Appendices to "Innovation-Led Transitions in Energy Supply"

Appendix A details the calibration and solution method. Appendix B reports further robustness checks. Appendix C contains a numerical example of the main analytic results. Appendix D considers the stability of each period's equilibrium. Appendix E contains proofs and derivations.

A Calibration, Climate Change Modeling, and Solution Method

Let resources 1, 2, and 3 represent coal, natural gas, and renewables, respectively. I use a 10-year timestep and a policy horizon of 400 years.

Begin by considering the supply of each type of resource. Marten et al. (2019) follow, among others, Haggerty et al. (2015) in using a long-run supply elasticity of 2.4 for coal. Marten et al. (2019) follow Arora (2014) in using a long-run supply elasticity of 0.5 for natural gas. Based on these, I use $\psi_1 = 2.4$ and $\psi_2 = 0.5$. Drawing in part on the work of others, Johnson et al. (2017) describe the supply of power from solar photovoltaics, concentrating solar power, onshore wind, and offshore wind available by region of the world and by resource quality. Costs are reported in dollars per unit power and resource potential is reported in units of energy. The desired parameter is the elasticity of the resource and not of its electricity. So I must convert dollars per units of delivered electric power to dollars per unit of energy in the resource (with my calibrated technology parameters including the conversion of energy to power). I convert costs to dollars per unit electrical energy by using the capacity factor reported for each resource quality bin in each region. This capacity factor adjusts for the fact that the power producible from renewable resources is not available throughout the day or throughout the year. In my setting, capacity factors are reflected in the technology and share parameters, and the elasticity of substitution σ can be interpreted as imposing a larger capacity factor penalty at higher penetrations. Finally, I convert dollars per unit of electrical energy to dollars per units of energy in the resource by using the efficiency of each type of generator. From the Energy Information Administration's Annual Energy Review 2011, the efficiencies are 12% for solar photovoltaics, 21% for solar thermal, and 26% for wind. Aggregating across resource types and regions, I estimate $\psi_3 = 3.00$.

Next consider the elasticities of substitution in the final-good and intermediate-good production functions. Papageorgiou et al. (2017) estimate an elasticity of substitution between clean and dirty energy capacity of around 1.8, and Stern (2012) estimates an elasticity of substitution between coal and gas of 1.426, with a standard error of 0.387. Version 6 of the EPPA model uses an elasticity of substitution of 1.5 (Chen et al., 2016), and the ADAGE model uses an elasticity of substitution of 1.25 (Ross, 2009). In line with these, I fix $\epsilon = 1.8$.

Much literature has estimated the elasticity of substitution between energy and other inputs, but there is not much literature on the elasticity of substitution between resources and other inputs in the production of energy. I fix $\sigma = 0.4$ based on several lines of evidence. The most directly relevant calibration is the calibration of the energy supply sector's production function in Lemoine (2020). This calibration assigns an elasticity of substitution of 0.42 to the energy supply sector, based on estimates in Koesler and Schymura (2015) implemented by Marten and Garbaccio (2018). Two other lines of evidence provide support from other industries. Okagawa and Ban (2008) estimate the elasticity of substitution between capital-labor and energy inputs as being around 0.5, and Atalay (2017, Appendix B) estimates an elasticity of substitution between a capital-labor composite and intermediates (potentially including energy) of 0.4–0.8. Some computable general equilibrium models of energy use assign an elasticity of substitution of 0.3 to nearly all sectors (see Turner, 2009), version 6 of the EPPA model uses an elasticity of substitution of 0.1 between resources and a capital-labor composite in electricity production (Chen et al., 2016), and ADAGE uses an elasticity of substitution of 0.6 between resources and a materials-value-added composite (Ross, 2009).

I fix two less important parameters, κ and α, at 0.5. The theory showed that the critical share parameters were the ν_j, not κ, and sensitivity tests support this conclusion.[35]

The remaining parameters are each A_{j0}, each Ψ_j, each ν_j, η, γ, and A_Y. I describe η and γ below. For given values of these two parameters, I calibrate the other ten parameters so that the first period's equilibrium Y_1, R_{j1}, s_{j1}, and p_{j1} match data (see Table A-2). World Bank data for global output from 2011–2015 imply that the value of the final good produced over the first ten-year timestep is 765 trillion year 2014 dollars. Initial resource consumption comes from summing consumption from 2011–2015, as reported in the BP Statistical Review of World Energy.[36] The International Energy Agency's World Energy Investment 2017 gives R&D spending on clean energy, on thermal generation, on coal production, and on oil and gas production. I divide thermal expenditures equally between coal and gas and attribute all oil and gas spending to gas. The first period must therefore have 12% of scientists working on coal, 65% of them working on gas, and 23% of them working on renewables. I calibrate each p_{j1} to be consistent with levelized costs from IEA (2015). Using the market discount rate of 7%, the median cost for coal is around 80 \$/MWh, for natural gas combined cycle plants is around 100 \$/MWh, and for solar photovoltaics is around 150 \$/MWh.[37]

These initial conditions and guesses for the A_{j0} and Ψ_j combine to yield the Y_{j1}. I then

[35] One might consider fixing $\kappa = 0.04$ based on Golosov et al. (2014). However, their parameter corresponds to the factor share of energy in Cobb-Douglas final good production, whereas here the relevant production function is for energy and is not Cobb-Douglas.

[36] Natural gas and coal are used for electricity generation, heating, and industrial processes. I here abstract from these differences. To obtain the energetic content of renewables from the reported tonnes of oil equivalent, use BP's assumed thermal efficiency of 38% to obtain the equivalent electrical energy and then use a 20% generator efficiency to convert electrical energy to energy in the renewable resource.

[37] These costs have changed over time and can be affected by pollution regulations. Experiments suggest that results are not highly sensitive to these choices.

use the ratio of the final-good firms' first-order conditions and the adding-up constraint on the share parameters to solve for the ν_j:

$$\begin{aligned}
\nu_3 =& \frac{1}{1 + \frac{p_{2,1}}{p_{3,1}}\left(\frac{Y_{2,1}}{Y_{3,1}}\right)^{1/\epsilon}\left(1 + \frac{p_{1,1}}{p_{2,1}}\left(\frac{Y_{1,1}}{Y_{2,1}}\right)^{1/\epsilon}\right)},\\
\nu_2 =& \frac{(1-\nu_3)}{1 + \frac{p_{1,1}}{p_{2,1}}\left(\frac{Y_{1,1}}{Y_{2,1}}\right)^{1/\epsilon}},\\
\nu_1 =& 1 - \nu_2 - \nu_3.
\end{aligned}$$

Now consider A_Y. The climate-economy integrated assessment literature typically models climate change as reducing total production. Letting T_t be surface temperature relative to 1900, we have, following Nordhaus (2017),

$$A_{Yt} = \left[1 - d\,T_t^2\right]\tilde{A}_{Yt}$$

with $d = 0.00236$. The robustness check with higher damages increases d to 0.0228, from the mean of the calibration to Pindyck (2019) in Appendix C.1 of Lemoine (2021), and caps damages at 85%. In a change of notation, A_{Yt} evolves over time in the numerical application. $\tilde{A}_{Yt}$ is total factor productivity, which follows DICE-2016R (Nordhaus, 2017) in growing initially at 1.48% annually, with the growth rate declining at a rate of 0.5% annually:

$$\tilde{A}_{Y(t+1)} = \tilde{A}_{Yt}\prod_{s=0}^{9}\left[1 + (0.0148)e^{-0.005*(10*(t-1)+s)}\right].$$

For the initial conditions and any given guesses for the A_{j0} and Ψ_j, I set $\tilde{A}_{Y1}$ to ensure that initial final good production matches Y_0.

We now have the ν_j, $\tilde{A}_{Y1}$ (and thus A_{Y1}), the initial conditions, and the guesses for the A_{j0} and the Ψ_j. The levels of the intermediate goods' prices then follow from the final-good firms' first-order conditions. We now require six conditions to pin down the A_{j0} and the Ψ_j. The zero-profit conditions for intermediate-good firms provide three conditions. The conditions on the initial research allocation provide two more conditions, as $\Pi_{1,1}/\Pi_{2,1} = 1$ and $\Pi_{1,1}/\Pi_{3,1} = 1$. These two conditions can be thought of as defining $A_{2,0}$ and $A_{3,0}$ as functions of $A_{1,0}$ and the Ψ_j. Final-good firms' zero-profit condition provides the remaining condition, which can be thought of as pinning down the level of the final-good firms' first-order conditions. I begin solving any given system by minimizing the sum of squared percentage deviations under constraints on the share parameters (as with Matlab fmincon) and then using that result to obtain a more precise solution by solving the system of equations exactly (as with Matlab fsolve).

Now consider the innovation function. Only the product of η and γ is important for improvements in technology over time. I therefore fix η at 1 and explore several values for γ, as described in the main text. Changes in γ do not affect the realized first-period technology as the calibration of the A_{j0} adjusts to offset γ. Instead, changes in γ affect how rapidly technology evolves after the first period. Different values of γ can be interpreted as different step sizes for research advances, as different probabilities of research successes, and/or as different sizes for the population of researchers.

Resource use generates carbon dioxide emissions that eventually cause warming. Time t emissions are

$$E_t = \bar{e} + \sum_{j=1}^{3} e_j R_{jt}.$$

I calculate the emission intensities of coal and gas by dividing emissions for each resource from 2010–2014 (from the Carbon Dioxide Information Analysis Center) by resource consumption over the initial timestep. Other emissions $\bar{e}$ come from summing emissions from all other reported categories, which includes emissions from oil.[38] The renewable resource does not generate emissions ($e_3 = 0$). The evolution of carbon and temperature over time follow DICE-2016R (Nordhaus, 2017). Stacking the time t atmospheric, upper ocean, and deep ocean stocks of carbon in a column vector $\boldsymbol{M}_t$, the dynamics of the carbon stocks are

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Lambda}^2 \boldsymbol{M}_t + \begin{bmatrix} E_t \\ 0 \\ 0 \end{bmatrix},$$

where $\boldsymbol{\Lambda}$ is a 3×3 matrix with positive entries that sum to 1 within each column and squaring it adjusts for using a 10-year timestep. Surface temperature evolves as:

$$T_{t+1} = T_t + C_1 \left[f \ln(M^{atm}_{t+1}/M_{pre})/\ln(2) - \frac{f}{s} T_t + C_3(T^o_t - T_t) \right],$$

where M^{atm}_{t+1} is the atmospheric stock of carbon (i.e., the first entry of $\boldsymbol{M}_{t+1}$), M_{pre} is the pre-industrial stock of atmospheric carbon, and T^o_t is time t ocean temperature. That ocean temperature in turn evolves as:

$$T^o_{t+1} = C_4 T_t + (1 - C_4) T^o_t.$$

I adjust C_4 and C_1 for a 10-year timestep.

In contrast to the DICE climate-economy model, abatement cost emerges endogenously within a period from the tradeoffs between fuels and evolves endogenously as technologies

[38] If there were no emissions from coal or gas, the (mostly oil) emissions $\bar{e}$ would raise global temperature by 1.4°C in 100 years and by 3.5°C in 400 years. Future analysis could allow $\bar{e}$ to vary over time. This extension is unlikely to affect the qualitative conclusions.

and resource depletion change over time. Using $\gamma = 1$ ($\gamma = 6$) and the initial period, a tax of 1 $/t$CO_2$ reduces emissions by 13% (10%), a tax of 10 $/t$CO_2$ reduces emissions by 17% (15%), a tax of 50 $/t$CO_2$ reduces emissions by 18% (18%), and a tax of 100 $/t$CO_2$ reduces emissions by 20% (19%). There is no one agreed-upon estimate for emission reductions from current taxes. The emission reductions from the smaller taxes are a bit larger than for industrial emissions in DICE-2016R, and the emission reductions from the larger taxes are a bit smaller than in DICE-2016R. The tax required to obtain a 5% reduction in emissions is very close to the estimate for the U.S. in Morris et al. (2012), which is encouraging because these emission reductions are close to the region of interest for optimal policy.

The base specification's preferences follow DICE-2016R. Per-period utility takes the familiar power form in per-capita consumption, with elasticity of intertemporal substitution EIS. The annual utility discount rate is ρ, set to 1.5%. Population L_t evolves as in DICE-2016R:

$$L_t = L_\infty \left(\frac{L_1}{L_\infty} \right)^{e^{-g_L (t-1)}},$$

where I convert the DICE-2016R equation into a differential equation (with time in decades) and solve it. The policymaker seeks to maximize utilitarian welfare W:

$$W = \sum_{t=1}^{40} \frac{L_t}{(1+\rho)^{10(t-1)}} \frac{(c_t/L_t)^{1-1/EIS}}{1-1/EIS}.$$

Table A-1 reports parameter values that are fixed across all specifications. Table A-2 reports market data used to calculate remaining parameters.

In the no-policy simulations, I solve each period's equilibrium by solving for the research allocation that maximizes scientists' expected profits within a search for the extraction allocation that clears the market for resources. For any given extraction allocation, I first check whether a case with all scientists in the renewable sector generates greater expected profits in that sector than in any other. If it does, the corner allocation is an equilibrium, but if it does not, I solve for the research allocation between the coal and gas sectors conditional on no scientists working in the renewable sector. If this allocation is also not an equilibrium, I solve for the equilibrium allocation between coal and gas conditional on any number of scientists in renewables and search for the number of scientists in working in renewables that equalizes that sector's expected profit to the expected profit from the other sectors that have nonzero scientists.

To optimize policy, I search for the policy and extraction trajectories that maximize welfare while clearing the market. This is a mathematical program with equilibrium constraints, which can be quite difficult to solve. I solve it by allowing the policymaker to control not only each period's tax and/or research subsidy but also each period's three extraction variables, three technology variables, and five climate variables, subject to the transition equations holding at every period and to the resource markets clearing in every period.[39] For each guess of controls, I solve for each period's equilibrium allocation of scientists using the rootfinding method described above. This problem is still a difficult bilevel programming problem, with the lower level programming problem often finding corner solutions (i.e., it is often true that some sector has no scientists). The key is that this form of the problem allows for the provision of analytic gradients for the objective and constraints.[40] Within those analytic gradients, I obtain the derivatives of equilibrium scientists by applying the implicit function theorem to the system of equations defined by equalized expected profits (for those sectors for which scientists are interior) and by the constraint on total scientists. I solve the model using the Knitro solver for Matlab (Byrd et al., 2006), trying several different algorithms and retaining the best of the solutions.

[39] In the cases with the research control, I model the policymaker as choosing the number of clean scientists directly, with the other two types of scientists clearing their markets conditional on this choice and with the level of the subsidy implied by the resulting research allocation. This works better than having the policymaker choose the subsidy directly.

[40] We essentially have a series of static problems once we condition on the expanded set of controls.

Table A-1: Parameters fixed across specifications.

Parameter	Value	Description
Market parameters		
ϵ	1.8	Elasticity of substitution in final-good production
σ	0.4	Elasticity of substitution in intermediate-good production
κ	0.5	Share parameter in intermediate-good production
α	0.5	Share parameter in machine service production
ψ_1, ψ_2, ψ_3	2.4, 0.5, 3	Resource supply elasticities
η	1	Probability of research success
Welfare parameters		
ρ	0.015	Annual utility discount rate
EIS	1/1.45	Elasticity of intertemporal substitution
L_1	7403	Year 2015 population (millions)
L_∞	11500	Asymptotic population (millions)
δ	0.7	Rate of approach to asymptotic population level
Climate parameters		
d	0.00236	Damage parameter
e_1, e_2, e_3	0.0250, 0.0139, 0	Emission intensity of resources (Gt C per EJ)
$\bar{e}$	37.7	Exogenous emissions per timestep (Gt C per decade)
M_{pre}	588	Pre-industrial atmospheric CO_2 (Gt C)
Λ_{11}	0.88	Carbon transfer coefficient for atmosphere to atmosphere
Λ_{12}	0.196	Carbon transfer coefficient for upper ocean to atmosphere
Λ_{13}	0	Carbon transfer coefficient for deep ocean to atmosphere
Λ_{21}	0.12	Carbon transfer coefficient for atmosphere to upper ocean
Λ_{22}	0.797	Carbon transfer coefficient for upper ocean to upper ocean
Λ_{23}	0.0015	Carbon transfer coefficient for lower ocean to upper ocean
Λ_{31}	0	Carbon transfer coefficient for atmosphere to lower ocean
Λ_{32}	0.007	Carbon transfer coefficient for upper ocean to lower ocean
Λ_{33}	0.9985	Carbon transfer coefficient for lower ocean to lower ocean
C_1	0.2010	Warming delay parameter
C_3	0.088	Parameter governing transfer of heat from ocean to surface
C_4	0.05	Parameter governing transfer of heat from surface to ocean
f	3.6813	Forcing from doubling of CO_2 (W/m^2)
s	3.1	Equilibrium temperature change from doubling CO_2 (°Celsius)
M_1^{atm}	861	Year 2015 atmospheric CO_2 (Gt C)
M_1^{up}	460	Year 2015 biosphere and upper ocean CO_2 (Gt C)
M_1^{lo}	1740	Year 2015 lower ocean CO_2 (Gt C)
T_1	0.85	Year 2015 atmospheric temperature (°Celsius from 1900)
T_1^o	0.0068	Year 2015 ocean temperature (°Celsius from 1900)

Table A-2: Market data matched by the first period's equilibrium (2011–2020). Resources are ordered as coal, gas, renewable.

Endogenous Outcome	Target	Description
Y_1	765	Global output in trillion year 2014 dollars
$\{R_{1,1}, R_{2,1}, R_{3,1}\}$	$\{1617, 1278, 224\}$	Resource consumption in EJ
$\{p_{1,1}, p_{2,1}, p_{3,1}\}$	$\{80, 100, 150\}$	Energy prices in $/MWh
$\{s_{1,1}, s_{2,1}, s_{3,1}\}$	$\{0.12, 0.65, 0.23\}$	Shares of research

B Additional Robustness Results

Tables A-3 and A-4 provide the analogues to the top four panels of Table 1 for the robustness checks presented in Section 5.3 of the main text. Table A-5 reports robustness of the relative value of policy to alternate calibrations. In all cases, the model is calibrated to match market equilibrium as described in Appendix A.[41]

The first rows in each panel of Table A-5 repeat results familiar from the main text.

The second rows increase σ from 0.4 to 0.7, near the upper end of values consistent with the literature (see Appendix A).[42] Bringing σ this close to 1 generates qualitatively different laissez-faire dynamics. In particular, coal now increases its share of resource use and research over time. In the case of small advances, this greater role for coal is largely at the expense of gas. Renewables actually do slowly increase their share of research over the next century but never claim much more than 60% of research before declining again. Their share of supply never exceeds 24%. In the case of large advances, renewables never surpass a 47% share of research or a 24% share of supply within 400 years. These dynamics are quite different from the main text's base case. As in that calibration, coal's initial technology is of higher quality than the technology used with either gas or renewables. With this larger value for σ, the supply expansion effect is too weak to drive a stark transition to either resource within the 400 year horizon. This case does not demonstrate the corner solutions typical of prior work on climate and directed technical change that implicitly fix $\sigma = 1$, but its high-coal, low-gas laissez-faire trajectories seem less realistic than the base specification.

Even though emission-intensive coal dominates the resource mix, the smaller scale of fossil resource use leads to slightly lower temperatures than in the base case. As a result of coal's greater entrenched advantage and of these lower temperatures, the policymaker now declines to use much policy, declining to shift all researchers to the clean sector when using a clean research subsidy and waiting at least 50 years to implement a nonnegligible standalone emission tax. In previous work with a slightly larger elasticity of substitution (e.g., the implementation of Acemoglu et al. (2012) in Greaker et al. (2018)), the extreme nature of lock-in allowed the policymaker to phase out an emission tax over time once technology advanced to the point where lock-in worked in favor of renewables. However, here the emission tax increases to high levels even after an early research subsidy helps renewables to dominate supply.

The third row in the top panel increases ϵ from 1.8 to 3.[43] Laissez-faire dynamics are qualitatively similar to the base case, except with a transition to renewables now not begin-

[41] I experimented with a value for σ of 0.1, but trajectories explode to over 35°C, well past the point at which losses from warming reach 100%.

[42] This value for σ is around the upper edge of values that can be successfully calibrated to market data. Even here the fit to market data is not perfect for the case of large advances. Coal starts out with a slightly greater share of research and resource supply than targeted, largely at the expense of renewables.

[43] Still-larger values lead to negative consumption per capita in laissez-faire in the case of small advances, and even a value of 3 does so in the case of large advances (which is why this case is omitted from the table).

Table A-3: Additional results for robustness to policymaking environment, for small advances ($\gamma = 1$).

	Policy Scenario			
Model Version	**No policy**	**Emission tax**	**Research subsidy**	**Both instruments**
Emission Tax in 2015 (\$ per tCO_2)				
Base	-	2.5	-	7.3
50-Year Delay	-	0.0	-	0.0
Less Discounting*	-	863.3	-	20.5
Higher Damages**	-	463.6	-	83.1
Renewables' Share of Resources in 2015 (%)				
Base	7.2	10.7	9.2	13.7
50-Year Delay	7.2	7.2	7.2	7.2
Less Discounting*	7.2	16.6	9.2	14.1
Higher Damages**	7.2	14.4	9.2	13.3
Renewables' Share of Scientists in 2015 (%)				
Base	23.3	24.1	100	100
50-Year Delay	23.3	23.3	23.3	23.3
Less Discounting*	23.3	64.6	100	100
Higher Damages**	23.3	49.0	100	60.7
Temperature in 2115 (°C, relative to 1900)				
Base	5.5	4.8	5.0	4.7
50-Year Delay	5.5	5.0	5.2	4.9
Less Discounting*	5.5	4.4	5.0	4.6
Higher Damages**	4.8	3.6	4.4	3.7

* Pure rate of time preference reduced from 1.5% to 0.01% per year, as in Stern (2007).

** Damages increased to calibration of Lemoine (2021), from survey evidence in Pindyck (2019).

Table A-4: Additional results for robustness to policymaking environment, for large advances ($\gamma = 6$).

	Policy Scenario			
Model Version	**No policy**	**Emission tax**	**Research subsidy**	**Both instruments**
Emission Tax in 2015 (\$ per tCO_2)				
Base	-	4.8	-	5.1
50-Year Delay	-	0.0	-	0.0
Less Discounting*	-	751.3	-	10.5
Higher Damages**	-	477.2	-	106.1
Renewables' Share of Resources in 2015 (%)				
Base	7.2	10.5	12.5	16.7
50-Year Delay	7.2	7.2	7.2	7.2
Less Discounting*	7.2	16.1	12.8	17.4
Higher Damages**	7.2	14.5	12.8	17.6
Renewables' Share of Scientists in 2015 (%)				
Base	23.3	23.8	96.7	95.7
50-Year Delay	23.3	23.3	23.3	23.3
Less Discounting*	23.3	37.3	100	100
Higher Damages**	23.3	33.3	100	89.8
Temperature in 2115 (°C, relative to 1900)				
Base	7.7	7.0	5.7	5.5
50-Year Delay	7.7	7.0	7.2	7.2
Less Discounting*	7.7	5.0	5.7	5.4
Higher Damages**	6.5	4.1	4.9	4.1

* Pure rate of time preference reduced from 1.5% to 0.01% per year, as in Stern (2007).

** Damages increased to calibration of Lemoine (2021), from survey evidence in Pindyck (2019).

Table A-5: Balanced growth equivalent gain under alternate specifications.

	Policy Scenario		
Specification	**Emission tax**	**Research subsidy**	**Both instruments**
Small Advances			
Base	1.6	2.6	3.1
Larger σ^*	0.1	0.1	0.2
Larger ϵ^{**}	1.5	0.2	5.2
Optimal Machine Subsidy	2.0	2.9	3.6
Large Advances			
Base	4.8	12.3	14.4
Larger σ^*	0.3	0.1	0.4
Optimal Machine Subsidy	7.1	12.9	15.2

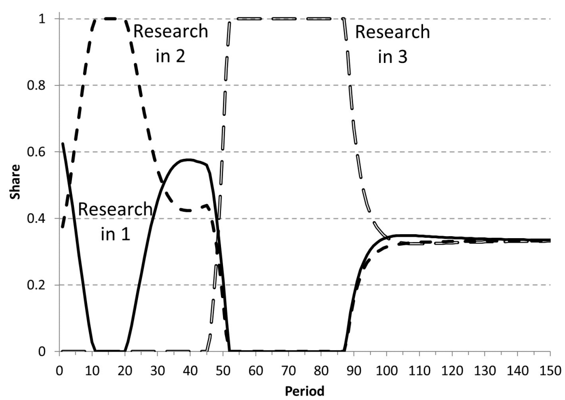
* σ increased from 0.4 to 0.7.
** ϵ increased from 1.8 to 3.

ning within 400 years. The main policy difference is that the policymaker declines to shift most scientists to the clean sector for around 200 years when using a standalone research subsidy. The standalone research subsidy therefore creates little value. The reason for the weak research subsidy is that a larger subsidy eventually drives consumption to extremely low (and even negative) levels by exacerbating long-run warming. In contrast, the policymaker does choose to immediately shift all scientists to the clean sector when using both a tax and a research subsidy. The emission tax controls long-run warming and the research subsidy quickly ignites a transition, as in the main text. The combined policy again generates substantially more value than does either standalone policy.

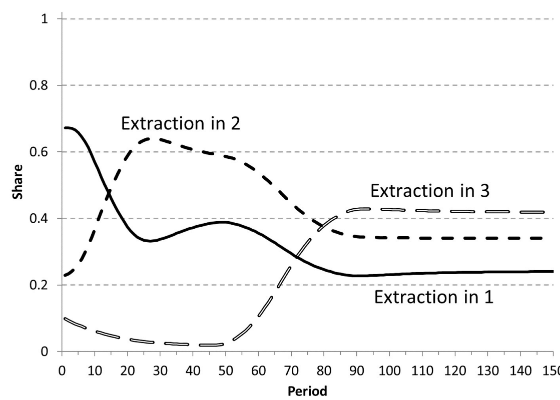
The final rows in each panel subsidize machine production to overcome distortions induced by monopolists' pricing. This subsidy reduces the consumer price p_{jxit} of machines from α to α^2. This subsidy is present in "laissez-faire" as well as in the models with climate policy. In order to preserve comparability with other specifications, the model is recalibrated to match market equilibrium even with this subsidy. Results are similar to the base specification.

C Numerical Example

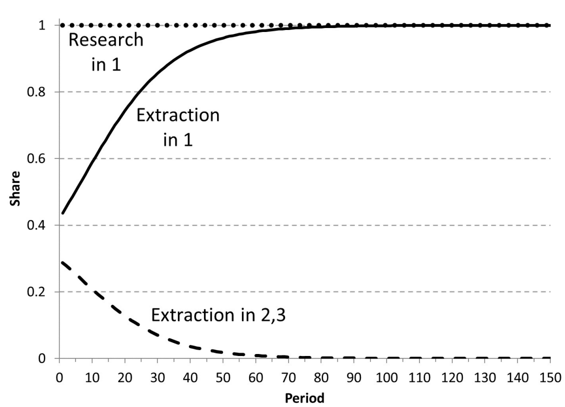
A numerical example will make the analytic results more concrete. Let there be three types of energy ($N = 3$), which differ only in their quality ν and in their initial technology. Let the first type of energy represent coal, the second represent oil, and the third represent gas. Looking back two hundred years, technologies for using coal were far more advanced than technologies for using oil, which in turn were more developed than technologies for using



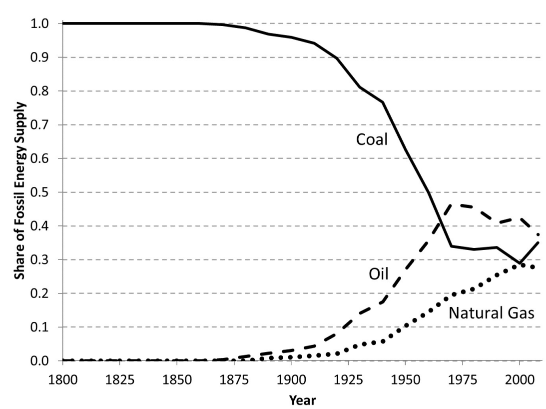
(a) Research Shares with $\sigma = 0.5$



(b) Extraction Shares with $\sigma = 0.5$



(c) Research and Extraction Shares with $\sigma = 1.5$



(d) Historical Extraction Shares

Figure A-1: Top: An example of an innovation-led transition, with $\sigma = 0.5$. Bottom left: An example of lock-in, with $\sigma = 1.5$. Resources 2 and 3 have nearly identical extraction shares. Bottom right: Shares of global fossil energy supply, from Smil (2010).

gas. I therefore fix the initial average quality of technology at 0.05 for coal, at 1% of this value for oil, and at 0.1% of this value for gas. We can think of the quality of fossil fuel resources as largely determined by the ratio of carbon to hydrogen bonds.[44] Energy derives from breaking hydrogen bonds. Fuels with a lot of carbon and little hydrogen are considered to be of lower quality because they are bulkier and more polluting. Coal is mostly carbon, oil has more hydrogen bonds per unit carbon, and natural gas has the most hydrogen bonds per unit carbon. I therefore set $\nu_1 = 0.27$ (for coal), $\nu_2 = 0.34$ (for oil), and $\nu_3 = 0.40$ (for gas).[45]

The top panels of Figure A-1 plot a case with $\sigma = 0.5$, and the lower left panel plots a case with $\sigma = 1.5$. The "coal" sector 1 begins with the majority of extraction and research activity. In the case of resource-saving technologies (bottom left), research activity and extraction are locked-in to the "coal" sector 1, which attracts all research effort in all periods and increases its share of resource extraction over time. In the case of resource-using technologies, we see innovation-led transitions. Research begins transitioning immediately towards the "oil" sector 2 (top left panel), and extraction eventually follows (top right panel). The "gas" sector 3 does not attract any research effort for a while and maintains a very small share of extraction even as oil displaces coal. However, after 20 periods, research effort shifts strongly towards the gas sector, and extraction shifts towards the gas sector after 60 periods. In the long run, all sectors attract identical shares of research effort and maintain stable shares of extraction, with their ordering determined by the quality ν of each resource.

The endogenous dynamics of our setting with resource-using machines are qualitatively similar to historical patterns. The bottom right panel of Figure A-1 plots resource shares since 1800. The historical patterns in these shares are similar to the patterns that emerge from our numerical simulations with resource-using machines: resource shares change rapidly as a transition occurs, and transitions do not drive formerly dominant resources out of the market. In fact, resource shares have been fairly stable since 1970. The historical patterns are nothing like the patterns that emerge from our simulations with resource-saving machines.

D Tâtonnement Stability

One may be concerned that interior equilibria are not "natural" equilibria in the presence of positive feedbacks from resource extraction to innovation and of potential complementarities. Indeed, Acemoglu (2002) and Hart (2012) have emphasized the role of knowledge spillovers

[44] Smil (2017, 245) describes how oil is of higher quality than coal because it has higher energy density, is cleaner, and is more transportable and storable. On page 270, he writes: "There has been a clear secular shift toward higher-quality fuels, that is, from coals to crude oil and natural gas, a process that has resulted in relative decarbonization (a rising H:C ratio) of global fossil fuel extraction..."

[45] The remaining parameters are $A_Y = 1$, $\epsilon = 3$, $\alpha = 0.5$, $\kappa = 0.5$, $\psi = 3$, $\Psi_1 = \Psi_2 = \Psi_3 = 1$, $\eta = 1$, and $\gamma = 0.5$. The qualitative results are not sensitive to the choice of these parameters.

in allowing interior research allocations to be stable in the long run. This appendix shows that interior equilibria are in fact "natural" equilibria in the present setting.

Assume $N = 2$ and label the two sectors j and k. Rearranging equation (10) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation.[46] Substituting into equations (12) and (13) then gives us two equations in two unknowns. This system defines the equilibrium R_{jt} and R_{kt} that clear the markets for each resource.

Define the tâtonnement adjustment process and stability as follows:

Definition A-1. *A* tâtonnement adjustment process *increases R_{jt} if equation* (12) *is not satisfied and its right-hand side is greater, decreases R_{jt} if equation* (12) *is not satisfied and its left-hand side is greater, and obeys analogous rules for R_{kt} using equation* (13). *I say that an equilibrium (R^*_{jt}, R^*_{kt}) is* tâtonnement-stable *if and only if the tâtonnement adjustment process leads to (R^*_{jt}, R^*_{kt}) from (R_{jt}, R_{kt}) sufficiently close to (R^*_{jt}, R^*_{kt}).*

The tâtonnement process changes R_{jt} and R_{kt} so as to eliminate excess supply or demand, and tâtonnement stability requires that this adjustment process converge to an equilibrium point from values close to the equilibrium. This process is the same as that in Samuelson (1941) and Arrow and Hurwicz (1958), except expressed in quantities rather than prices. The following proposition shows that our equilibrium is tâtonnement-stable:

Proposition A-1. *The equilibrium is tâtonnement-stable.*

Proof. See Appendix E.2. □

Now use equations (12) and (13) to define R_{jt} and R_{kt} as functions of s_{jt},[47] and then restate equation (10) as a function only of s_{jt}:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)}}{A_{k(t-1)}} \left(\frac{A_{j(t-1)} + \eta\gamma s_{jt} A_{j(t-1)}}{A_{k(t-1)} + \eta\gamma(1 - s_{jt}) A_{k(t-1)}} \right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}} \left(\frac{R_{jt}(s_{jt})}{R_{kt}(s_{jt})} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} . \tag{A-1}$$

The following corollary gives us the total derivative of Π_{jt}/Π_{kt} with respect to s_{jt}:

Corollary A-2. *The right-hand side of equation* (A-1) *strictly decreases in s_{jt}.*

Proof. See Appendix E.3 □

[46] Technically, this function should be written to allow for corner solutions in the research allocation. The proof of stability will account for corner solutions.

[47] Rearrange equations (12) and (13) to put all terms on the right-hand side. For given s_{jt}, the Jacobian of this system in R_{jt} and R_{kt} is negative definite.

The supply expansion effect makes the relative incentive to research in sector j decline in the number of scientists working in sector j. However, when sector j's share of resource extraction increases in the relative quality of its technology, a positive feedback between research and extraction maintains sector j's research incentives even as more scientists move to sector j. The proof shows, as is intuitive, that whether the relative incentive to research in sector j declines in the number of scientists working in sector j is identical to whether the equilibrium is tâtonnement-stable: tâtonnement-stability is not consistent with positive feedbacks that are strong enough to overwhelm the supply expansion effect. And we have already seen that interior equilibria are in fact tâtonnement-stable.

E Proofs and Derivations

This appendix derives useful intermediate results before providing proofs and derivations omitted from the main text.

E.1 Useful Lemmas

First, note that equations (8) and (9) imply

$$X_{jt} = \left[\frac{1-\kappa}{\kappa} p_{jRt}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}. \tag{A-2}$$

Rearranging equation (10) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation:

$$s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right) = \frac{(1+\eta\gamma)\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma} - 1}{\eta\gamma + \eta\gamma\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}}. \tag{A-3}$$

Let $\Sigma_{x,y}$ represent the elasticity of x with respect to y, and let $\Sigma_{x,y|z}$ represent the elasticity of x with respect to y holding z constant. The following lemma establishes signs and bounds for elasticities that will prove useful:

Lemma A-3. *The following hold, with analogous results for sector k:*

1. $\Sigma_{Y_t,Y_{jt}}, \Sigma_{Y_t,Y_{kt}} \in [0,1]$ *and* $\Sigma_{Y_t,Y_{jt}} + \Sigma_{Y_t,Y_{kt}} = 1$.
2. $\Sigma_{Y_{jt},R_{jt}|X_{jt}}, \Sigma_{Y_{jt},X_{jt}} \in [0,1]$ *and* $\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} = 1$.
3. *If* $\sigma < 1$, *then* $\Sigma_{Y_{jt},X_{jt}} \to 0$ *as* $A_{j(t-1)} \to \infty$ *and* $\Sigma_{Y_{kt},X_{kt}} \to 0$ *as* $A_{k(t-1)} \to \infty$.
4. $\Sigma_{X_{jt},A_{jt}} = \frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha} \in (0,1)$

5. $\Sigma_{X_{jt},R_{jt}} = \frac{\alpha\sigma/\psi+\alpha}{\sigma(1-\alpha)+\alpha} \in (0,1]$
6. $\Sigma_{A_{jt},s_{jt}} = \frac{\eta\gamma s_{jt}}{1+\eta\gamma s_{jt}} \in [0,1)$
7. $\Sigma_{s_{jt},R_{jt}} = \frac{\psi+\sigma}{\psi}\frac{2+\eta\gamma}{\eta\gamma s_{jt}} Z_t > 0$, *where* $Z_t \in \left[\frac{1+\eta\gamma}{(2+\eta\gamma)^2}, \frac{1}{4}\right]$. $\Sigma_{s_{jt},R_{kt}} = -\Sigma_{s_{jt},R_{jt}}$.
8. $\Sigma_{s_{jt},A_{j(t-1)}} = -\frac{(1-\sigma)(1-\alpha)}{A_{j(t-1)}}\frac{(2+\eta\gamma)}{\eta\gamma} Z_t$, *which is* < 0 *if and only if* $\sigma < 1$. Z_t *is as above.* $\Sigma_{s_{jt},A_{k(t-1)}} = -\Sigma_{s_{jt},A_{j(t-1)}}$.
9. $\Sigma_{s_{jt},s_{kt}} = -s_{kt}/s_{jt} \leq 0$

Proof. Most of the results follow by differentiation and the definition of an elasticity. #1 follows from differentiating the final-good production function $Y_t(Y_{jt}, Y_{kt})$; #2 follows from differentiating the intermediate-good production function $Y_{jt}(R_{jt}, X_{jt})$; #4 follows from differentiating equation (A-2); #5 follows from differentiating equation (A-2) after using equation (2) to substitute for p_{jRt} and using $\psi \geq \alpha/(1-\alpha)$; #6 follows from differentiating equation (5); #7 and #8 follow from differentiating equation (A-3); and #9 follows from the research constraint.

To derive #3, note that

$$\Sigma_{Y_{jt},X_{jt}} = \frac{(1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}}{\kappa R_{jt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}}.$$

From (7), (8), and (2), we have:

$$\begin{aligned} X_{jt} =& A_{jt}\left(\frac{1-\kappa}{\kappa}\left[\frac{R_{jt}}{X_{jt}}\right]^{1/\sigma}\Psi_j^{-1/\psi}R_{jt}^{1/\psi}\right)^{\frac{\alpha}{1-\alpha}} \\ =& A_{jt}\left(\frac{1-\kappa}{\kappa}\Psi_j^{-1/\psi}R_{jt}^{\frac{1}{\psi}+\frac{1}{\sigma}}\right)^{\frac{\sigma\alpha}{\sigma(1-\alpha)+\alpha}}. \end{aligned}$$

$X_{jt} \to \infty$ as $A_{j(t-1)} \to \infty$, which implies with $\sigma < 1$ that $\Sigma_{Y_{jt},X_{jt}} \to 0$ as $A_{j(t-1)} \to \infty$. Analogous results hold for sector k.

To derive #7 and #8, define

$$Z_t \triangleq \frac{\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)}\frac{R_{jt}}{R_{kt}}\left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}}{\left[1+\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)}\frac{R_{jt}}{R_{kt}}\left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}\right]^2}$$

and recognize that $s_{jt} \in (0,1)$ implies

$$\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)}\frac{R_{jt}}{R_{kt}}\left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma} \in \left(\frac{1}{1+\eta\gamma},\ 1+\eta\gamma\right)$$

from equation (10).

□

Note that $\Sigma_{X,A}$ and $\Sigma_{X,R}$ are the same in each sector. I therefore often omit the sector subscripts on these terms.

Using $s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right)$, the equilibrium is defined by equations (12) and (13), which are functions only of R_{jt} and R_{kt}. Rewrite these equations as (suppressing the predetermined technology arguments in s_{jt}):

$$1 = \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{Y_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{Y_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq G_j(R_{jt}, R_{kt}),$$

$$1 = \kappa\, (1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{Y_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{Y_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq G_k(R_{jt}, R_{kt}).$$

We have:

Lemma A-4. *$\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$ and $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt} < 0$.*

Proof. Differentiating yields:

$$\begin{aligned}
\frac{\partial G_j(R_{jt}, R_{kt})}{\partial R_{jt}} =& G_j\Bigg\{ -\left(\frac{1}{\psi}+\frac{1}{\sigma}\right)\frac{1}{R_{jt}} + \left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\frac{1}{Y_{jt}}\left[\frac{\partial Y_{jt}}{\partial R_{jt}} + \frac{\partial Y_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right] \\
&+ \frac{1}{\epsilon}\frac{1}{Y_t}\left[\frac{\partial Y_t}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial Y_{kt}}\frac{\partial Y_{kt}}{\partial s_{kt}}\frac{\partial s_{kt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right]\Bigg\} \\
=& \frac{G_j}{R_{jt}}\Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\left[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
&- \frac{1}{\epsilon}\Bigg[\left(1-\Sigma_{Y_t,Y_{jt}}\right)\left(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
&- \Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg]\Bigg\}.
\end{aligned}$$

If the economy is at a corner in s_{jt}, then $\Sigma_{s_{jt},R_{jt}} = 0$ and, using Lemma A-3, the above expression is clearly negative. So consider a case with interior s_{jt}. The final two lines are negative. So the overall expression is negative if the third-to-last line is negative, which is the case if and only if

$$\begin{aligned}
0 \geq& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\frac{\sigma+\psi}{\psi}\frac{\alpha + \sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha}\right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[-1 + \frac{\sigma+\psi}{\psi}\frac{\alpha + \sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha}\right],
\end{aligned} \tag{A-4}$$

where I use results from Lemma A-3. Note that $\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t \leq 3/4$, which implies that $\Sigma_{Y_{jt},X_{jt}} \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\sigma(1-\alpha)+\alpha} < 1$. Using this, inequality (A-4) holds if and only if

$$\frac{\sigma}{\psi} \geq \Sigma_{Y_{jt},X_{jt}} \frac{-1 + \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\alpha+\sigma(1-\alpha)}}{1 - \Sigma_{Y_{jt},X_{jt}} \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\alpha+\sigma(1-\alpha)}}. \tag{A-5}$$

$\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t \leq 3/4$ implies that $\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\alpha+\sigma(1-\alpha)} < 1$, which implies that the right-hand side of inequality (A-5) is negative. Thus, inequality (A-5) always holds and $\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$.

The analysis of $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt}$ is virtually identical.

□

Now define the matrix G:

$$G \triangleq \begin{bmatrix} \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}} \\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}} \end{bmatrix}.$$

We have:

Lemma A-5. *The determinant of G is positive.*

Proof. Analyze $det(G)$:

$$
\begin{aligned}
det(G) \propto & \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \left[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \right) \right] \Bigg\} \\
& \quad \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \left[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \left(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \right) \right] \Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \right) \\
& \qquad - \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg] \Bigg\} \\
& \quad \Bigg\{ \frac{1}{\epsilon} \Bigg[\Sigma_{Y_t,Y_{kt}} \left(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},R_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \right) \\
& \qquad + \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg] \Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \left(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \right) \\
& \qquad - \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg] \Bigg\} \\
& \quad \Bigg\{ \frac{1}{\epsilon} \Bigg[\Sigma_{Y_t,Y_{jt}} \left(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \right) \\
& \qquad + \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg] \Bigg\} \\
& - \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)^2 \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}},
\end{aligned}
$$

where I factored $G_j G_k / R_{jt} R_{kt}$. Use $\Sigma_{Y_t,Yjt} + \Sigma_{Y_t,Y_{kt}} = 1$ from Lemma A-3 and cancel terms with $1/\epsilon^2$ to obtain:

$$
\begin{aligned}
det(G) \propto & \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg)\Bigg]\Bigg\} \\
& \quad \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{kt},R_{kt}|X_{kt}} - \Sigma_{Y_{kt},X_{kt}}\Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg)\Bigg]\Bigg\} \\
& - \frac{1}{\sigma}\left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& + \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}} \\
& \quad \Bigg[-\Bigg(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg) \\
& \qquad + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
& + \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}} \\
& \quad \Bigg[-\Bigg(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg) \\
& \qquad + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg)\Bigg] \\
& \quad \Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg)\Bigg]. \qquad \text{(A-6)}
\end{aligned}
$$

All lines after the first three are positive by results from Lemma A-3. Expanding the products in those first three lines and rearranging, those first three lines become:

$$
\begin{aligned}
& \frac{1}{\psi^2} \\
& + \frac{1}{\sigma^2}\Big[1 - \Sigma_{X,R}\Big]\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\Bigg(1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg) \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\Big[1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Big] \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\Big[1 - \Sigma_{X,R} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Big] \\
& + \frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right), \qquad \text{(A-7)}
\end{aligned}
$$

where I write $\Sigma_{X,R}$ because this elasticity is the same in each sector. At corner allocations of research, $\Sigma_{s_{jt},R_{jt}} = \Sigma_{s_{jt},R_{kt}} = 0$. In this case, (A-7) is clearly positive. Now assume an interior allocation of research, so that $\Pi_{jt} = \Pi_{kt}$. Note that

$$\begin{aligned}&1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\\ =&\frac{1}{\psi}\frac{\sigma}{\sigma(1-\alpha)+\alpha}\left\{\psi[1-\alpha] - \alpha - (1-\alpha)[\sigma+\psi]\frac{(2+\eta\gamma)^2}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})}Z_t\right\}. \qquad \text{(A-8)}\end{aligned}$$

Substituting for Z_t and using equation (10) at $\Pi_{jt}/\Pi_{kt} = 1$, we have

$$\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}.$$

Equation (A-8) then becomes

$$1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} = -\frac{\sigma}{\psi}.$$

Substituting into (A-7), the first three lines of (A-6) are equal to

$$\begin{aligned}&\frac{1}{\psi^2}\\ &-\frac{1}{\psi}\frac{1}{\sigma}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\\ &+\frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\left[1-\Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right]\\ &+\frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[1-\Sigma_{X,R} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right]\\ &+\frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right). \qquad \text{(A-9)}\end{aligned}$$

The final line is positive. Factoring $1/\psi$, the first four lines are jointly positive if and only if:

$$\begin{aligned}0 \leq&\frac{1}{\psi} + \frac{1}{\sigma}\Bigg[(1-\Sigma_{X,R})\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\qquad - \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} - \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg]\\ =&\frac{1}{\psi} + \frac{1}{\sigma}\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &-\frac{1}{\sigma}\frac{\sigma+\psi}{\psi}\frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\qquad + \sigma(1-\alpha)\Big(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Big)\frac{1}{2+\eta\gamma}\Bigg], \qquad \text{(A-10)}\end{aligned}$$

where we use $\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}$. Note that $\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}$ increases in $\Sigma_{Y_{jt},X_{jt}}$ and thus reaches a maximum at $\Sigma_{Y_{jt},X_{jt}} = 1$. Therefore,

$$\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}} \leq 1 + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{kt},X_{kt}} = 1.$$

Also note that $\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})$ increases in each elasticity, and each elasticity is ≤ 1. Thus,

$$\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt}) \leq (1+\eta\gamma s_{kt}) + (1+\eta\gamma s_{jt}) = 2+\eta\gamma,$$

which implies

$$\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma} \leq 1.$$

These results together imply that

$$\begin{aligned}&\alpha+\sigma(1-\alpha)\\ \geq&\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) + \sigma(1-\alpha)\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}.\end{aligned} \tag{A-11}$$

Using this, we have that inequality (A-10) holds if and only if

$$\begin{aligned}\frac{\sigma}{\psi} \geq&\Bigg\{-\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) + \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\qquad + \sigma(1-\alpha)\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\}\\ &\Bigg\{1 - \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\qquad + \sigma(1-\alpha)\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\}^{-1}.\end{aligned} \tag{A-12}$$

The denominator on the right-hand side is positive via inequality (A-11). The numerator on the right-hand side is equal to:

$$\begin{aligned}&\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)\\ &\left\{-1 + \frac{1}{\sigma(1-\alpha)+\alpha}\left[\alpha + \sigma(1-\alpha)\frac{\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)}{(2+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)}\right]\right\}.\end{aligned} \tag{A-13}$$

Consider the fraction in brackets. If that fraction is ≤ 1, then the whole expression is negative and we are done. I will now prove that the fraction cannot be > 1. Assume that the fraction is > 1. Then:

$$\left(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right) > (2+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)$$

$$\Leftrightarrow \eta\gamma s_{kt}\Sigma_{Y_{jt},X_{jt}} + \eta\gamma s_{jt}\Sigma_{Y_{kt},X_{kt}} \geq (1+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}}\right) - (2+\eta\gamma)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}.$$

Assume without loss of generality that $\Sigma_{Y_{jt},X_{jt}} > \Sigma_{Y_{kt},X_{kt}}$. Then the left-hand side of the last line attains its largest possible value when $s_{kt} = 1$. The inequality on the last line is then satisfied only if

$$0 > \Sigma_{Y_{jt},X_{jt}} + (1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}. \tag{A-14}$$

The right-hand side is monotonic in $\Sigma_{Y_{jt},X_{jt}}$. At $\Sigma_{Y_{jt},X_{jt}} = 1$, the right-hand side is

$$1 + (1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} = 1 - \Sigma_{Y_{kt},X_{kt}} \geq 0.$$

But this contradicts inequality (A-14). Now consider the other extremum: $\Sigma_{Y_{jt},X_{jt}} = 0$. The right-hand side of inequality (A-14) becomes:

$$(1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} \geq 0,$$

which again contradicts inequality (A-14). Because the right-hand side of inequality (A-14) was monotonic in $\Sigma_{Y_{jt},X_{jt}}$ and was not satisfied for either the greatest or smallest possible values for $\Sigma_{Y_{jt},X_{jt}}$, the inequality is not satisfied for any values of $\Sigma_{Y_{jt},X_{jt}}$. Thus, the fraction in brackets in (A-13) is ≤ 1, which means that the right-hand side of inequality (A-12) is ≤ 0 and inequality (A-12) is satisfied. As a result, the first three lines of (A-6) are positive, which means that $det(G) > 0$.

□

The next two lemmas establish how relative extraction and relative profit change with the average quality of technology in sector j:

Lemma A-6. *Define* $\mathbf{R}(A_{jt}, A_{kt}) \triangleq [R_{jt}(A_{jt}, A_{kt})/R_{kt}(A_{jt}, A_{kt})]$. *Then (i)* $\partial\mathbf{R}/\partial A_{jt} > 0$ *and (ii)* $\partial\mathbf{R}/\partial A_{jt} \to 0$ *as* $A_{jt} \to \infty$.

Proof. I begin by using the implicit function theorem on the two-dimensional system obtained from equations (12) and (13). Rewriting previous expressions for G_j and G_k to hold s_{jt} fixed at some value s, the two-dimensional system becomes:

$$1 = \kappa\,\nu_j A_Y^{\frac{\epsilon-1}{\epsilon}}\left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}=s\right)}{Y_{jt}\left(R_{jt}, s_{jt}=s\right)}\right]^{1/\epsilon}\left[\frac{Y_{jt}\left(R_{jt}, s_{jt}=s\right)}{R_{jt}}\right]^{1/\sigma}\left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq H_j(R_{jt}, R_{kt}; s_{jt}=s),$$

$$1 = \kappa\,(1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}}\left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}=s\right)}{Y_{kt}\left(R_{kt}, s_{jt}=s\right)}\right]^{1/\epsilon}\left[\frac{Y_{kt}\left(R_{kt}, s_{jt}=s\right)}{R_{kt}}\right]^{1/\sigma}\left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq H_k(R_{jt}, R_{kt}; s_{jt}=s).$$

Fixing $s_{jt} = s$ makes A_{jt} a parameter. I analyze the following:

$$\begin{aligned}
\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} =& \frac{R_{jt}}{R_{kt}} \left\{ \frac{\partial R_{jt}}{\partial A_{jt}} \frac{1}{R_{jt}} - \frac{\partial R_{kt}}{\partial A_{jt}} \frac{1}{R_{kt}} \right\} \\
=& \frac{R_{jt}}{R_{kt}} \left\{ \frac{1}{R_{jt}} \frac{-\frac{\partial H_j}{\partial A_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}} \frac{\partial H_k}{\partial A_{jt}}}{det(H)} - \frac{1}{R_{kt}} \frac{-\frac{\partial H_k}{\partial A_{jt}} \frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}} \frac{\partial H_j}{\partial A_{jt}}}{det(H)} \right\} \\
=& \frac{R_{jt}}{R_{kt}} \frac{1}{det(H)} \left\{ -\frac{\partial H_j}{\partial A_{jt}} \left[\frac{1}{R_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_k}{\partial R_{jt}} \right] + \frac{\partial H_k}{\partial A_{jt}} \left[\frac{1}{R_{jt}} \frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_j}{\partial R_{jt}} \right] \right\}.
\end{aligned} \tag{A-15}$$

Differentiation and algebraic manipulations (including applying relationships from Lemma A-3) yield:

$$-\frac{\partial H_j}{\partial A_{jt}} = - H_j \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \right\} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} \frac{1}{A_{jt}},$$

$$\frac{\partial H_k}{\partial A_{jt}} = H_k \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} \frac{1}{A_{jt}},$$

$$\begin{aligned}
\frac{1}{R_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_k}{\partial R_{jt}} =& \frac{H_k}{R_{jt} R_{kt}} \left\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{kt}, X_{kt}} \left[1 - \Sigma_{X,R} \right] \right. \\
& \left. + \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{Y_{jt}, X_{jt}} - \Sigma_{Y_{kt}, X_{kt}} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{R_{jt}} \frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_j}{\partial R_{jt}} =& \frac{H_j}{R_{jt} R_{kt}} \left\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{jt}, X_{jt}} \left[1 - \Sigma_{X,R} \right] \right. \\
& \left. + \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{Y_{kt}, X_{kt}} - \Sigma_{Y_{jt}, X_{jt}} \right] \right\}.
\end{aligned}$$

Using these in equation (A-15), we obtain:

$$\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} = \frac{1}{A_{jt}} \frac{1}{det(H)} \frac{R_{jt}}{R_{kt}} \frac{H_j H_k}{R_{jt} R_{kt}} \Sigma_{X,A} \left(\frac{1}{\sigma} - \frac{1}{\epsilon} \right) \Sigma_{Y_{jt}, X_{jt}} \left(\frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{kt}, X_{kt}} [1 - \Sigma_{X,R}] \right). \tag{A-16}$$

Now consider $det(H)$. It follows from our analysis of $det(G)$ with $\Sigma_{s,R} = 0$. Make this change in equation (A-6):

$$\begin{aligned}
det(H) = \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},R_{jt}} \Bigg] \Bigg\} \\
& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{Y_{kt},R_{kt}|X_{kt}} - \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},R_{kt}} \Bigg] \Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Bigg\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Bigg[- \Bigg(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},R_{kt}} \Bigg) \Bigg] \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Bigg\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Bigg[- \Bigg(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},R_{jt}} \Bigg) \Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma} \Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X_{jt},R_{jt}} \Bigg] \Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X_{kt},R_{kt}} \Bigg] \Bigg).
\end{aligned}$$

Now analyze, using relations in Lemma A-3:

$$\begin{aligned}
det(H) = \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(& \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} \Bigg[1 - \Sigma_{X,R} \Bigg] \Bigg\} \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} \Bigg[1 - \Sigma_{X,R} \Bigg] \Bigg\} \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Bigg\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Bigg(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X,R} \Bigg) \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Bigg\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Bigg(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X,R} \Bigg) \\
& + \frac{1}{\epsilon}\frac{1}{\sigma} \Bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \Sigma_{X,R} \Bigg] \Bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \Sigma_{X,R} \Bigg] \Bigg) \\
= \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(& \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} \Bigg[1 - \Sigma_{X,R} \Bigg] \Bigg\} \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} \Bigg[1 - \Sigma_{X,R} \Bigg] \Bigg\} \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Bigg\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Bigg[1 - \Sigma_{Y_{kt},X_{kt}} (1 - \Sigma_{X,R}) \Bigg] \\
& + \Bigg\{ \frac{1}{\psi} + \frac{1}{\sigma} \Bigg\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Bigg[1 - \Sigma_{Y_{jt},X_{jt}} (1 - \Sigma_{X,R}) \Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma} \Bigg[1 - \Sigma_{Y_{jt},X_{jt}} (1 - \Sigma_{X,R}) \Bigg] \Bigg[1 - \Sigma_{Y_{kt},X_{kt}} (1 - \Sigma_{X,R}) \Bigg] \Bigg).
\end{aligned}$$

From Lemma A-3, $1 - \Sigma_{X,R} = \frac{\sigma}{\psi} \frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}$. Substituting $det(H)$ into equation (A-16), we have:

$$
\begin{aligned}
\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} =& \frac{1}{A_{jt}} \frac{R_{jt}}{R_{kt}} \Sigma_{X,A} \left(\frac{1}{\sigma} - \frac{1}{\epsilon} \right) \Sigma_{Y_{jt},X_{jt}} \left(\frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} [1 - \Sigma_{X,R}] \right) \\
& \left(\left\{ \frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} \left[1 - \Sigma_{X,R} \right] \right\} \left\{ \frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} \left[1 - \Sigma_{X,R} \right] \right\} \right. \\
& + \left\{ \frac{1}{\psi} + \frac{1}{\sigma} \right\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \left[1 - \Sigma_{Y_{kt},X_{kt}} (1 - \Sigma_{X,R}) \right] \\
& + \left\{ \frac{1}{\psi} + \frac{1}{\sigma} \right\} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \left[1 - \Sigma_{Y_{jt},X_{jt}} (1 - \Sigma_{X,R}) \right] \\
& \left. + \frac{1}{\epsilon} \frac{1}{\sigma} \left[1 - \Sigma_{Y_{jt},X_{jt}} (1 - \Sigma_{X,R}) \right] \left[1 - \Sigma_{Y_{kt},X_{kt}} (1 - \Sigma_{X,R}) \right] \right)^{-1} \\
>& 0.
\end{aligned} \tag{A-17}
$$

We have established the first part of the lemma. To establish the second part, use Lemma A-3 in equation (A-17).

□

Lemma A-7. *Fix $s_{jt} = s$. If $\sigma > 1$ or σ is not too much smaller than 1, then Π_{jt}/Π_{kt} increases in $A_{j(t-1)}$. As $A_{j(t-1)} \to \infty$, Π_{jt}/Π_{kt} decreases in $A_{j(t-1)}$ for all $\sigma < 1$.*

Proof. To a first-order approximation, we have, with s_{jt} fixed at s,

$$
\begin{aligned}
& \frac{\mathrm{d} \ln[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d} A_{j(t-1)}} \\
\approx & \frac{1}{A_{j(t-1)}} \left[1 - \frac{1}{\sigma + \alpha(1-\sigma)} \right] + \frac{1 + \sigma/\psi}{\sigma + \alpha(1-\sigma)} \frac{\partial A_{jt}}{\partial A_{j(t-1)}} \frac{\partial [R_{jt}/R_{kt}]}{\partial A_{jt}} \frac{R_{kt}}{R_{jt}} \\
= & \frac{1}{A_{j(t-1)}} \left[1 - \frac{1}{\sigma + \alpha(1-\sigma)} \right] + \frac{1}{\psi} \frac{\psi + \sigma}{\sigma + \alpha(1-\sigma)} (1 + \eta\gamma s) \frac{\partial [R_{jt}/R_{kt}]}{\partial A_{jt}} \frac{R_{kt}}{R_{jt}} \\
= & \frac{1}{A_{j(t-1)}} \frac{(1-\alpha)(\sigma-1)}{\sigma + \alpha(1-\sigma)} + \frac{1}{\psi} \frac{\psi + \sigma}{\sigma + \alpha(1-\sigma)} (1 + \eta\gamma s) \frac{\partial [R_{jt}/R_{kt}]}{\partial A_{jt}} \frac{R_{kt}}{R_{jt}}.
\end{aligned}
$$

The first term is positive if and only if $\sigma > 1$ and, using Lemma A-6, the second term is positive. Therefore the whole expression is positive if $\sigma > 1$. The first term becomes small for σ close to 1. Therefore the second term dominates (and the whole expression is positive) for σ not too much smaller than 1. Finally, Lemma A-6 shows that the second term goes to 0 as $A_{j(t-1)} \to \infty$ if $\sigma < 1$. Therefore the whole expression is negative if $\sigma < 1$ and $A_{j(t-1)} \to \infty$.

□

Finally, consider the evolution of relative extraction and thus of market size effects. From equation (14), R_{jt}/R_{kt} increases in s_{jt}. Define $\hat{s}_{t+1}$ as the unique value of $s_{j(t+1)}$ such that sector j's share of resource extraction increases from time t to $t+1$ if and only if $s_{j(t+1)} \geq \hat{s}_{t+1}$. Lemma A-6 implies that $\hat{s}_{t+1} \in (0, 1)$.

Lemma A-8. *If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \geq [\Psi_j/\Psi_k]^{1/[(1-\alpha)(1+\psi)]}$. If $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \leq [\Psi_j/\Psi_k]^{1/[(1-\alpha)(1+\psi)]}$.*

Proof. The change in R_{jt}/R_{kt} from time t to $t+1$ is

$$\begin{aligned}\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} =& \frac{(R_{j(t+1)} - R_{jt})R_{kt} - (R_{k(t+1)} - R_{kt})R_{jt}}{R_{k(t+1)}R_{kt}} \\ \propto& \frac{R_{j(t+1)} - R_{jt}}{R_{jt}} - \frac{R_{k(t+1)} - R_{kt}}{R_{kt}},\end{aligned}$$

where the first equality adds and subtracts $R_{jt}R_{kt}$ in the numerator and the second line factors $R_{jt}/R_{k(t+1)}$. To a first-order approximation, this is proportional to

$$\frac{1}{R_{jt}}\left(\frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right) - \frac{1}{R_{kt}}\left(\frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right),$$

with the derivatives evaluated at the time t allocation. Note that s_{jt} is included in A_{jt} when differentiating with respect to A_{jt}, which reflects that we will seek the allocation of scientists that holds R_{jt}/R_{kt} constant. Defining $H_j(R_{jt}, R_{kt}; s_{jt} = s)$ and $H_k(R_{jt}, R_{kt}; s_{jt} = s)$ as in the proof of Lemma A-6 and using the implicit function theorem, the previous expression becomes:

$$\begin{aligned}&\frac{1}{R_{jt}}\left(\frac{-\frac{\partial H_j}{\partial A_{jt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}}\frac{\partial H_k}{\partial A_{jt}}}{det(H)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial H_j}{\partial A_{kt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}}\frac{\partial H_k}{\partial A_{kt}}}{det(H)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ &- \frac{1}{R_{kt}}\left(\frac{-\frac{\partial H_k}{\partial A_{jt}}\frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}}\frac{\partial H_j}{\partial A_{jt}}}{det(H)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial H_k}{\partial A_{kt}}\frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}}\frac{\partial H_j}{\partial A_{kt}}}{det(H)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ \propto& \left[-\frac{\partial H_j}{\partial A_{jt}}s_{j(t+1)}A_{jt} - \frac{\partial H_j}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial H_k}{\partial R_{jt}}\right] \\ &+ \left[\frac{\partial H_k}{\partial A_{jt}}s_{j(t+1)}A_{jt} + \frac{\partial H_k}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial H_j}{\partial R_{jt}}\right], \qquad \text{(A-18)}\end{aligned}$$

where the second expression factors $\eta\gamma/det(H)$, which is readily seen to be positive by altering the proof of Lemma A-5 to set the $\Sigma_{s,R}$ terms to zero. Differentiation and algebraic manipulations (including applying relationships from Lemma A-3) yield:

$$
\begin{aligned}
-\frac{\partial H_j}{\partial A_{jt}} s_{j(t+1)} A_{jt} - \frac{\partial H_j}{\partial A_{kt}} s_{k(t+1)} A_{kt} =& - H_j \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \right\} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} s_{j(t+1)} \\
&- H_j \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} \Sigma_{X_{kt}, A_{kt}} (1 - s_{j(t+1)}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H_k}{\partial A_{jt}} s_{j(t+1)} A_{jt} + \frac{\partial H_k}{\partial A_{kt}} s_{k(t+1)} A_{kt} =& H_k \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \right\} \Sigma_{Y_{kt}, X_{kt}} \Sigma_{X_{kt}, A_{kt}} (1 - s_{j(t+1)}) \\
&+ H_k \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} s_{j(t+1)}.
\end{aligned}
$$

Substitute these and expressions derived in the proof of Lemma A-6 into (A-18) and factor

$\Sigma_{X,A} H_j H_k / [R_{jt} R_{kt}]$:

$$
\begin{aligned}
&\bigg\{ - s_{j(t+1)} \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \bigg\} \Sigma_{Y_{jt}, X_{jt}} - (1 - s_{j(t+1)}) \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} \bigg\} \\
&\qquad \bigg\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{kt}, X_{kt}} \bigg[1 - \Sigma_{X,R} \bigg] \bigg\} \\
&+ \bigg\{ (1 - s_{j(t+1)}) \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \bigg\} \Sigma_{Y_{kt}, X_{kt}} + s_{j(t+1)} \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \bigg\} \\
&\qquad \bigg\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{jt}, X_{jt}} \bigg(1 - \Sigma_{X,R} \bigg) \bigg\} \\
&+ \frac{1}{\epsilon} \bigg[1 - \Sigma_{X,R} \bigg] \bigg[\Sigma_{Y_{kt}, X_{kt}} - \Sigma_{Y_{jt}, X_{jt}} \bigg] \\
&\qquad \bigg\{ - s_{j(t+1)} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \bigg] \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} - (1 - s_{j(t+1)}) \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \bigg\} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} \bigg\} \\
&- \frac{1}{\epsilon^2} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_t, Y_{kt}} \bigg[1 - \Sigma_{X,R} \bigg] \bigg[\Sigma_{Y_{kt}, X_{kt}} - \Sigma_{Y_{jt}, X_{jt}} \bigg] \bigg\{ (1 - s_{j(t+1)}) \Sigma_{Y_{kt}, X_{kt}} + s_{j(t+1)} \Sigma_{Y_{jt}, X_{jt}} \bigg\} \\
=& s_{j(t+1)} \Sigma_{Y_{jt}, X_{jt}} \bigg\{ \frac{1}{\psi} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \bigg] \\
&\qquad\qquad + \frac{1}{\sigma} \bigg(1 - \Sigma_{X,R} \bigg) \bigg[\frac{1}{\sigma} \Sigma_{Y_{kt}, X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \bigg] \bigg\} \\
&- (1 - s_{j(t+1)}) \Sigma_{Y_{kt}, X_{kt}} \bigg\{ \frac{1}{\psi} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \bigg] \\
&\qquad\qquad\qquad + \frac{1}{\sigma} \bigg(1 - \Sigma_{X,R} \bigg) \bigg[\frac{1}{\sigma} \Sigma_{Y_{jt}, X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} \bigg] \bigg\} \\
&+ \frac{1}{\epsilon} \bigg[1 - \Sigma_{X,R} \bigg] \bigg[\Sigma_{Y_{kt}, X_{kt}} - \Sigma_{Y_{jt}, X_{jt}} \bigg] \\
&\qquad \bigg\{ - s_{j(t+1)} \bigg[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \bigg] \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} - (1 - s_{j(t+1)}) \bigg\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \bigg\} \Sigma_{Y_t, Y_{kt}} \Sigma_{Y_{kt}, X_{kt}} \\
&\qquad\quad - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_t, Y_{kt}} \bigg[(1 - s_{j(t+1)}) \Sigma_{Y_{kt}, X_{kt}} + s_{j(t+1)} \Sigma_{Y_{jt}, X_{jt}} \bigg] \bigg\}
\end{aligned}
$$

$$
\begin{aligned}
=&s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\right]\Sigma_{Y_{kt},X_{kt}}\right\}\\
&-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\right]\Sigma_{Y_{jt},X_{jt}}\right\}\\
&-s_{j(t+1)}\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,Y_{jt}}\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}+(1-s_{j(t+1)})\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\right]\\
&+\frac{1}{\sigma^2}\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)-\frac{1}{\sigma}\frac{1}{\epsilon}\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\left(2s_{j(t+1)}-1\right)\\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\right]+\frac{1}{\sigma}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)
\end{aligned}
$$

Substituting for $\Sigma_{X,R}$ and rearranging, we obtain

$$
\begin{aligned}
&\frac{1}{\psi}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{kt},X_{kt}}\right)\right.\\
&\qquad\left.-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{jt},X_{jt}}\right)\right].
\end{aligned}
\tag{A-19}
$$

This expression is positive if and only if the term in brackets is positive. Define $\hat{s}_{t+1}$ as the $s_{j(t+1)}$ such that $R_{jt}/R_{kt} = R_{j(t+1)}/R_{k(t+1)}$. Then $\hat{s}_{t+1}$ is the root of the term in brackets. Solving for that root, we have:

$$
\hat{s}_{t+1}=\frac{\Sigma_{Y_{kt},X_{kt}}C_{jt}}{\Sigma_{Y_{jt},X_{jt}}C_{kt}+\Sigma_{Y_{kt},X_{kt}}C_{jt}},
\tag{A-20}
$$

where $\Sigma_{w,z}$ is the elasticity of w with respect to z and where

$$
\begin{aligned}
C_{jt}\triangleq&1+\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\left[\psi-\frac{\alpha}{1-\alpha}\right]\Sigma_{Y_{jt},X_{jt}}>0,\\
C_{kt}\triangleq&1+\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\left[\psi-\frac{\alpha}{1-\alpha}\right]\Sigma_{Y_{kt},X_{kt}}>0.
\end{aligned}
$$

Thus,

$$
\left\{\hat{s}_{t+1}\geq\frac{1}{2}\right\}\Leftrightarrow\left\{\Sigma_{Y_{kt},X_{kt}}\geq\Sigma_{Y_{jt},X_{jt}}\right\},
$$

where the right-hand side is evaluated at $\hat{s}_{t+1}$. Using the explicit expressions for the elasticities, for intermediate-good production, and for X_{jt} and X_{kt} (see equation (A-2)), we have:

$$\begin{aligned}
\Sigma_{Y_{kt},X_{kt}} \geq& \Sigma_{Y_{jt},X_{jt}} \\
\Leftrightarrow 0 \leq& \frac{(1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}} Y_{kt}^{\frac{\sigma-1}{\sigma}}}{Y_{kt}^{\frac{\sigma-1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}}} \qquad \text{(A-21)} \\
\Leftrightarrow 0 \leq& X_{kt}^{\frac{\sigma-1}{\sigma}} Y_{jt}^{\frac{\sigma-1}{\sigma}} - X_{jt}^{\frac{\sigma-1}{\sigma}} Y_{kt}^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 0 \leq& \kappa R_{jt}^{\frac{\sigma-1}{\sigma}} X_{kt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa) X_{jt}^{\frac{\sigma-1}{\sigma}} X_{kt}^{\frac{\sigma-1}{\sigma}} - \kappa R_{kt}^{\frac{\sigma-1}{\sigma}} X_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa) X_{kt}^{\frac{\sigma-1}{\sigma}} X_{jt}^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq& \left(\frac{R_{jt} \left[\frac{1-\kappa}{\kappa} \left(\frac{R_{kt}}{\Psi_k} \right)^{1/\psi} \right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{kt}}{A_{kt}} \right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{kt}}{R_{kt} \left[\frac{1-\kappa}{\kappa} \left(\frac{R_{jt}}{\Psi_j} \right)^{1/\psi} \right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}} \right)^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq& \left[\left(\frac{\Psi_j}{\Psi_k} \right)^{\frac{\alpha\,\sigma/\psi}{\sigma(1-\alpha)+\alpha}} \left(\frac{R_{jt}}{R_{kt}} \right)^{\frac{\sigma(1-\alpha-\alpha/\psi)}{\sigma(1-\alpha)+\alpha}} \left(\frac{A_{kt}}{A_{jt}} \right)^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq& \left(\frac{\Psi_j}{\Psi_k} \right)^{\chi \frac{1}{\psi}[\alpha+\sigma(1-\alpha)]} \left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}} \right)^{-\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]} \left(\frac{A_{j(t-1)}}{A_{k(t-1)}} \right)^{\chi(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi)-(1+\sigma/\psi)]}, \qquad \text{(A-22)}
\end{aligned}$$

where the final line substitutes for R_{jt}/R_{kt} from equation (10) (which must hold for $\hat{s}_{t+1}$ interior) and where

$$\chi \triangleq \frac{\sigma-1}{[\sigma(1-\alpha)+\alpha][1+\sigma/\psi]} < 0 \text{ iff } \sigma < 1.$$

The right-hand side of inequality (A-22) is increasing in s_{jt} if and only if $\sigma < 1$. Therefore, if $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the strict version of the inequality does not hold at $s_{jt} = 0.5$, and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the inequality holds at $s_{jt} = 0.5$. If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \geq \left[\frac{\Psi_j}{\Psi_k} \right]^{\theta},$$

and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \leq \left[\frac{\Psi_j}{\Psi_k} \right]^{\theta},$$

where

$$\theta \triangleq \frac{-\frac{1}{\psi}[\alpha + \sigma(1-\alpha)]}{(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi) - (1+\sigma/\psi)]} = \frac{1}{(1-\alpha)(1+\psi)} > 0.$$

□

E.2 Proof of Proposition A-1

The tâtonnement adjustment process generates, to constants of proportionality, the following system for finding the equilibrium within period t:

$$\begin{aligned}\dot{R}_{jt} =& h\Big(G_j(R_{jt}, R_{kt}) - 1\Big),\\ \dot{R}_{kt} =& h\Big(G_k(R_{jt}, R_{kt}) - 1\Big),\end{aligned}$$

where dots indicate time derivatives (with the fictional time for finding an equilibrium here flowing within a period t), $h(0) = 0$, and $h'(\cdot) > 0$. The system's steady state occurs at the equilibrium values, which I denote with stars. Linearizing around the steady state, we have

$$\begin{bmatrix}\dot{R}_{jt}\\ \dot{R}_{kt}\end{bmatrix} \approx h'(0) \begin{bmatrix}\frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}}\\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}}\end{bmatrix} \begin{bmatrix}R_{jt} - R^*_{jt}\\ R_{kt} - R^*_{kt}\end{bmatrix} = h'(0)\, G \begin{bmatrix}R_{jt} - R^*_{jt}\\ R_{kt} - R^*_{kt}\end{bmatrix},$$

where G is the 2×2 matrix of derivatives, each evaluated at (R^*_{jt}, R^*_{kt}). Lemma A-4 implies that the trace of G is strictly negative, in which case at least one of the two eigenvalues must be strictly negative. Lemma A-5 shows that $det(G) > 0$, which means that both eigenvalues must have the same sign. Therefore both eigenvalues are strictly negative. The linearized system is therefore globally asymptotically stable, and, by Lyapunov's Theorem of the First Approximation, the full nonlinear system is locally asymptotically stable around the equilibrium.

E.3 Proof of Corollary A-2

Now treat equations (12) and (13) as functions of R_{jt}, R_{kt}, and s_{jt} (recognizing that $s_{kt} = 1 - s_{jt}$):

$$\begin{aligned}1 = \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{Y_{jt}(R_{jt}, s_{jt})}\right]^{1/\epsilon} \left[\frac{Y_{jt}(R_{jt}, s_{jt})}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \quad &\triangleq \hat{G}_j(R_{jt}, R_{kt}; s_{jt}),\\ 1 = \kappa\, (1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{Y_{kt}(R_{kt}, s_{jt})}\right]^{1/\epsilon} \left[\frac{Y_{kt}(R_{kt}, s_{jt})}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \quad &\triangleq \hat{G}_k(R_{jt}, R_{kt}; s_{jt}).\end{aligned}$$

This system of equations implicitly defines R_{jt} and R_{kt} as functions of the parameter s_{jt}. Define the matrix $\hat{G}$ analogously to the matrix G. Using the implicit function theorem, we have

$$\frac{\partial R_{jt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} \text{ and } \frac{\partial R_{kt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})}.$$

Interpreting equation (10) as implicitly defining s_{jt} as a function of R_{jt} and R_{kt}, we have:

$$\frac{\partial s_{jt}}{\partial R_{jt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}} \text{ and } \frac{\partial s_{jt}}{\partial R_{kt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}},$$

and thus

$$\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \text{ and } \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}.$$

Using these expressions, consider how the right-hand side of equation (A-1) changes in s_{jt}:

$$\begin{aligned}
\frac{\mathrm{d}[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d}s_{jt}} =& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}\frac{\partial R_{jt}}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}\frac{\partial R_{kt}}{\partial s_{jt}} \\
=& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} \\
& - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})} \\
\propto& - \frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} \\
& - \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}} - \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}} \\
=& - \left(\frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\right) \\
& + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \\
=& - det(G).
\end{aligned}$$

The third expression factored $det(\hat{G})$, which is positive by the proof of Proposition A-1 for a corner solution in s_{jt}, and it also factored $\partial[\Pi_{jt}/\Pi_{kt}]/\partial s_{jt}$, which is negative. The final equality recognizes that the only difference between the equations with a hat and the equations without a hat are that the equations without a hat allow s_{jt} to vary with R_{jt} and R_{kt}. Lemma A-5 showed that $det(G) > 0$. Thus the right-hand side of equation (A-1) strictly decreases in s_{jt}.

E.4 Proof of Lemma 1

Under the given assumption that $\nu = 0.5$ and $\Psi_j = \Psi_k$, we have $R_{jt} = R_{kt}$ when $A_{j(t-1)} = A_{k(t-1)}$ and $s_{jt} = 0.5$. Therefore, it is easy to see that $\Pi_{jt}/\Pi_{kt} = 1$ at $s_{jt} = 0.5$ when $A_{j(t-1)} = A_{k(t-1)}$. By Lemma A-7, increasing $A_{j(t-1)}$ increases Π_{jt}/Π_{kt} if either $\sigma > 1$ or σ is not too much smaller than 1. In those cases, Corollary A-2 gives us that $A_{j(t-1)} > A_{k(t-1)}$ implies $s^*_{jt} > 0.5$. The lemma follows from observing that $A_{j(t-1)} > A_{k(t-1)}$ and $\Psi_j = \Psi_k$ imply that $A_{j(t-1)}/A_{k(t-1)} > (\Psi_j/\Psi_k)^{1/[(1-\alpha)(1+\psi)]}$.

E.5 Proof of Proposition 2

To start, let Assumption 1 hold. From Lemma A-8, $\hat{s}_{t+1} < 0.5$. Therefore $s_{jt_0} > \hat{s}_{t+1}$. Assume that $s_{j(t_0+1)} < s_{jt_0}$. From equation (10), $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)}$ increases in A_{jt_0}/A_{kt_0} for any given $s_{j(t_0+1)}$ if $\sigma > 1$. Therefore, for the equilibrium to have $s_{j(t_0+1)} < s_{jt_0}$, it must be true that $R_{jt_0}/R_{kt_0} > R_{j(t_0+1)}/R_{k(t_0+1)}$ and thus $s_{j(t_0+1)} < \hat{s}_{t_0+1}$. From Corollary A-2 and $s_{jt_0} > \hat{s}_{t_0+1}$, it must be true that $\Pi_{jt_0}/\Pi_{kt_0} > 1$ when evaluated at $\hat{s}_{t_0+1}$. Because $R_{jt_0}/R_{kt_0} = R_{j(t_0+1)}/R_{k(t_0+1)}$ if $s_{j(t_0+1)} = \hat{s}_{t_0+1}$ and $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$ by $s_{jt_0} > 0.5$, it therefore must be true that $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)} > 1$ when evaluated at $\hat{s}_{t_0+1}$. By Corollary A-2, it then must be true that $s_{j(t_0+1)} > \hat{s}_{t_0+1}$. We have a contradiction. It must be true that $s_{j(t_0+1)} \geq s_{jt_0}$.

Because $s_{j(t_0+1)} \geq s_{jt_0} > 0.5 > \hat{s}_{t+1}$, it follows that $R_{jt_0}/R_{kt_0} \leq R_{j(t_0+1)}/R_{k(t_0+1)}$ and $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$. Therefore Assumption 1 still holds at time $t_0 + 1$. Proceeding by induction, sector j's shares of research and extraction increase forever: resource j is locked-in from time t_0 if $\sigma > 1$ and Assumption 1 holds at time t_0. We have established the first part of the proposition.

Now consider the remaining parts of the proposition, no longer imposing Assumption 1. We know that $\Pi^*_{jt}/\Pi^*_{kt} = 1$ when $s^*_{jt} \in (0, 1)$. Assume that $s^*_{jt} \in (0.5, 1)$. By Lemma A-7, $\Pi_{j(t+1)}/\Pi_{k(t+1)} > 1$ when evaluated at s^*_{jt}. Therefore, by Corollary A-2, $s^*_{j(t+1)} > s^*_{jt}$. Analogous arguments apply when $s^*_{jt} \in (0, 0.5)$. We have established the second part of the proposition.

By the foregoing, the only possible steady states are at $s^*_{jt} = 0.5$, $s^*_{jt} = 0$, and $s^*_{jt} = 1$. We just saw that a steady state at $s^*_{jt} = 0.5$ cannot be stable (should it even exist). When $s^*_{jt} = 1$, only $A_{j(t-1)}$ changes over time, increasing by $\eta\gamma A_{j(t-1)}$ at each time t. By Lemma A-7, $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)} > \Pi_{jt_0}/\Pi_{kt_0}$ if $s_{j(t_0+1)} \geq s_{jt_0}$. If $s_{jt_0} = 1$, then $\Pi_{jt_0} > \Pi_{kt_0}$, in which case $\Pi_{j(t_0+1)} > \Pi_{k(t_0+1)}$ if $s_{j(t_0+1)} = s_{jt_0}$. It is then an equilibrium for s^*_{jt} to equal 1 for all $t \geq t_0$. An analogous proof covers the case where $s^*_{jt} = 0$.

E.6 Proof of Proposition 3

First consider whether a corner allocation can persist indefinitely. If $s^*_{jt} = 1$ for all $t \geq t_0$, then $A_{j(t-1)} \to \infty$ as $t \to \infty$ and, by Lemma A-6, R_{jt}/R_{kt} goes to a constant. In that case, from equation (10), Π_{jt}/Π_{kt} goes to zero for all s_{jt}. But Π_{jt}/Π_{kt} cannot be zero if $s^*_{jt} = 1$ because $s^*_{jt} = 1$ implies that $\Pi_{jt}/\Pi_{kt} \geq 1$. We have contradicted the assumption that $s^*_{jt} = 1$ for all $t \geq t_0$. Analogous arguments show that it cannot be true that $s^*_{kt} = 1$ for all $t \geq t_0$. It therefore must be true that, for all t_0, there exists some $t > t_0$ such that $s^*_{jt} \in (0, 1)$.

Because a corner research allocation cannot persist indefinitely, A_{jt} and A_{kt} both become arbitrarily large as t becomes large. From equations (8), (9), and (2), we have

$$\begin{aligned} X_{jt} &= \left\{ \left[\left(\frac{R_{jt}}{\Psi_j} \right)^{1/\psi} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}} \right\}^{\frac{\alpha}{1-\alpha}} A_{jt} \\ &= \left[\Psi_j^{-1/\psi} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} R_{jt}^{\frac{\alpha(1+\sigma/\psi)}{\sigma(1-\alpha)+\alpha}}. \end{aligned}$$

X_{jt} and X_{kt} thus also become arbitrarily large as t becomes large. This in turn implies that $Y_{jt} \to \kappa^{\frac{\sigma}{\sigma-1}} R_{jt}$ and $Y_{kt} \to \kappa^{\frac{\sigma}{\sigma-1}} R_{kt}$ as t becomes large. From equation (14), we have:

$$\left[\frac{R_{jt}}{R_{kt}} \right]^{\frac{1}{\sigma}+\frac{1}{\psi}} \to \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \left[\frac{R_{jt}}{R_{kt}} \right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}}$$

as t becomes large. Therefore, as $t \to \infty$,

$$\frac{R_{jt}}{R_{kt}} \to \left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}}. \tag{A-23}$$

Define $\Omega_t \triangleq A_{jt}/A_{kt}$, so that

$$\Omega_t = \frac{1 + \eta\gamma s_{jt}}{1 + \eta\gamma(1 - s_{jt})} \Omega_{t-1}. \tag{A-24}$$

Because a corner allocation cannot persist indefinitely, $\Pi^*_{jt}/\Pi^*_{kt} = 1$ for some t sufficiently large. Using this and equation (A-23) in equation (10), we have:

$$\frac{1 + \eta\gamma s^*_{jt}}{1 + \eta\gamma(1 - s^*_{jt})} = \Omega_{t-1}^{-(1-\sigma)(1-\alpha)} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi}.$$

Therefore, from equation (A-24),

$$\Omega_t = \Omega_{t-1}^{1-(1-\sigma)(1-\alpha)} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi}.$$

Define $\tilde{\Omega}_t \triangleq \ln[\Omega_t]$. We then have:

$$\tilde{\Omega}_t = [1-(1-\sigma)(1-\alpha)]\tilde{\Omega}_{t-1} + \ln \left[\left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi} \right].$$

This is a linear difference equation. For $\sigma < 1$, the coefficient on $\tilde{\Omega}_{t-1}$ is strictly between 0 and 1. The linear difference equation is therefore stable. The system approaches a steady state in $\tilde{\Omega}_t$ and therefore in Ω_t. From equation (A-24), any steady state in Ω_t must have $s_{jt}^* = 0.5$. Therefore as $t \to \infty$, $s_{jt}^* \to 0.5$. We have established the first result.

Equation (A-23) implies that if $\nu_j = 0.5$ and $\Psi_j = \Psi_k$ then $R_{jt}^* = R_{kt}^*$. Further, if $\nu_j \geq 0.5$ and $\Psi_j \geq \Psi_k$ with at least one inequality being strict, then $R_{jt}^* > R_{kt}^*$. Now substitute into equation (10) and use $s_{jt} = 0.5$:

$$\begin{aligned} \frac{\Pi_{jt}}{\Pi_{kt}} &\to \left(\frac{A_{j(t-1)}}{A_{k(t-1)}} \right)^{\frac{-(1-\sigma)(1-\alpha)}{\sigma+\alpha(1-\sigma)}} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \\ &= \left(\frac{A_{j(t-1)}}{A_{k(t-1)}} \right)^{\frac{-(1-\sigma)(1-\alpha)}{\sigma+\alpha(1-\sigma)}} \left(\frac{\nu_j}{1-\nu_j} \right)^{\frac{\sigma+\psi}{\sigma+\alpha(1-\sigma)} \frac{\epsilon}{\epsilon+\psi}} \left(\frac{\Psi_j}{\Psi_k} \right)^{\frac{\epsilon-\sigma}{\sigma+\alpha(1-\sigma)} \frac{1}{\epsilon+\psi}}, \end{aligned}$$

and this must equal 1 because $s_{jt}^* = 0.5$. Therefore, if $\nu_j = 0.5$ and $\Psi_j = \Psi_k$ then $A_{jt} = A_{kt}$, and if $\nu_j \geq 0.5$ and $\Psi_j \geq \Psi_k$ with at least one inequality being strict, then $A_{jt} > A_{kt}$. We have established the second and third results.

Finally, as t becomes large along a path with $s_{jt}^* = 0.5$, using previous results in equation (12) yields:

$$\begin{aligned}
\left[\frac{R_{jt}}{\Psi_j}\right]^{1/\psi} \to& \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma} \\
=&\kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{\kappa^{\frac{\sigma}{\sigma-1}} R_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
=&\kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{\kappa^{\frac{\sigma}{\sigma-1}} R_{jt}}{A_Y Y_{jt}\left(\nu_j + (1-\nu_j)\left(\frac{Y_{kt}}{Y_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
=&\kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{1}{A_Y \left(\nu_j + (1-\nu_j)\left(\frac{R_{kt}}{R_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
=&\nu_j \kappa^{\frac{\sigma}{\sigma-1}} A_Y \left[\nu_j + (1-\nu_j)\left(\frac{R_{kt}}{R_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1}{\epsilon-1}}.
\end{aligned} \tag{A-25}$$

From equation (A-23), R^*_{jt}/R^*_{kt} becomes constant as t becomes large. Then from (A-25), R^*_{jt} approaches a constant. An analogous derivation establishes that R^*_{kt} approaches a constant. We have established the final result.

E.7 Proof of Proposition 4

Let time $w \geq t_0$ be the first time after t_0 at which sector j's share of extraction begins decreasing, so that $R_{jx}/R_{kx} \leq R_{j(x+1)}/R_{k(x+1)}$ for all $x \in [t_0, w-1]$ and $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$, which in turn requires $s_{jx} \geq \hat{s}_x$ for all $x \in [t_0+1, w]$ and $s_{j(w+1)} < \hat{s}_{w+1}$. Note that $s_{jt_0} > 0.5$ implies that $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$. Assume that sector j's share of research begins declining sometime after its share of extraction does, so that $s_{jx} \leq s_{j(x+1)}$ for all $x \in [t_0, w]$. Then we have $A_{jx}/A_{kx} > A_{j(x-1)}/A_{k(x-1)}$ for all $x \in [t_0, w+1]$, and thus $A_{jx}/A_{kx} > [\Psi_j/\Psi_k]^\theta$ for all $x \in [t_0, w+1]$. Using this with Lemma A-8 and $\sigma < 1$ then implies $\hat{s}_{x+1} \geq 0.5$ for all $x \in [t_0, w+2]$. Combining this with the requirement that $s_{jw} \geq \hat{s}_w$, we have $s_{jw} \geq 0.5$. From equation (10) and $\sigma < 1$, we then have $s_{j(w+1)} \geq s_{jw}$ only if $R_{jw}/R_{kw} \leq R_{j(w+1)}/R_{k(w+1)}$. But that contradicts the definition of w, which required $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$. Sector j's share of research must have begun declining no later than time w. We have shown that a transition in extraction occurs only after a transition in research.

We now have two possibilities. We will see that the first one implies that $s_{jx} \geq 0.5$ at all times $x \in [t+1, w]$ and the second one generates a contradiction.

First, we could have $A_{j(x-2)}/A_{k(x-2)} \geq [\Psi_j/\Psi_k]^\theta$ at all times $x \in [t_0+1, w]$. Then by Lemma A-8, $\hat{s}_x \geq 0.5$ at all times $x \in [t_0+1, w]$. The definition of time w then requires $s_{jx} \geq 0.5$ at all times $x \in [t_0+1, w]$.

Second, we could have $A_{j(x-2)}/A_{k(x-2)} < [\Psi_j/\Psi_k]^\theta$ at some time $x \in [t_0+1, w]$. In order for this to happen, it must be true that $s_{jx} < 0.5$ at some times $x \in [t_0+2, w]$.[48] Let z be the first time at which $s_{jx} < 0.5$. $A_{j(t_0-1)}/A_{k(t_0-1)} > [\Psi_j/\Psi_k]^\theta$ and $s_{jx} \geq 0.5$ for all $x \in [t_0, z-1]$ imply that $A_{j(z-2)}/A_{k(z-2)} > [\Psi_j/\Psi_k]^\theta$, which implies by Lemma A-8 and $\sigma < 1$ that $\hat{s}_z \geq 0.5$. So we have $s_{jz} < \hat{s}_z$, which means that $R_{j(z-1)}/R_{k(z-1)} > R_{jz}/R_{kz}$. But this contradicts the definition of time w as the first time at which sector j's share of extraction begins decreasing.

Therefore, we must have $A_{j(x-2)}/A_{k(x-2)} \geq [\Psi_j/\Psi_k]^\theta$ and $s_{jx} \geq 0.5$ at all times $x \in [t_0+1, w]$. Observe that $s_{jx} \geq 0.5$ at all times $x \in [t_0, w]$ implies $A_{jx}/A_{kx} \geq A_{j(x-1)}/A_{k(x-1)}$ at all times $x \in [t_0, w]$. We have shown that a transition in technology happens only after a transition in extraction. We have established the first part of the proposition.

Now consider the first time $z > t_0$ at which $R_{jz} < R_{kz}$. Assume that $\Psi_j \geq \Psi_k$ and that $s_{jx} \geq 0.5$ for $x \in [t_0, z]$. Assumption 1, $\Psi_j \geq \Psi_k$, and $s_{jx} \geq 0.5$ imply $A_{jx} \geq A_{kx}$ for $x \in [t_0, z]$. Using $\sigma < 1$, we see that $A_{j(z-1)} \geq A_{k(z-1)}$, $\Psi_j \geq \Psi_k$, and $R_{jz} < R_{kz}$ imply that the right-hand side of equation (A-1) is < 1 when evaluated at $s_{jz} = 0.5$. So by Corollary A-2, time z equilibrium scientists must be less than 0.5. But $s_{jz} < 0.5$ contradicts $s_{jx} \geq 0.5$ for $x \in [t_0, z]$. Therefore, if $\Psi_j \geq \Psi_k$, then there must be some time $x \in [t_0, z]$ at which $s_{jx} < 0.5$. We have shown that if $\Psi_j \geq \Psi_k$, then sector k must begin dominating research before it begins dominating extraction. We have established the second part of the proposition.

Finally, let $\nu_j = \nu_k$ and $\Psi_j = \Psi_k$. By Proposition 3, $A_{jt} = A_{kt}$ in the steady-state research allocation. But Assumption 1 ensures that $A_{jt_0} > A_{kt_0}$. Thus there exists $t_1 > t_0$ such that $s_{jt_1} < 0.5$. By the foregoing parts of this proposition, a transition in research, a transition in extraction, and a transition in technology must happen between t_0 and t_1. We have established the third part of the proposition.

E.8 Intermediate steps for Leontief special case

From equation (8),

$$p_{jXt} X_{jt} = X_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}.$$

And from equation (2),

$$p_{jRt} R_{jt} = \Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}}.$$

[48]Recall that $s_{jt} \geq 0.5$ and $s_{j(t+1)} \geq s_{jt}$ imply $s_{j(t+1)} \geq 0.5$.

Intermediate good producers' zero-profit condition is

$$p_{jt}Y_{jt} = \Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}} + X_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}.$$

Substituting for p_{jt} from the final good producers' first-order condition and then setting $X_{jt} = R_{jt}$ and $Y_{jt} = R_{jt}$, we have:

$$\nu_j Y_t^{1/\epsilon} = A_Y^{\frac{1-\epsilon}{\epsilon}} R_{jt}^{\frac{1-\epsilon}{\epsilon}} \left[\Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}} + R_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}} \right].$$

Using $\psi = \alpha/(1-\alpha)$, we have:

$$\nu_j Y_t^{1/\epsilon} = A_Y^{\frac{1-\epsilon}{\epsilon}} R_{jt}^{\frac{1-\epsilon}{\epsilon}+\frac{1}{\alpha}} \left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}} \right].$$

An analogous result holds for sector k. Equation (18) follows.

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