z^{t+1}) \geqslant V_{t+1}(h, x, z^{t+1}), \forall (x, z^t) \in \mathcal{X} \times \mathcal{Z}^{t+1}. \tag{46}$$

Therefore,

$$\begin{aligned} V_t(h', x, z^t) &= u(x_t^{h'}, x_{t+1}^{h'}, z_t) + \beta Q_\tau \left[ V_{t+1}(h', x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right] \\ &\geqslant u(x_t^{h'}, x_{t+1}^{h'}, z_t) + \beta Q_\tau \left[ V_{t+1}(h, x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right] \\ &= u(x_t^{h}, x_{t+1}^{h}, z_t) + \beta Q_\tau \left[ V_{t+1}(h, x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right] \\ &= V_t(h, x, z^t), \end{aligned}$$

where the first and last equalities come from the recursive equation (13), the first inequality comes from (46) and Lemma A.1(vi), while the equality in the third line comes from the fact that the plans aggree on all times up to $t$, that is, $x_t^{h'} = x_t^h$ and $x_{t+1}^{h'} = h'_t(x_t^h, z^t) = h_t(x_t^h, z^t) = x_{t+1}^h$. This establishes the claim. □

**Proof of Theorem 3.9:** We organize the proof in a series of Lemmas.

**Lemma A.7.** *If $v \in \mathcal{C}$, the map $(y, z) \mapsto Q_\tau[v(y, w)|z]$ is continuous.*

*Proof.* Consider a sequence $(y^n, z^n) \to (y^*, z^*)$. Since $v$ and $f$ are continuous, $v(y^n, w) \to v(y^*, w)$ and

$$m^n(\alpha) \equiv \Pr\left(\{w : v(y^n, w) \leqslant \alpha\} | z^n\right) \to \Pr\left(\{w : v(y^*, w) \leqslant \alpha\} | z^*\right) \equiv m^*(\alpha). \tag{47}$$

Let $\alpha^n \equiv \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geqslant \tau\} = Q_\tau[\nu(y^n,\cdot)|z^n]$ and $\alpha^* \equiv \inf\{\alpha \in \mathbb{R} : m^*(\alpha) \geqslant \tau\} = Q_\tau[\nu(y^*,\cdot)|z^*]$. We want to show that $\alpha^n \to \alpha^*$.

In general, $m^n(\cdot)$ and $m^*(\cdot)$ may fail to be continuous, but they are right-continuous and (weakly) increasing by Lemma A.1. Moreover, $m^*$ and $m^n$ are strictly increasing in the range of $\nu$. More precisely, for each $y$, define $R(y) \equiv \{\alpha \in \mathbb{R} : \exists w \text{ such that } \nu(y,w) = \alpha\}$. We claim that if $\alpha < \alpha', \alpha, \alpha' \in R(y)$, then $m^*(\alpha') > m^*(\alpha)$, and similarly for $m^n$.[27]

Indeed, assume that $\exists w, w'$ such that $\nu(y,w) = \alpha$ and $\nu(y,w') = \alpha'$. The set $P = \{\alpha w + (1-\alpha)w' : \alpha \in [0,1]\}$ is contained in $\mathcal{Z}$ because this is convex. Thus, $\{\nu(y,p) : p \in P\}$ is connected, that is, a nonempty interval. We conclude that, since $\nu$ is continuous, the set $\{w \in \mathcal{Z} : \alpha < \nu(y,w) < \alpha'\}$ is a nonempty and open interval. (This implies, in particular, that $R(y)$ is an interval.) Since $f(\cdot|z)$ is strictly positive in $\mathcal{Z}$, we conclude that

$$m^*(\alpha') - m^*(\alpha) \geqslant \Pr\left(\{w \in \mathcal{Z} : \alpha < \nu(y,w) < \alpha'\}|z\right) > 0,$$

which establishes the claim. By Lemma A.1$(iv)$, we have

$$m^n(\alpha^n) \geqslant \tau \text{ and } m^*(\alpha^*) \geqslant \tau. \tag{48}$$

We will show that $\alpha^n \to \alpha^*$ by first establishing $\liminf_n \alpha^n \geqslant \alpha^*$ and then $\alpha^* \geqslant \limsup_n \alpha^n$.

Suppose that $\underline{\alpha} \equiv \liminf_n \alpha^n < \alpha^*$. This means that there exists $\epsilon > 0$ and for each $j$, $n_j > j$ such that $\alpha^{n_j} < \underline{\alpha} + \epsilon < \alpha^*$. By the definition of $\alpha^*$, $\underline{\alpha} < \alpha^*$ implies $m^*(\underline{\alpha}) < \tau$. However, by (48), $m^{n_j}(\alpha^{n_j}) \geqslant \tau$, which implies $m^{n_j}(\underline{\alpha}) \geqslant \tau$ and $m^*(\underline{\alpha}) \geqslant \tau$, by (47). This contradiction establishes that $\liminf_n \alpha^n \geqslant \alpha^*$.

If $\bar{\alpha} \equiv \limsup_n \alpha^n > \alpha^*$, there exists $\epsilon > 0$ and for each $j$, $n_j > j$ such that

$$\bar{\alpha} + \epsilon > \alpha^{n_j} > \bar{\alpha} - \epsilon > \bar{\alpha} - 2\epsilon > \alpha^* + \epsilon. \tag{49}$$

Recall that $\alpha^n = \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geqslant \tau\}$. Therefore, $\alpha^{n_j} > \bar{\alpha} - \epsilon$ implies $m^{n_j}(\bar{\alpha} - \epsilon) < \tau$. Thus, $m^{n_j}(\alpha^* + \epsilon) < m^{n_j}(\bar{\alpha} - \epsilon) < \tau$. This implies that

$$m^*(\alpha^*) \leqslant m^*(\bar{\alpha} - 2\epsilon) \leqslant m^*(\bar{\alpha} - \epsilon) = \lim_n m^{n_j}(\bar{\alpha} - \epsilon) \leqslant \tau \leqslant m^*(\alpha^*).$$

Therefore, $m^*$ is constant between $\alpha^*$ and $\bar{\alpha} - 2\epsilon$. This will be a contradiction if we show that $\alpha^*, \bar{\alpha} - 2\epsilon \in R(y^*)$.

Since $m^*(\alpha^*) = \Pr\left(\{w : \nu(y^*,w) \leqslant \alpha^*\}|z^*\right) \geqslant \tau > 0$, $\{w : \nu(y^*,w) \leqslant \alpha^*\} \neq \varnothing$ and there exists some $\alpha \in R(y^*) \cap (-\infty, \alpha^*]$. On the other hand, if $\{w : \bar{\alpha} - 2\epsilon \leqslant \nu(y^*,w) \leqslant \bar{\alpha} + 2\epsilon\} = \varnothing$, then for sufficiently high $j$, $\{w : \bar{\alpha} - \epsilon \leqslant \nu(y^{n_j},w) \leqslant \bar{\alpha} + \epsilon\} = \varnothing$. In this case, $m^{n_j}(\bar{\alpha} - \epsilon) = m^{n_j}(\bar{\alpha} + \epsilon) \equiv \tau^{n_j}$. But this would imply either $\alpha^{n_j} \leqslant \bar{\alpha} - \epsilon$, if $\tau^{n_j} \geqslant \tau$ or $\alpha^{n_j} \geqslant \bar{\alpha} + \epsilon$, if $\tau^{n_j} < \tau$. In either case, we have a contradiction with $\alpha^{n_j} \in (\bar{\alpha} - \epsilon, \bar{\alpha} + \epsilon)$ as required in (49). This contradiction shows that there exists $\alpha' \in R(y^*) \cap [\bar{\alpha} - 2\epsilon, \bar{\alpha} + 2\epsilon]$. Since $\alpha, \alpha' \in R(y^*)$, we have $[\alpha^*, \bar{\alpha} - 2\epsilon] \subset [\alpha, \alpha'] \subset R(y^*)$. This concludes the proof. $\square$

[27] Note that $m^n$ and $m^*$ are the corresponding c.d.f. functions for $\nu$. Thus, proving that those functions are strictly increasing in the range of $\nu$ leads to continuity of the quantile with respect to $\tau$, by (an adaptation of) Lemma A.1(viii). But this is not what we need: we want continuity in $(y,z)$. We prefer to offer here a direct and detailed argument, although long.

**Lemma A.8.** *For each $v \in \mathcal{C}$ the supremum in (22) is attained and $\mathbb{M}^{\tau}(v) \in \mathcal{C}$. Moreover, the optimal correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by*

$$\Upsilon(x, z) \equiv \arg \max_{y \in \Gamma(x,z)} Q_{\tau}[u(x, y, z) + \beta v^{\tau}(y, w)|z] \tag{50}$$

*is nonempty and upper semi-continuous.*

*Proof.* Let

$$g(x, y, z, w) = u(x, y, z) + \beta v(y, w). \tag{51}$$

By Lemma A.2, $Q_{\tau}[g(x, y, z, \cdot)|z] = u(x, y, z) + \beta Q_{\tau}[v(y, \cdot)|z]$. By Lemma A.7, $Q_{\tau}[g(x, y, z, \cdot)|z]$ is continuous in $(x, y, z)$. From Berge's Maximum Theorem, the maximum is attained, the value function $\mathbb{M}^{\tau}(v)$ is continuous and $\Upsilon$ is nonempty and upper semi-continuous. $\mathbb{M}^{\tau}(v)$ is bounded because $u$ and $v$, hence $g$, are bounded. Therefore, $\mathbb{M}^{\tau}(v) \in \mathcal{C}$. □

We conclude the proof of Theorem 3.9 by showing that $\mathbb{M}^{\tau}$ satisfies Blackwell's sufficient conditions for a contraction.

**Lemma A.9.** $\mathbb{M}^{\tau}$ *satisfies the following conditions:*

*(a) For any $v, v' \in \mathcal{C}$, $v \leqslant v'$ implies $\mathbb{M}^{\tau}(v) \leqslant \mathbb{M}^{\tau}(v')$.*

*(b) For any $a \geqslant 0$ and $x \in X$, $\mathbb{M}(v + a)(x) \leqslant \mathbb{M}(v)(x) + \beta a$, with $\beta \in (0, 1)$.*

*Then, $\|\mathbb{M}(v) - \mathbb{M}(v')\| \leqslant \beta \|v - v'\|$, that is, $\mathbb{M}$ is a contraction with modulus $\beta$. Therefore, $\mathbb{M}^{\tau}$ has a unique fixed-point $v^{\tau} \in \mathcal{C}$.*

*Proof.* To see (a), let $v, v' \in \mathcal{C}$, $v \leqslant v'$ and define $g$ as in (51) and analogously for $g'$, that is, $g'(x, y, z, w) = u(x, y, z) + \beta v'(y, w)$. It is clear that $g \leqslant g'$. Then, by Lemma A.1(vi), $Q_{\tau}[g(\cdot)|z] \leqslant Q_{\tau}[g'(\cdot)|z]$, which implies (a).

To verify (b), we use the monotonicity property (Lemma A.2):

$$Q_{\tau}[u(x, y, z) + \beta(v(x, z) + a)|z] \quad = \quad Q_{\tau}[u(x, y, z) + \beta v(x, z)|z] + \beta a.$$

Thus, $\mathbb{M}^{\tau}(v + a) = \mathbb{M}^{\tau}(v) + \beta a$, that is, (b) is satisfied with equality. □

**Proof of Theorem 3.10:** Let Assumption 2 hold. It is convenient to introduce the following notation. Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ which are concave in its first argument. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly concave functions. If we show that $\mathbb{M}^{\tau}(\mathcal{C}') \subset \mathcal{C}''$, then the fixed-point of $\mathbb{M}^{\tau}$ will be strictly concave in $x$. (See, for instance, Stokey, Lucas, and Prescott (1989, Corollary 1, p. 52).)

**Lemma A.10.** *Let Assumption 2 hold. $\mathbb{M}^{\tau}(\mathcal{C}') \subseteq \mathcal{C}''$. Therefore, $v^{\tau} \in \mathcal{C}''$. Moreover, the optimal correspondence $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by (50) is single-valued. Therefore, we can denote it by a function $y^*(x, z)$.*

*Proof.* Let $\alpha \in (0,1)$, and consider $x_0, x_1 \in \mathcal{X}$, $x_0 \neq x_1$. For $i = 0, 1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, that is,

$$\mathbb{M}^\tau(v)(x_i, z) = u(x_i, y_i, z) + \beta Q_\tau[v(y_i, w)|z] = Q_\tau[g(x_i, y_i, z, w)|z].$$

Let $x_\alpha \equiv \alpha x_0 + (1-\alpha)x_1$ and $y_\alpha \equiv \alpha y_0 + (1-\alpha)y_1$. First, let us observe that

$$\begin{aligned}
g(x_\alpha, y_\alpha, z, w) &= u(x_\alpha, y_\alpha, z) + \beta v(y_\alpha, w) \\
&> \alpha u(x_0, y_0, z) + (1-\alpha)u(x_1, y_1, z) \\
&\quad + \beta v(y_\alpha, w) \\
&\geqslant \alpha u(x_0, y_0, z) + (1-\alpha)u(x_1, y_1, z) \\
&\quad + \beta\left[\alpha v(y_0, w) + (1-\alpha)v(y_1, w)\right] \\
&= \alpha g(x_0, y_0, z, w) + (1-\alpha)g(x_1, y_1, z, w),
\end{aligned}$$

where the first inequality comes from the strict concavity of $u$ and the second, from the concavity of $v$. That is, $g$ is strictly quasiconcave, which establishes that $\Upsilon(x, z)$ is single-valued. Therefore,

$$Q_\tau[g(x_\alpha, y_\alpha, z, w)|z] > Q_\tau\left[\alpha g(x_0, y_0, z, w) + (1-\alpha)g(x_1, y_1, z, w)|z\right].$$

Note that the variables $X = g(x_0, y_0, z, w)$ and $Y = g(x_1, y_1, z, w)$ satisfy the assumption of Proposition A.4 since $v$ is nondecreasing in $w$ (holding $z$ fixed). Therefore,

$$\begin{aligned}
Q_\tau[g(x_\alpha, y_\alpha, z, w)|z] &> \alpha Q_\tau\left[g(x_0, y_0, z, w)|z\right] + (1-\alpha)Q_\tau\left[g(x_1, y_1, z, w)|z\right] \\
&= \alpha \mathbb{M}^\tau(v)(x_0, z) + (1-\alpha)\mathbb{M}^\tau(v)(x_1, z). \qquad (52)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{M}^\tau(v)(x_\alpha, z) &\geqslant Q_\tau[g(x_\alpha, y_\alpha, z, w)|z] \\
&> \alpha \mathbb{M}^\tau(v)(x_0, z) + (1-\alpha)\mathbb{M}^\tau(v)(x_1, z),
\end{aligned}$$

This establishes strict concavity, concluding the proof. ☐

**Lemma A.11.** *Let Assumption 2 hold. If* $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leqslant z'$*, then* $Q_\tau[h(w)|z] \leqslant Q_\tau[h(w)|z']$.

*Proof.* From Assumption 2(ii), if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leqslant z'$:

$$\int_{\mathcal{Z}} h(\alpha)\left[-1_{\{\alpha \in \mathcal{Z}: \alpha \leqslant w\}}\right] f(\alpha|z)d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha)\left[-1_{\{\alpha \in \mathcal{Z}: \alpha \leqslant w\}}\right] f(\alpha|z')d\alpha.$$

Thus,

$$\int_{\{\alpha \in \mathcal{Z}: \alpha \leqslant w\}} h(\alpha) f(\alpha|z)d\alpha \geqslant \int_{\{\alpha \in \mathcal{Z}: \alpha \leqslant w\}} h(\alpha) f(\alpha|z')d\alpha. \qquad (53)$$

If we define $H(w|z) = \Pr([h(W) \leqslant h(w)]|Z = z)$, then (53) can be written as:

$$H(w|z) \geqslant H(w|z').$$

Observe that $Q_\tau[h(w)|z] = \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\}$ and, whenever $z \leqslant z'$, $H(w|z') \leqslant H(w|z)$, for all

$w$. Therefore, if $z \leqslant z'$, then

$$\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\} \supset \{\alpha \in \mathbb{R} : H(\alpha|z') \geqslant \tau\},$$

which implies that

$$Q_\tau[h(w)|z] \quad = \quad \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\} \leqslant \inf\{\alpha \in \mathbb{R} : H(\alpha|z') \geqslant \tau\} = Q_\tau[h(w)|z'],$$

as we wanted to show. $\square$

**Lemma A.12.** *Let Assumption 2 hold. If $v \in \mathcal{C}$ is increasing in $z$ then $\mathbb{M}^\tau(v)$ is strictly increasing in $z$.*

*Proof.* Let $z_1, z_2 \in \mathcal{Z}$, with $z_1 < z_2$. For $i = 1, 2$, let $y_i \in \Gamma(x, z_i)$ realize the maximum, that is,

$$\mathbb{M}^\tau(v)(x_i, z) = u(x, y_i, z_i) + \beta Q_\tau[v(y_i, w)|z_i].$$

Since $u$ is strictly increasing in $z$, we have:

$$\mathbb{M}^\tau(v)(x, z_1) = u(x, y_1, z_1) + \beta Q_\tau[v(y_1, w)|z_1] < u(x, y_1, z_2) + \beta Q_\tau[v(y_1, w)|z_1].$$

From Lemma A.11, we have $Q_\tau[v(y_1, w)|z_1] \leqslant Q_\tau[v(y_1, w)|z_2]$, which gives:

$$\mathbb{M}^\tau(v)(x, z_1) < u(x, y_1, z_2) + \beta Q_\tau[v(y_1, w)|z_2].$$

From Assumption 2, $\Gamma(x, z) \subseteq \Gamma(x, z')$, that is, $y_1 \in \Gamma(x, z_2)$. Optimality thus implies that:

$$u(x, y_1, z_2) + \beta Q_\tau[v(y_1, w)|z_2] \leqslant u(x, y_2, z_2) + \beta Q_\tau[v(y_2, w)|z_2] = \mathbb{M}^\tau(v)(x, z_2).$$

Therefore, $\mathbb{M}^\tau(v)(x, z_1) < \mathbb{M}^\tau(v)(x, z_2)$, which shows strict increasingness in $z$. $\square$

We conclude the proof of Theorem 3.10 by showing differentiability of $v$, which follows from an easy adaptation of Benveniste and Scheinkman (1979)'s argument. For completeness and reader's convenience, we reproduce it here. Given $(x, z)$, let $y^*(x, z) \in \Gamma(x, z)$ be unique maximum as established in Lemma A.10. Thus, for all $(x, z)$, we have:

$$v(x, z) = u(x, y^*(x, z), z) + \beta Q_\tau[v(y^*(x, z), w)|z].$$

Fix $x_0$ in the interior of $X$ and define:

$$\bar{w}(x, z) = u(x, y^*(x_0, z), z) + \beta Q_\tau[v(y^*(x_0, z), w)|z].$$

From the optimality, for a neighborhood of $x_0$, we have $\bar{w}(x, z) \leqslant v(x, z)$, with equality at $x = x_0$, which implies $\bar{w}(x, z) - \bar{w}(x_0, z) \leqslant v(x, z) - v(x_0, z)$. Note that $\bar{w}$ is concave and differentiable in $x$ because $u$ is. Thus, any subgradient $p$ of $v$ at $x_0$ must satisfy

$$p \cdot (x - x_0) \geqslant v(x, z) - v(x_0, z) \geqslant \bar{w}(x, z) - \bar{w}(x_0, z).$$

Thus, $p$ is also a subgradient of $\bar{w}$. But since $\bar{w}$ is differentiable, $p$ is unique. Therefore, $\nu$ is a concave function with a unique subgradient. Therefore, it is differentiable in $x$ (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to $x$ is the same as that of $\bar{w}$, that is,

$$\frac{\partial \nu^\tau}{\partial x_i}(x,z) = \frac{\partial \bar{w}}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x, y^*(x,z), z),$$

as we wanted to show. $\square$

**Proof of Lemma 3.12:** By Stokey, Lucas, and Prescott (1989, Theorem 7.6), $\Gamma$ has a measurable selection. Therefore, the argument in Stokey, Lucas, and Prescott (1989, Lemma 9.1) establishes the result. $\square$

We need the following notation in the next proof. Let $T \in \mathbb{N} \cup \{\infty\}$ and $S : \mathcal{Z}^T \to \mathcal{Z}^{T-1}$ be the shift operator, that is, given $z = (z_1, z_2, ..., z_T) \in \mathcal{Z}^T$, $S(z) = (z_2, ..., z_T) \in \mathcal{Z}^{T-1}$. Abusing notation, let $S : H \to H$ also denote the shift operator for plans, that is, given $h \in H$, $h^s = S(h) \in H$ is defined as follows: for each given $z^\infty \in \mathcal{Z}^\infty$, $h^s_t(x, S(z^{t+1})) = h_{t+1}(x, z^{t+1})$. Let $S_t : H \to H$ be the composition of $S$ with itself $t$ times.

**Proof of Lemma 3.13:** Let $t \geqslant 2$ (otherwise there is nothing to prove). Since $H_t(x,z) \subset H_1(x,z) = H(x,z)$ by definition, we have $\nu^*_t(x,z) \leqslant \nu_1(x,z)$. Suppose, for an absurd, that there exists $h \in H(x,z)$ such that

$$V_1(h,x,z) > \nu^*_t(x,z). \tag{54}$$

Let $\tilde{h}$ and $(\tilde{x}, \tilde{z}^t)$ be such that $S_{t-1}(\tilde{h}) = h$, $x^{\tilde{h}}_t(\tilde{x}, \tilde{z}^t) = x$ and $\tilde{z}_t = z$. Then, $V_t(\tilde{z}, \tilde{x}, \tilde{z}^t) = V_1(h,x,z)$. Since $\nu^*_t(x,z) \geqslant V_t(\tilde{z}, \tilde{x}, \tilde{z}^t)$, this establishes a contradiction with (54). $\square$

**Proof of Lemma 3.14:** If $\nu$ is bounded and satisfies (24), then it is the unique fixed-point of the contraction $\mathbb{M}^\tau$. Thus, the proof of Theorem 3.9 establishes, via the Maximum Theorem, the claims. $\square$

**Proof of Theorem 3.15:** Assume that $\nu$ satisfies (24). It is sufficient to show that $(i)$ $\nu(x,z) \geqslant V_1(h,x,z)$ for any $h \in H(x,z)$ and $(x,z) \in \mathcal{X} \times \mathcal{Z}$; and $(ii)$ $\nu(x,z) = V_1(h^\psi, x, z)$. Let $h \in H(x,z)$. We have:

$$\begin{aligned}
\nu(x,z) &= \sup_{y \in \Gamma(x^h_1, z_1)} u\left(x^h_1, y, z_1\right) + \beta Q_\tau[\nu(y, z_2)|z_1] \\
&\geqslant u\left(x^h_1, x^h_2, z_1\right) + \beta Q_\tau\left[\nu(x^h_2, z_2)|z_1\right] \\
&= u\left(x^h_1, x^h_2, z_1\right) + \beta Q_\tau\left[\sup_{y \in \Gamma(x^h_2, z_2)} \left\{u\left(x^h_2, y, z_2\right) + \beta Q_\tau\left[\nu(y, z_3)\middle|z_2\right]\right\}\middle|z_1\right] \\
&\geqslant u\left(x^h_1, x^h_2, z_1\right) + Q_\tau\left[\beta u\left(x^h_2, x^h_3, z_2\right) + Q_\tau\left[\beta^2 \nu(x^h_3, z_3)\middle|z_2\right]\middle|z_1\right],
\end{aligned}$$

where the two inequalities come from the definition of sup, and the equalities from (24) and Corollary

A.3. Repeating the same arguments, we obtain:

$$\begin{aligned} v(x,z) \quad &\geqslant \quad u(x_1^h, x_2^h, z_1) + Q_\tau\Big[\beta u(x_2^h, x_3^h, z_2) + Q_\tau\Big[\beta^2 u(x_3^h, x_4^h, z_3) + \ldots \\ &\ldots + Q_\tau\Big[\beta^n u(x_{n+1}^h, x_{n+2}^h, z_n) + \beta^{n+1} v(x_n^h, Z_n)\Big] \Big| Z_n = z_n\Big] \ldots \Big| Z_1 = z\Big]. \end{aligned}$$

Repeating the arguments in the proof of Proposition 3.5, we can conclude that the limit of the right hand size when $n \to \infty$ is $V^h(x,z) = V_1(h,x,z)$. Thus, we have established that $v(x,z) \geqslant V_1(h,x,z)$. Since $h$ was arbitrary, then $v(x,z) \geqslant v^*(x,z)$. On the other hand, for $h^\psi$ the inequalities above hold with equality and we obtain $v(x,z) = v^*(x,z)$. □

**Proof of Theorem 3.16:** Let $g(x,y,z,w) \equiv u(x,y,z) + \beta Q_\tau[v^\tau(y,w)|z]$ and $y^*(x,z)$ be an interior solution of the problem (24). Observe that $v^\tau$ is increasing in $w$, differentiable in its first variable and for $0 < x_i' - x_i < \epsilon$, for some small $\epsilon > 0$,

$$v^\tau(x_i', x_{-i}, z) - v^\tau(x_i, x_{-i}, z) = \int_x^{x'} \frac{\partial v^\tau}{\partial x_i}(\alpha, x_{-i}, z) d\alpha = \int_x^{x'} \frac{\partial u}{\partial x_i}(\alpha, x_{-i}, z) d\alpha$$

is increasing in $z$ because $\frac{\partial u}{\partial x_i}$ is. Therefore, the assumptions of Proposition 3.17 are satisfied and we conclude that $\frac{\partial Q_\tau}{\partial x_i}[v^\tau(x,z)] = Q_\tau\left[\frac{\partial v^\tau}{\partial x_i}(x,z)\right]$. Since $u$ is differentiable in $y$, so is $g$. Since $y^*(x,z)$ is interior, the following first order condition holds:

$$\frac{\partial g}{\partial y_i}(x, y^*(x,z), z, Q_\tau[w|z]) = \frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta Q_\tau\left[\frac{\partial v^\tau}{\partial x_i}(y^*(x,z), w)|z\right] = 0.$$

Now we apply Theorem 3.10 and its expression: $\frac{\partial v^\tau}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x, y^*(x,z), z)$, to conclude that

$$\frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta Q_\tau\left[\frac{\partial u}{\partial x_i}(y^*(x,z), y^*(y^*(x,z), w), w)\Big| z\right] = 0. \tag{55}$$

Now, we have just to put the notation of a sequence. For this, let $h = (x_t)$ denote an optimal path beginning at $(x_0, z_0)$, (55) can be rewritten, substituting $x$ for $x_t^h$, $y^*(x,z)$ for $x_{t+1}^h$, $y^*(y^*(x,z),w)$ for $x_{t+2}^h$, $z$ for $z_t$ and $w$ for $z_{t+1}$, as:

$$\frac{\partial u}{\partial y_i}\left(x_t^h, x_{t+1}^h, z_t\right) + \beta Q_\tau\left[\frac{\partial u}{\partial x_i}\left(x_{t+1}^h, x_{t+2}^h, z_{t+1}\right)\Big| z_t\right] = 0. \tag{56}$$

which we wanted to establish. □

**Proof of Proposition 3.17:** Fix a sufficiently small $\delta > 0$ and $x = (x_i, x_{-i})$, with the usual meaning.[28] Define $X = d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ and $Y = \tilde{g}(z) = h(x_i, x_{-i}, z)$. Since $h$ and $d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ are increasing in $z$ by assumption, the random variables $X$ and $Y$ satisfy

[28] "Sufficiently small" here means that $\delta > 0$ is taken so that $d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ is increasing, as required by the assumption of the proposition. This "smallness" condition will be left implicit below.

the assumptions of Proposition A.4, which allows us to conclude that

$$\begin{aligned}
Q_\tau[h(x_i+\delta, x_{-i}, z)] &= Q_\tau[X+Y] = Q_\tau[X] + Q_\tau[Y] \\
&= Q_\tau[h(x_i+\delta, x_{-i}, z) - h(x_i, x_{-i}, z)] + Q_\tau[h(x_i, x_{-i}, z)].
\end{aligned}$$

Therefore, for all sufficiently small $\delta > 0$,

$$\frac{Q_\tau[h(x_i+\delta, x_{-i}, z)] - Q_\tau[h(x_i, x_{-i}, z)]}{\delta} = Q_\tau\left[\frac{h(x_i+\delta, x_{-i}, z) - h(x_i, x_{-i}, z)]}{\delta}\right].$$

Since $\delta \mapsto \frac{h(x_i+\delta, x_{-i}, z) - h(x_i, x_{-i}, z)]}{\delta}$ is continuous, Lemma A.7 implies that

$$\begin{aligned}
\lim_{\delta \downarrow 0} \frac{Q_\tau[h(x_i+\delta, x_{-i}, z)] - Q_\tau[h(x_i, x_{-i}, z)]}{\delta} &= \lim_{\delta \downarrow 0} Q_\tau\left[\frac{h(x_i+\delta, x_{-i}, z) - h(x_i, x_{-i}, z)]}{\delta}\right] \\
&= Q_\tau\left[\lim_{\delta \downarrow 0} \frac{h(x_i+\delta, x_{-i}, z) - h(x_i, x_{-i}, z)]}{\delta}\right] \\
&= Q_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right].
\end{aligned}$$

We can adapt the above arguments for $\delta > 0$ and $X = d(z) = h(x_i, x_{-i}, z) - h(x_i - \delta, x_{-i}, z)$ and $Y = \tilde{g}(z) = h(x_i, x_{-i}, z)$ to conclude that:

$$\begin{aligned}
\lim_{\delta \downarrow 0} \frac{Q_\tau[h(x_i, x_{-i}, z)] - Q_\tau[h(x_i-\delta, x_{-i}, z)]}{\delta} &= \lim_{\delta \downarrow 0} Q_\tau\left[\frac{h(x_i, x_{-i}, z) - h(x_i-\delta, x_{-i}, z)]}{\delta}\right] \\
&= Q_\tau\left[\lim_{\delta \downarrow 0} \frac{h(x_i, x_{-i}, z) - h(x_i-\delta, x_{-i}, z)]}{\delta}\right] \\
&= Q_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right].
\end{aligned}$$

By changing $\delta > 0$ above by $\tilde{\delta} = -\delta < 0$, we obtain

$$\lim_{\tilde{\delta} \uparrow 0} \frac{Q_\tau[h(x_i+\tilde{\delta}, x_{-i}, z)] - Q_\tau[h(x_i, x_{-i}, z)]}{\tilde{\delta}} = Q_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right].$$

This shows that the right and left derivative of $x_i \mapsto Q_\tau[h(x_i, x_{-i}, z)]$ exist and are equal. Therefore, $x_i \mapsto Q_\tau[h(x_i, x_{-i}, z)]$ is differentiable and its derivative is $Q_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right]$, as we wanted to show. $\square$

## A.4 Proofs of Section 4

**Proof of Lemma 4.1:** Assumption 1 (i)–(iii) and (v) are immediate. Since $\mathcal{Z}$ and $\mathcal{X}$ are bounded, and $U$ and $z \mapsto z + p(z)$ are $C^1$, $u$ is $C^1$ and bounded. Thus, Assumption 1 is satisfied. Similarly, Assumptions 2 are easily seen to be satisfied. It remains to verify the assumption of Theorem 3.16, namely that $\frac{\partial u}{\partial x_i}\left(x_t^h, x_{t+1}^h, z_t\right)$ is strictly increasing in $z_t$, which happens if and only if $\log \frac{\partial u}{\partial x_i}\left(x_t^h, x_{t+1}^h, z_t\right)$ is strictly increasing in $z_t$. Since

$$\log \frac{\partial u}{\partial x}(x, y, z) = -\gamma \log\left[z \cdot x + p(z) \cdot (x - y)\right] + \log\left(z + p(z)\right),$$

and $x_t^h = x_{t+1}^h = 1$, we need to verify only that $-\gamma [\log(z)]' + [\log(z + p(z))]' > 0$. This is equivalent to $\gamma < z [\log(z + p(z))]'$, which is contained in Assumption 3(iv). $\square$

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