Competing Sellers in Security-Bid Auctions under Risk-Averse Bidders *

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Abstract

We analyze security-bid auctions in which two risk-neutral sellers compete for a group of risk-averse bidders. Sellers face a trade-off in the steepness of the security design because steeper families (i) extract more surplus ex-post but attract lower participation ex-ante, and (ii) provide bidders with a higher insurance, making them bid more aggressively. We show that when bidders are homogeneously risk-averse, all equilibria are symmetric. Meanwhile, when they are heterogeneously risk-averse, there is always an equilibrium in which one seller chooses a steeper family to serve the more-risk-averse bidders, while the other chooses a flatter family to serve the less-risk-averse bidders.

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1 Introduction

In many economic and financial markets, the rights to implement and control projects (assets) that generate future cash flows are sold through auctions. Examples are ubiquitous, including highway-building auctions, corporate takeovers, venture-capital financing, governments' sales of oil, gas, timber, and spectrum leases, and individuals' sales of publishing rights. Auctions of this type share three fundamental characteristics. First, the future revenue of the auctioned project can be verified ex-post, and, thus, can be used to determine the winner's payment to the seller. In other words, the project's revenue can be used to *securitize* bids. Second, the allocation of the project requires a long-term contractual commitment from the winning bidder, which involves significant uncertainty regarding future cash flows. And third, multiple sellers compete for a limited number of bidders due to the large-scale nature of the projects auctioned and the high costs associated with due diligence.

The complexity of these auctions demands several strategic decisions from bidding firms because they need to decide whether or not to enter a given auction, how much to bid upon their entry, and when to implement the project in case they win the auction. Additionally, the revenue generated by the project typically depends on uncertain economic interactions in a downstream market. For example, in oil-lease auctions, bidders face the uncertainty entailed by the fluctuation of oil prices during the term of the lease; and in highway-building auctions, construction firms face the uncertainty of potential cost overruns in several of the contracted tasks.

All of these elements make firms' participation in such auctions potentially risky investments. Hence, it is crucial to incorporate firms' risk attitudes to analyze their strategic decisions in the different stages of the induced game. Furthermore, the presence of risk aversion makes sellers' competition through families of securities more relevant, as sellers need to consider both the ex-post extraction of each family and the inherent interim uncertainty introduced to bidders.

In this paper, we analyze a model in which two *risk-neutral* sellers compete for (possibly heterogeneously) *risk-averse* bidders to allocate two ex-ante identical projects. Sellers choose a family of securities and commit to run a *second-price* auction under the selected family. We assume auctions are run simultaneously, and, thus, bidders decide which auctions to enter (possibly randomly) given their risk aversion and the sellers' choice of design. Then, given the endogenous number of competitors in each auction, bidders choose a security bid once they learn their own signals about the future revenue of the project. Our focus on a second-price auction simplifies the equilibrium analysis in the sequential game because the bidding game of the second stage is dominance solvable, enabling us to concentrate on studying the effects that drive bidders' entry decisions in a more transparent way.[1]

When bidders are risk-averse, one crucial aspect that sellers need to consider is the effect that the *steepness* of the chosen family has on bidders' interim payoffs.[2] On the one hand, steeper securities extract greater surplus because they are more sensitive to bidders' true types, and, thus, are less attractive to bidders ex-ante DeMarzo et al. (2005). On the other hand, they provide bidders with greater insurance because they allow them to smooth payoffs across realizations, asking for lower payments when the marginal disutility of money is high and for higher payments when the marginal disutility is low. Such insurance is relatively more valuable for more-risk-averse bidders, making them bid relatively more aggressively (Fioriti and Hernandez-Chanto, 2022). Therefore, the implications of *extraction* and *insurance* effects for bidders' entry and bidding decisions depend on bidders' risk aversion.

Sellers also face a trade-off in the steepness of the chosen family, but in the opposite direction. Under risk-neutral bidders, if a seller chooses a steeper family than the one selected by his opponent, he will attract fewer bidders to the auction but will extract greater surplus ex-post (Gorbenko and Malenko, 2011). When bidders are risk-averse, there are other effects to consider. On the one hand, the insurance provided by steeper securities mitigates the negative impact of the extraction effect on entry; on the other hand, the bidders more prone to enter would be the most risk-averse, which are also the less competitive, since they are the ones that shave their bids by more relative to what they would have bid had they been risk-neutral. Therefore, when selecting a security design, sellers need to assess the impact that a given family of securities has on (i) the number and composition of bidders in the auction; (ii) the aggressiveness of bidders; and (iii) the surplus extraction and insurance provided. These effects are multi-causal and take place at different stages of the game: the first one takes place at the

[1] Although the post-entry game is dominance solvable, the presence of risk aversion plays a fundamental role for the competition of the project because its revenue is given by a lottery that depends on a "price" determined by the losing bidder.

[2] Steepness refers to a partial order introduced by DeMarzo et al. (2005) to compare different securities. A security is steeper than another if the expected payment that it yields to the seller grows more rapidly with the winner's signal—i.e, with a steeper slope—starting at a signal level at which the seller's expected revenue is the same under both securities.

entry stage, the second one at the bidding stage, and the third one during the implementation of the project.

Our contribution is to characterize bidders' and sellers' equilibrium behavior in all stages of the game, providing sufficient conditions for sellers to separate (concentrate) in the steepness spectrum to take advantage of bidders' incentives. That is, we provide conditions for when it is optimal for bidders to soften competition by using "product differentiation," linking fundamental results in the industrial organization literature to the competition for financial assets by risk-averse bidders.

The (a)symmetry of risk aversion among bidders plays a fundamental role in the equilibrium characterization. When bidders are homogeneously risk-averse, we show that, in every equilibrium, bidders' entry strategies are totally mixed and seller's strategies are symmetric, *despite the fact that we allow for asymmetric strategies*. Furthermore, we show that any family can be chosen in equilibrium. Nonetheless, because the presence of risk aversion reinforces sellers' preferences for choosing steeper families of securities—as the higher insurance provided mitigates the negative effect on bidders' entry—we need to impose additional conditions on the ones considered by Gorbenko and Malenko (2011) to obtain an equilibrium under families flatter than a call-option.

In turn, when bidders are heterogeneously risk-averse, we can prove a result that is novel to the auction literature. We show the existence of different equilibria in which sellers separate in the spectrum of steepness to serve different populations of risk-averse bidders. Specifically, one seller chooses a flatter family to serve the less-risk-averse bidders, while the other chooses a steeper family to serve the more-risk-averse bidders. We argue that this equilibrium resembles a *Hotelling-location equilibrium* in the order induced by steepness. In addition, we note that when bidders are sufficiently heterogeneous in their risk-aversion, sellers do not tend to concentrate in the middle of their steepness spectrum, in the spirit of the classical paper by d'Aspremont et al. (1979). In fact, one seller always chooses the steepest family allowed by the strategy that his opponent plays.

The distinctive features of security-bid auctions make sellers' optimal strategies more complicated vis-à-vis the strategies in standard second-price auctions under risk-neutral bidders. In the latter, sellers compete by choosing different reserve prices, and bidders care only about the expected probability of winning (McAfee, 1993; Peters, 1997; Burguet and Sákovics, 1999;

Hernando-Veciana, 2005; Damianov, 2012). In contrast, in our environment sellers have to choose their security design from a complex space and bidders face competition through different channels. Furthermore, following Gorbenko and Malenko (2011), we show that reserve prices are less useful in this realm because whenever they are binding, sellers can always optimally adjust their security design to a steeper family with no reserves, provided that they are choosing a family flatter than a call option.

Firms' risk aversion One key element that drives our results is the presence and heterogeneity of risk aversion among bidding firms, which is normally neglected in auction theory. Hence, it is worth discussing the factors that determine firms' risk aversion. One strand of the finance literature relates firms' risk aversion to their willingness and ability to engage in corporate risk management. Firms might engage in risk management to overcome credit rationing (Froot et al., 1993; Holmström and Tirole, 2000), hedge against volatile cash flows (Smith and Stulz, 1985), and avoid convex structures in taxation (Graham and Smith, 1999). Managers can also use hedging as a vehicle to signal their private ability to the market (DeMarzo and Duffie, 1995; Breeden and Viswanathan, 2015).

Another strand of the literature focuses on analyzing how firms' risk aversion is associated to their executives' risk aversion. Here, there could be a potential misalignment of incentives if managerial claims to the firm are not easily diversifiable, leading risk-averse managers to reduce their exposure by hedging, even when this decision is not optimal from the perspective of well-diversified shareholders. Moreover, Bodnar et al. (2019) show empirically that the relationship between managers' risk aversion and corporate hedging strategies is more prominent among young, highly-educated, and short-tenure-job executives. In turn, Schosser (2019) shows that when executives have uncertain tenure horizons, they tend to hedge by making investments that yield returns in the short run but that are frequently misaligned with the firm's long-term objectives.

Therefore, given the diversity of channels that can drive firms' risk aversion, it is important to analyze how risk aversion affects firms' behavior in the bidding of large-scale projects, which could be integrated in a broader analysis of firms' investment portfolios.

1.1 Examples of auctions with competing sellers

To illustrate the interplay among all described channels when multiple sellers compete for risk-averse bidders, we elaborate on the details of three examples.

Oil-lease auctions Governments frequently use security auctions to sell the rights to exploit their oil fields, competing with each other for a limited number of drilling firms. The predominant mechanism used is a *royalty plus bonus* auction—also known as a fixed-equity auction. In this mechanism, bidding firms compete in cash; the lease is allocated to the highest bidder; and the final payment to the government corresponds to an upfront cash payment—i.e., a bonus determined by the auction format—plus a fixed equity of future revenues. Once a firm is allocated a lease, its decision to invest and drill depends on many factors, including how variable costs and prices evolve. It also depends on the upfront cash paid in the auction, which becomes a *sunk cost* in the implementation stage. Hence, in an environment with high volatility of oil prices, higher cash payments leave the firm more exposed to unfavorable shocks. Consequently, if a firm is sufficiently risk-averse, it may prefer to join an auction with a higher equity, even though the seller will extract greater surplus ex-post.[3]

Highway- and bridge-building auctions Another context that features competition in auctions with risk-averse bidders corresponds to procurement auctions to build highways. In the US, departments of transportation use either standard cash auctions or *scaling auctions* to procure the provision of these goods (Bolotnyy and Vasserman, 2023). In the latter, bidders submit unit-price bids for each task specified in a contract to complete a given project.[4] The winning bid corresponds to the *lowest* sum of each unit bid multiplied by the corresponding quantity estimated by the designers of the project. After the project is completed, the winner is paid the unit bid times the *effective* quantity used. The design of the auction incentivizes bidders to *skew* their bids according to their beliefs: bidding higher prices for tasks that they believe will overrun the designer's estimates, and bidding lower prices for tasks that they believe will underrun the estimates. Hence, scaling auctions give bidders the flexibility to adjust costs

[3]Although oil firms are large, they are not as well diversified as similar firms in other industries. Furthermore, oil-extraction projects are large and complex, and, thus, have a considerable effect in the portfolio position of the firms that undertake them.

[4]For instance, they might submit a bid for the structural design, another bid for the pavement settlement, and so on.

across tasks, providing them more *insurance* and inducing them to bid more competitively. In contrast, when the designer uses a standard auction with cash upfront payments, bidders are exposed to all shocks in unexpected costs. This induces risk-averse bidders to charge higher prices because the project is less attractive.

The pool of bidders that compete for these contracts is limited and normally stable. This is so because there are increasing returns to scale of having contracts in the same geographical area due to the costs of mobilizing machinery and the need to save on seismic studies. The incentive of bidders to establish in certain areas induces fierce competition among local governments to attract those companies to contract their services.

Publishing-rights auctions The previous two examples involve governments allocating large projects among few firms. However, security-bid auctions are also used in contexts in which sellers are individuals and the pool of biding firms is large and heterogeneous. An example of this are auctions used to sell the rights for publishing a book (Skrzypacz, 2013). This happens when an agent representing an author with celebrity status organize an auction among publishing houses to sell the rights of a book that stands out for its marketability or newsworthiness. Different agents manage large lists of clients, and, thus, it is not uncommon to observe agents competing to entice publishing houses (with heterogeneous sizes and diverse project portfolios) to enter their auction. To soften the competition and make their auction distinctive, agents differ in their choice of security designs: some of them choose royalties (i.e., equity), while others choose flatter securities like debt, and some others prefer upfront payments in cash.

1.2 Related literature

The steepness of the security design plays a dual role when bidders are risk averse: on the one hand, it allows seller to extract higher surplus, and, on the other hand, it provides bidders with higher insurance. The extraction role of steepness has been well studied in the literature. For instance, DKS show that, under identical opportunity costs, the use of *steeper securities* increases sellers' expected revenue because they are tied more tightly to the winner's true type. This finding generalizes the early results by Hansen (1985), Riley (1988), and Rhodes-Kropf and Viswanathan (2000), who were the first to discover this relationship by comparing cash and equity auctions. Other studies have analyzed the relationship between the steepness of the security design used and the seller's expected revenue in alternate frameworks, these include: Che and Kim (2010) and Liu and Bernhardt (2019) under adverse selection; Kogan and Morgan (2010) under moral hazard; Sogo et al. (2016) when entry is endogenous and costly; and Hernandez-Chanto and Fioriti (2019) under negative externalities.

The study of security-bid auctions with risk-averse bidders is more recent and less abundant. Abhishek et al. (2015) show that when bidders are homogeneously risk-averse and signals are ranked according to the first-stochastic-order, the notion of steepness is not sufficient to rank securities in revenue, and, instead, a notion called "strong steepness" is needed. Meanwhile, Fioriti and Hernandez-Chanto (2022) show that the insurance provided by steeper securities *levels the field* for more-risk-averse agents, inducing them to bid more aggressively and, thus, increasing their competitiveness in the auction they participate.

Regardless of bidders' risk attitudes, most of the literature has focused on the analysis of monopolistic auctions; yet the most natural applications occur under competition. To account for this possibility, Gorbenko and Malenko (2011) extended the framework in DeMarzo et al. (2005) to include the possibility of having multiple sellers competing for a group of risk-averse buyers.

We add to this literature by considering *both* competing sellers and risk-averse bidders. In this context, sellers compete with each other by implicitly inducing different combinations of insurance and surplus extraction, and bidders endogenously enter to each auction based on their differentiated values of the insurance provided and the extraction exercised.

This paper differs from Gorbenko and Malenko (2011) in a number of ways. First, we consider *two* risk-neutral sellers that compete for a set of *risk-averse* bidders, whereas they study the case of multiple risk-neutral sellers that compete for a set of risk-neutral bidders. Hence, in general, our setting is not contained, neither contains, their framework; except when the number of sellers is restricted to two, in which case, their framework is a particular case of our setting. Second, we allow for the possibility of having asymmetric equilibria, instead of exclusively focusing on symmetric equilibria. Third, we consider two-dimensional types that are sequentially learned by bidders, which are fundamental for the existence of asymmetric strategies in the entry stage. And fourth, we add the insurance and aggressiveness effects to the competition and extraction effects present in their model.

Methodologically, we prove the uniqueness of symmetric equilibria when there are only two sellers by means of an increasing-differences argument, instead of comparing the marginal value of attracting an additional bidder vis-à-vis the marginal cost of having lower ex post extraction, as in Gorbenko and Malenko (2011).

It is worth highlighting that Gorbenko and Malenko (2011) do not dismiss the existence of asymmetric equilibria. Nonetheless, they are agnostic about the type of equilibria that could emerge when sellers choose different security designs. We instead *prove* the uniqueness of a symmetric equilibria with two sellers, which can be seen as a strong robust check for the symmetry of equilibria when competition is duopolistic. The restriction to two sellers is important because with more than two sellers the increasing-difference condition between bidders' entry probability and the steepness of the security design is not sufficient to rule out the existence of asymmetric equilibria, and, thus, more research is needed to characterize such potential equilibria.

Contemporaneously to our paper, Breig et al. (2023) experimentally implement the competition of sellers in second-price, security-bid auctions when sellers can choose their security designs between debt and equity, and buyers select auctions based on sellers' choices. The authors focus on the comparison between monopolistic and competitive auctions. They find that an auction's security design has limited influence on revenue in monopolistic auctions, whereas equity substantially increases revenue in the competitive auctions. This is mainly due to equity's effectiveness in attracting more bidders, given the preference of bidders for a linear payment structure. Despite this fact, sellers' rate of choosing equity does not differ between the monopolistic and competitive treatments.

More broadly, our framework constitutes a general approach to the study of risk-averse bidders in security-bid auctions, and, thus, complements the analysis conducted under monopolistic cash auctions in previous papers, such as, Matthews (1987). In this case, we show that risk aversion has first-order effects as it induces sellers to specialize in their security designs. This result contributes, thus, to the literature that studies the allocation of indivisible objects under competing designers, including, Biais et al. (2000), Ellison et al. (2004), and Epstein and Peters (1999).

Organization of the paper The rest of the paper is organized as follows. Section 2 lays out the model. Section 3 determines sellers' and bidders' optimal strategies. Section 4 discusses the different channels of competition in the auction. Section 5 characterizes the equilibrium when bidders are homogeneously risk-averse, while Section 6 does the same when bidders are heterogeneously risk-averse. Section 7 illustrates the main results of the paper through a numerical simulation. Section 8 discusses several different extensions of our model, including the presence of reserve prices, and proposes future research avenues. Section 9 concludes. All proofs are relegated to the Appendix.

2 The model

Two *risk-neutral* sellers seek to allocate two *ex-ante identical* and indivisible projects. Each seller uses an auction to allocate exactly one project.[5] Any winning bidder $i \in \{1, \ldots, I\}$ must incur a cost $\kappa > 0$ for the acquired project to generate a stochastic revenue Z_i. The implementation cost κ is commonly known by all agents.

After selecting an auction, bidders receive a private signal $V \in \mathcal{V} = [v_L, v_H]$ of the stochastic revenue Z.[6] Bidders learn their signals after entering the auction since due diligence is costly. For any signal realization $V = v$, the revenue Z has a conditional distribution $Q(z|v)$ with a positive everywhere and continuous density $q(Z|v)$ on $\mathcal{Z} = [0, \infty)$. We assume that densities can be ordered by the strict Monotone-Likelihood Ratio (sMLR). Thus, the ratio $q(z|v)/q(z|v')$ is strictly increasing in z for all $v > v'$.[7]

All bidders are expected-utility maximizers with Bernoulli utility functions $u(\cdot, r)$ parameterized by the risk-aversion level $r \in \mathcal{R} = [r_N, r_H]$ with $r_H < \infty$. Here, the lower bound r_N denotes risk neutrality. We assume that, for all $r > r_N$, $u(\cdot, r)$ is strictly increasing and strictly concave, and it satisfies $u(0, r) = 0$. We say that for any two bidders $i \neq j$, bidder i is more risk-averse than bidder j if $r_i > r_j$. In particular, for any $r_i > r_j$, there exists a strictly concave transformation $\varphi(r_i, r_j)$ such that $u(\cdot, r_i) = \varphi(r_i, r_j) \circ u(\cdot, r_j)$. If $r_i = r_j$ bidders are equally risk-averse and $\varphi(r_i, r_j)$ becomes the identity function. A bidder's type is represented by the

[5] For the sake of simplicity, we will refer to the seller as "he" and the bidder as "she."

[6] The distribution of signals is independent of the auction selected by bidders.

[7] Furthermore, we assume that (i) the conditional density function $q(z|v)$ is twice differentiable in z and v; and (ii) the functions $zq(z|v)$, $|zq_v(z|v)|$, and $|q_{vv}(z|v)|$ are integrable on $z \in (0, \infty)$. These assumptions guarantee that the expectations and derivatives discussed in the paper are well-defined.

pair $\theta = (v, r) \in \mathcal{V} \times \mathcal{R} \triangleq \Theta$. Each bidder's type is drawn *independently* from a full-support distribution $H = (F \times G)$, which is assumed to be absolutely continuous, bounded, and atomless. All distributions are common knowledge.

We assume that both sellers use a second-price security-bid auction as a selling mechanism. Hence, bids correspond to securities that are tied to the project's ex-post revenue Z. We consider a class $\mathscr{S}$ of indexed *families of securities* $\mathcal{S} \triangleq \{S(\cdot, s) : s \in [s_L, s_H]\}$, where $S : \mathcal{Z} \times [s_L, s_H] \to \mathbb{R}_+$ maps the revenue of the project Z and the index s to the payment received by the seller.[8]

Assumption 1. *For all* $s \in [s_L, s_H]$,

i) $S(z, s)$ *and* $z - S(z, s)$ *are continuous and increasing in* z.

ii) $0 \leqslant S(z, s) \leqslant z$ *for all* z.

Assumption 1 states that the expected payoffs obtained by sellers and bidders are increasing in the revenue of the project for all security bids. Moreover, bidders cannot promise to pay more than the revenue of the project, and the seller cannot finance its implementation. Thus, sellers and bidders face *double limited liability*, since they can share only the resources generated by the project.[9]

Assumption 2. *For any bidder with type* $\theta = (v, r)$,

i) $EU_{\mathcal{S}}(\theta, s) \triangleq \mathbb{E}[u(Z - \kappa - S(Z, s), r)|v]$ *is continuous and strictly decreasing in* s, *and non-positive for* $s = s_H$.

ii) $E\mathcal{S}(\theta, s) \triangleq \mathbb{E}[S(Z, s)|v]$ *is continuous and strictly increasing in* s.

Assumption 2 states that each family of securities is completely ordered from the perspective of bidders and sellers. Securities that satisfy both assumptions are called *feasible securities*. Our definition is comprehensive and encompasses standard types of securities, which include:

- Equity: In this case, the seller receives a fraction $s \in [0, 1]$ of the project's revenue. Thus, the seller's payment is given by $S(z, s) = sz$.

[8] As an illustration, the family $\mathcal{S}$ could refer to the family of, say, equity functions that are parameterized by the indices $s \in [0, 1]$. For other securities (e.g., debt and call options) it is always possible to find a bijective mapping between $[s_L, s_H]$ and the relevant index space (e.g., $\mathbb{R}^+$).

[9] This last assumption is crucial to rule out the critique of Crémer (1987), who shows that if the seller could finance the project's implementation cost, he would be able to extract the whole surplus.

- Debt: With debt securities, the seller receives a fixed amount $s \in \mathbb{R}$ if the project's revenue surpasses such threshold. Otherwise, the seller retains the entire revenue. Hence, the payment is given by $S(z, s) = \min\{z, s\}$.

- call options: In the case of call options, the seller has the right to "call back" the project when its revenue exceeds a specified strike price $s \in \mathbb{R}$. If the call option is exercised, the seller must pay the strike price to the bidder and retains the remaining revenue. However, if the project's revenue is below the strike price, the seller does not call back the project and obtains a payoff of zero. Thus, the payment is given by $S(z, s) = \max\{0, z - s\}$.

  These examples illustrate the different types of securities that fall within the scope of our definition.

Timing Figure 1 depicts the timeline of the game. First, each seller chooses a feasible family of securities $\mathcal{S} \in \mathscr{S}$ to run his second-price auction. Then, bidders observe the sellers' choice, learn their risk aversion—but not their signal—and make auction-entry decisions. Once bidders enter an auction, they learn their signal about the future revenue of the project, observe the number of competitors in their chosen auction, and submit their bids. Then, a winner is determined in each auction. Finally, projects' revenues are realized and payments are made.

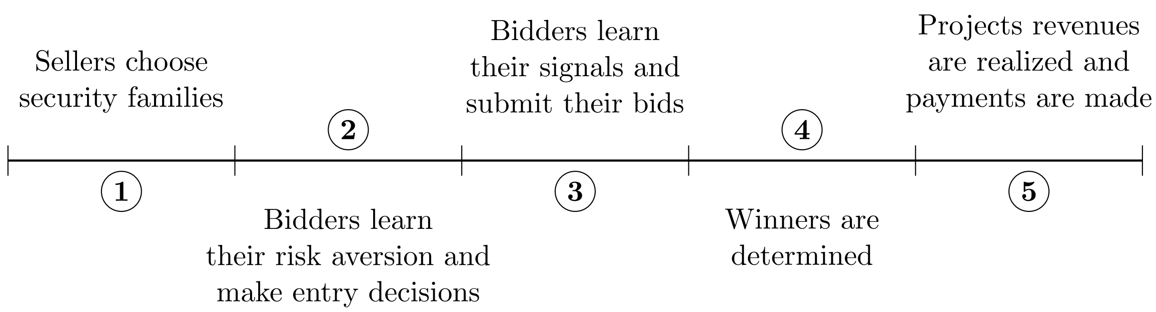


Figure 1: Timing of the auction.

2.1 Steepness

Following DeMarzo et al. (2005), we rank securities using the notion of steepness. For that purpose, let $E\mathcal{S}_1(v, s)$ be the partial derivative of $E\mathcal{S}(v, s)$ with respect to the bidder's signal v.

Definition 1 (DeMarzo et al. (2005)). *The security $S'(\cdot, s')$ is steeper than the security $S''(\cdot, s'')$, if $E\mathcal{S}'(v, s') = E\mathcal{S}''(v, s'')$ implies that $E\mathcal{S}'_1(v, s') > E\mathcal{S}''_1(v, s'')$.*[10]

That is, a security is steeper than another if the seller's expected payment has a "greater slope" under the steeper security, starting at the signal level at which both securities yield the seller the same expected payment. If all securities $S'(\cdot, s')$ in family $\mathcal{S}'$ are steeper than all securities $S''(\cdot, s'')$ in family $\mathcal{S}''$, we say that family $\mathcal{S}'$ is steeper than $\mathcal{S}''$ and write, occasionally, $\mathcal{S}' \succ \mathcal{S}''$. Intuitively, steeper securities compensate the lower payments under low realizations with higher payments under high realizations, providing risk-averse bidders with a better way to smooth payments.

Lemma 1 (DeMarzo et al. (2005)). *We can say that $S'(\cdot, s')$ is steeper than $S''(\cdot, s'')$ if for any $s' \in [s_L, s_H]$ and any $s'' \in [s_L, s_H]$, the function $S'(s', z) - S''(s'', z)$ is quasi-monotone in z—that is, if there exists a point z^* such that $S'(z, s') \leqslant S''(z, s'')$ for all $z \leqslant z^*$ and $S'(z, s') > S''(z, s'')$ for all $z > z^*$.*

Notably, for the vast majority of securities—in particular for standard securities—steepness implies the existence of a single crossing point, and, thus, this relationship is a two-way street (see DeMarzo et al., 2005).

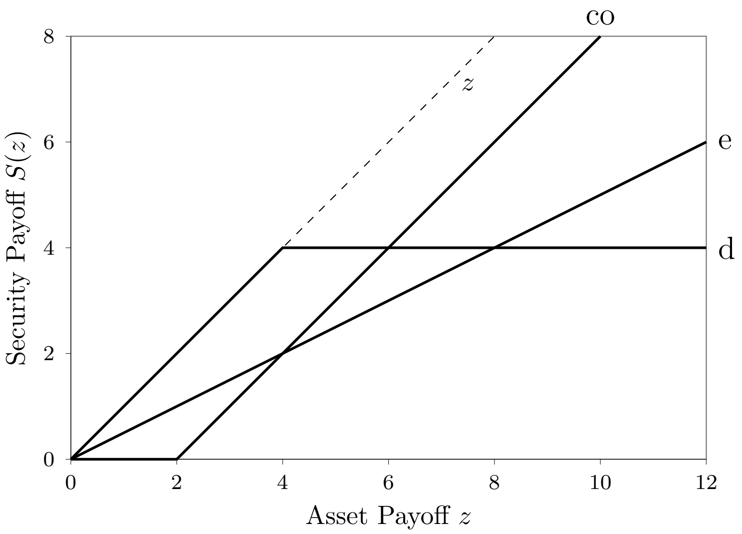


Figure 2: Single-crossing property across standard securities. The figure depicts the payment structure of debt (d), equity (e), and a call option (co).

[10]For any family $\mathcal{S}$ and any bid s, the partial derivative $E\mathcal{S}_1(v, s)$ is well-defined by virtue of the assumptions on the conditional density $q(z|v)$ and the application of the Dominated Convergence Theorem.

Steepness induces a partial order to compare families of securities. In particular, a call option is steeper than equity, which is steeper than debt. Moreover, debt and call option are, respectively, the *flattest and steepest* feasible securities. As we will show later, the notion of steepness drives many of the fundamental results in the paper.

3 Optimal strategies

In this section, we determine the optimal strategies of bidders and sellers in the sequential game of incomplete information induced by the environment of Section 2.

3.1 Bidding strategies

Consider a bidder who enters an $\mathcal{S}$-auction in which there are already $k-1 \geqslant 0$ bidders. Given that there will be k active bidders in the auction after she joins, her bidding strategy corresponds to a mapping from her type space into the relevant family of securities, conditional on the number of participants k; that is, $\sigma_{\mathcal{S}}(\cdot|k) : \Theta \to \mathcal{S}$.

Proposition 1. *[Fioriti and Hernandez-Chanto (2022)] The unique symmetric Bayes-Nash equilibrium in the second-price auction, when there are k active bidders, is for each bidder with type $\theta = (v, r)$ to submit a security-bid $\sigma_{\mathcal{S}}(\theta|k)$ such that*

$$EU_{\mathcal{S}}(\theta, \sigma_{\mathcal{S}}(\theta|k)) = \mathbb{E}[u(Z - \kappa - S(Z, \sigma_{\mathcal{S}}(\theta|k)), r)|v] = 0 \tag{1}$$

when $EU_{\mathcal{S}}(\theta, s_L) \geqslant 0$, and to submit a bid $\sigma_{\mathcal{S}}(\theta|k) = s_L$ otherwise.

Given a seller's family choice, the optimal strategy in (1) corresponds to a bid that sets to zero the certainty equivalent of the lottery induced by the bidder's type when the expected utility is non-negative at the lowest security index of the corresponding family. As a convention, we assume that if the bidder obtains a negative utility when she gets the project at the lowest security, she bids the lowest possible security. Nonetheless, the latter case would be off the equilibrium path, as we will see later.

Lemma 2. *[Fioriti and Hernandez-Chanto (2022)] For any family $\mathcal{S}$ and any $1 \leqslant k \leqslant I$, the equilibrium bid $\sigma_{\mathcal{S}}(\theta|k)$ determined in (1) is continuous, strictly increasing in the signal v, and strictly decreasing in r.*

Lemma 2 allows us to characterize equilibrium bids in a second-price auction by a family of indifference curves in the space (v, r). As such, it is possible to obtain the marginal rate of substitution between the signal and the risk-aversion level such that a bidder submits the same equilibrium bid. The continuity of bids in risk-aversion is crucial to determine the existence of mixed strategies when bidders are homogeneously risk-averse and cutoff strategies when bidders are heterogeneously risk-averse.

Because the selling mechanism is a second-price auction, the optimal strategy is independent of the number of participants. Hence, in what follows, we suppress the dependence of strategies on the number of bidders when there is no risk of confusion for the reader.

3.2 Entry strategies

We assume that bidders are ex-ante identical and focus on a symmetric equilibrium in the entry stage. Thus, when sellers' family choices are $\mathcal{S}'$ and $\mathcal{S}''$, bidders' entry strategy is given by the mapping $\gamma(\cdot|\mathcal{S}', \mathcal{S}'') : \mathcal{R} \to \Delta(\{\mathcal{S}', \mathcal{S}''\})$. Write $\gamma'(r|\mathcal{S}', \mathcal{S}'')$ as the probability that a bidder with risk aversion r joins the $\mathcal{S}'$-auction and $\gamma''(r|\mathcal{S}', \mathcal{S}'')$ as the probability that she joins the $\mathcal{S}''$-auction.

Definition 2. *For any two families $\mathcal{S}'$ and $\mathcal{S}''$, the entry strategy $\gamma(\cdot|\mathcal{S}', \mathcal{S}'')$ satisfies individual rationality (IR) if $\gamma'(r|\mathcal{S}', \mathcal{S}'') = 0$ whenever*

$$\mathbb{E}_V[EU_{\mathcal{S}'}(\theta, s_L)] = \int_{\mathcal{V}} EU_{\mathcal{S}'}((v, r), s_L) dF(v) < 0, \tag{IR}$$

and likewise for $\gamma''(r|\mathcal{S}', \mathcal{S}'')$.

That is, the (IR) condition requires that, for any sellers' family choices, bidders will never join an auction in which they would obtain a negative expected utility when paying the lowest possible security-bid within the corresponding family.

We can define the set of risk-averse types that can become active in an auction by means of the correspondence $R^+ : \mathscr{S} \rightrightarrows \mathcal{R}$, where $R^+(\mathcal{S}) = \{r \in \mathcal{R} : \mathbb{E}_V[EU_{\mathcal{S}}((v, r), s_L)] \geqslant 0\}$.

Assumption 3. *For all $\mathcal{S} \in \mathscr{S}$, $E\mathcal{S}(\theta, s_L) = 0$ and the seller retains the project when all bids are equal to s_L.*

Assumption 3 is satisfied by all standard securities (e.g., debt, equity, and call options). As such, this assumption implies that $R^+(\mathcal{S})$ is the same for any family $\mathcal{S} \in \mathscr{S}$. Hence, let $\mathcal{R}^+ \triangleq R^+(\mathcal{S})$ and relabel $r_H = \sup \mathcal{R}^+$ and $\Theta^+ \triangleq \mathcal{V} \times \mathcal{R}^+$.[11]

Using the definition of conditional entry probabilities, we can compute the unconditional probabilities as follows:

$$\gamma'(\mathcal{S}', \mathcal{S}'') = \int_{\mathcal{R}^+} \gamma'(r|\mathcal{S}', \mathcal{S}'')dG(r) \tag{2}$$
$$\gamma''(\mathcal{S}', \mathcal{S}'') = \int_{\mathcal{R}^+} \gamma''(r|\mathcal{S}', \mathcal{S}'')dG(r) = 1 - \gamma'(\mathcal{S}', \mathcal{S}'').$$

Then, the expected number of bidders that would join each auction in equilibrium is given by $I'(\mathcal{S}', \mathcal{S}'') = I\gamma'(\mathcal{S}', \mathcal{S}'')$ and $I''(\mathcal{S}', \mathcal{S}'') = I\gamma''(\mathcal{S}', \mathcal{S}'') = I - I'(\mathcal{S}', \mathcal{S}'')$.

Losing types Denote $\boldsymbol{\theta}_k = (\theta_1, \ldots, \theta_k) \in \Theta^k$ as a profile of types with length k. For any type θ, the set of types that lose against it in the $\mathcal{S}$-auction is denoted as

$$\mathcal{L}^k_{\mathcal{S}}(\theta) \triangleq \{\boldsymbol{\theta}_k \in {\Theta^+}^k | \sigma_{\mathcal{S}}(\theta_j) < \sigma_{\mathcal{S}}(\theta) : \theta_j \in \boldsymbol{\theta}_k\}.$$

Furthermore, for a given profile $\boldsymbol{\theta}_k \in \mathcal{L}^k_{\mathcal{S}}(\theta)$ define

$$\hat{\sigma}_{\mathcal{S}}(\boldsymbol{\theta}_k) \triangleq \max\{\sigma_{\mathcal{S}}(\theta_j) : \theta_j \in \boldsymbol{\theta}_k\}$$

as the maximum bid among the k-losing types against type θ.

We follow the convention used in the mechanism design literature and decompose a given profile of types $\boldsymbol{\theta}_k$, containing θ_i as $(\theta_i, \boldsymbol{\theta}_{k-1,-i})$, for any $k \leqslant I$ and any $i \in \mathcal{I}$. Furthermore, we write the distribution of types in Θ^k as $H_k(\boldsymbol{\theta}_k) = \prod_{j=1}^k F(v_j)G(r_j)$. By the vector decomposition introduced above, we can also write $H_k(\boldsymbol{\theta}_k) = F_k(v_i, \boldsymbol{v}_{k-1,-i})G_k(r_i, \boldsymbol{r}_{k-1,-i}) = \prod_{j=1}^k F(v_j)G(r_j)$.

The interim expected utility that bidder i with risk aversion r_i obtains by joining the $\mathcal{S}'$-

[11] Equation (1) and Assumption 3 imply that, for any family $\mathcal{S}$, a bidder who submits the lowest security index s_L never wins the auction.

auction, conditional on facing k opponents, is given by

$$\mathcal{EU}(\mathcal{S}'|r_i,k) = \begin{cases} \int_{\mathcal{V}} EU_{\mathcal{S}'}(\theta_i, s_L)dF(v_i) & \text{if } k=0 \\ \int_{\mathcal{V}} \int_{\mathcal{L}^k_{\mathcal{S}'}(\theta_i)} EU_{\mathcal{S}'}(\theta_i, \hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_{k,-i}))dH_k(\boldsymbol{\theta}_{k,-i})dF(v_i) & \text{if } 1 \leqslant k \leqslant I-1. \end{cases} \tag{3}$$

The interim expected utility can be separated in two cases. The first case corresponds to the utility that bidder i obtains when she faces zero opponents—i.e., when she is the only participant in the $\mathcal{S}'$-auction. In such a case, bidder i pays the lowest feasible security s_L, and the utility of implementing the project becomes $EU_{\mathcal{S}'}(\theta_i, s_L)$ for any realization of the signal v_i. In the second case, bidder i faces $k \geqslant 1$ opponents, and, thus, the auction features competition. Here, bidder i's expected utility is obtained by integrating the profile of types $\boldsymbol{\theta}_{k,-i} = (\boldsymbol{v}_{k,-i}, \boldsymbol{r}_{k,-i})$ over the region $\mathcal{L}^k_{\mathcal{S}'}(\theta_i)$ in which the types of her opponents lose against her type $\theta_i = (v_i, r_i)$. Now, because bidder i only knows her risk-aversion level r_i before joining the auction, the region $\mathcal{L}^k_{\mathcal{S}'}(\theta_i)$ depends on the draw of the signal v_i. By independence, the probability of obtaining the profile $(\theta_i, \boldsymbol{\theta}_{k,-i})$, conditional on r_i, is given by $dF_{k+1}(v_i, \boldsymbol{v}_{k,-i})dG_k(\boldsymbol{r}_{k,-i}) = dH_k(\boldsymbol{\theta}_{k,-i})dF(v_i)$. Therefore, the expression inside the integral gives the interim expected utility for bidder i when she pays the second-highest bid $\hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_{k,-i})$.

The unconditional interim expected utility of joining the $\mathcal{S}'$-auction can be written as

$$\mathbb{EU}(\mathcal{S}'|r_i, \mathcal{S}'') = \sum_{k=0}^{I-1} \binom{I-1}{k} \gamma'(\mathcal{S}', \mathcal{S}'')^k \gamma''(\mathcal{S}', \mathcal{S}'')^{I-1-k} \mathcal{EU}(\mathcal{S}'|r_i, k), \tag{4}$$

and similarly for the $\mathcal{S}''$-auction.

Therefore, if bidder i with risk aversion r_i uses a non-degenerate mixed strategy $\gamma'(r_i|\mathcal{S}', \mathcal{S}'')$, the following equality must be satisfied:

$$\textstyle\sum_{k=0}^{I-1} \binom{I-1}{k} (\gamma')^k (1-\gamma')^{I-1-k} \mathcal{EU}(\mathcal{S}'|r_i,k) = \sum_{k=0}^{I-1} \binom{I-1}{k} (1-\gamma')^k (\gamma')^{I-1-k} \mathcal{EU}(\mathcal{S}''|r_i,k). \tag{MS}$$

Here, we dropped the dependence of γ on $\mathcal{S}'$ and $\mathcal{S}''$ for simplicity.

3.3 Sellers' security design

The expected payment to a seller who chooses family $\mathcal{S}'$, conditional on k bidders joining his auction, is given by

$$\mathcal{ES}(\mathcal{S}'|k) = \begin{cases} \int_{\Theta^+} E\mathcal{S}'(\theta_i, s_L) dH(\theta_i) & \text{if } k = 1 \\ k \int_{\Theta^+} \int_{\mathcal{L}^{k-1}_{\mathcal{S}'}(\theta_i)} E\mathcal{S}'(\theta_i, \hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_{k-1,-i})) dH_k(\theta_i, \boldsymbol{\theta}_{k-1,-i}) & \text{if } k \geqslant 2. \end{cases} \tag{5}$$

The expected payment to a seller who chooses $\mathcal{S}'$, when k bidders joins his auction, can be separated into two cases. The first term corresponds to the case in which only one bidder joins the auction, and, thus, the seller receives the lowest feasible security s_L. The second term corresponds to the case in which exactly $k \geqslant 2$ bidders participate. To compute the seller's expected payment in the second case, we obtain the expected payment from bidder i and then multiply it by k. This can be done because types are independent and the mechanism is anonymous. In turn, to compute the seller's expected payment for bidder i, we compute the probability that she wins the auction by integrating the profile of types $\boldsymbol{\theta}_{k-1,-i} = (\boldsymbol{v}_{k-1,-i}, \boldsymbol{r}_{k-1,-i})$ over the region $\mathcal{L}^{k-1}_{\mathcal{S}'}(\theta_i)$. In this region, the seller receives $E\mathcal{S}'(\theta_i, \hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_{k-1,-i}))$. We integrate over bidder's i type in the exterior integral because payoffs are computed from an ex-ante perspective.

The expected payment for a seller that chooses the family $\mathcal{S}'$ when his opponent chooses the family $\mathcal{S}''$ is given by

$$\mathbb{ES}(\mathcal{S}'|\mathcal{S}'') = \sum_{k=1}^{I} \binom{I}{k} \gamma'(\mathcal{S}', \mathcal{S}'')^k \gamma''(\mathcal{S}', \mathcal{S}'')^{I-k} \mathcal{ES}(\mathcal{S}'|k). \tag{6}$$

For any seller $j \in \{1, 2\}$, a mixed strategy is denoted by $\zeta_j \in \Delta(\mathscr{S})$. Hence, the expected payoff for seller j of choosing family $\mathcal{S}'$, when his opponent $-j$ follows the strategy ζ_{-j}, corresponds to

$$\hat{\mathbb{ES}}(\mathcal{S}'|\zeta_{-j}) = \sum_{\mathcal{S}'' \in \mathscr{S}} \mathbb{ES}(\mathcal{S}'|\mathcal{S}'') \zeta_{-j}(\mathcal{S}''). \tag{7}$$

Sellers' family choices take into account the interplay across the different competition effects that are produced by bidders' optimal behavior. In order to ensure the existence of an interior equilibrium strategy, it is necessary that $\mathcal{EU}(\mathcal{S}^d|r, I-1) - \mathbb{E}_{\mathcal{V}}[EU_{\mathcal{S}^{co}}((v,r), s_L)] < 0$, where $\mathcal{S}^d$ is the debt family and $\mathcal{S}^{co}$ is the call-option family. This condition guarantees that the value of the ex-ante insurance offered by the steepest family—when participation is maximal—does not compensate the lower extraction featured by the flattest family under no competition.

4 Channels of competition

The family choices of both sellers affect bidders' equilibrium behavior through three channels: (i) the insurance and surplus extraction; (ii) the induced bidders' aggressiveness; and (iii) the number and composition of participating bidders. For instance, auctions under steeper securities extract higher surplus ex-post, but also provide higher insurance from an interim perspective. Furthermore, depending on bidders' level of risk aversion, they attract less (more) entry, which, in turn, features lower (higher) competition. In particular, if bidders are sufficiently risk-averse, so that the insurance provided by the steeper-family auction more than offsets its higher surplus extraction, they might prefer to join such an auction to take advantage of the low competition and to exercise their higher aggressiveness, as we will see later. Conversely, if bidders have low risk aversion, then, even if they have to face higher competition and get lower insurance, they might prefer to join a flatter-family auction to avoid the higher surplus extraction exercised by the alternate seller.

4.1 Extraction and insurance

Because bidders are risk-averse, it is important to decompose the *extraction* and the *insurance* effects embedded in each security design when comparing the expected utility that a bidder will get by joining alternate auctions under different families of securities.

For any region $\tilde{\mathcal{Z}} \subset \mathcal{Z}$, and families $\mathcal{S}'$ and $\mathcal{S}''$ with $s', s'' \in [s_L, s_H]$, define

$$\Delta_{\tilde{\mathcal{Z}}} EU(s', s'', \theta) \triangleq EU_{\mathcal{S}'}(\theta, s'|z \in \tilde{\mathcal{Z}}) - EU_{\mathcal{S}''}(\theta, s''|z \in \tilde{\mathcal{Z}})$$

as the difference in the expected utility between securities $S'(\cdot, s')$ and $S''(\cdot, s'')$ over $\tilde{\mathcal{Z}}$.

Definition 3. *Fix a bidder with type $\theta = (v, r)$ and consider the families $\mathcal{S}'$ and $\mathcal{S}''$, such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$. Let $s'_1, s'_2, s'' \in [s_L, s_H]$ be the security bids such that $ES'(v, s'_1) > ES''(v, s'')$ and $ES'(v, s'_2) = ES''(v, s'')$. Additionally, let z^*_1 be the single-crossing point between $S(\cdot, s'_1)$ and $S(\cdot, s'')$, and z^*_2 be the single-crossing point between $S(\cdot, s'_2)$ and $S(\cdot, s'')$. The insurance* *effect between $S'(\cdot, s')$ and $S''(\cdot, s'')$ is defined as*

$$\Delta_{[0,z_2^*]}EU(\theta, s_2', s'') - \Delta_{(z_2^*,\infty)}EU(\theta, s'', s_2'), \tag{IE}$$

and the extraction effect between $S'(\cdot, s')$ and $S''(\cdot, s'')$ is defined as

$$\Delta_{\mathcal{Z}}EU(\theta, s_1', s_2'). \tag{EE}$$

Notice that the bid $s_2' \in [s_L, s_H]$ equalizes the seller's expected payment under the steeper security S' with the expected payment under the flatter security S'' when the bid is $s'' \in [s_L, s_H]$. By Theorem 1 in Stiglitz and Rothschild (1970), any risk-averse bidder would prefer to deliver the same expected payment to the seller using a steeper security because flatter securities induce lotteries that are mean-preserving spreads of lotteries induced by steeper securities (Fioriti and Hernandez-Chanto, 2022). Then, the insurance effect in (IE) is determined by the difference between the gain in utility over the region where the flatter security extracts more surplus, $\Delta_{[0,z_2^*]}EU(\theta, s_2', s'')$, net of the loss in utility over the region where the steeper security extracts more surplus, $\Delta_{(z_2^*,\infty)}EU(\theta, s'', s_2')$. The strict concavity of the utility function guarantees that the marginal utility over the first region is greater than the marginal disutility over the second region. Meanwhile, because $S'(\cdot, s_1')$ and $S'(\cdot, s_2')$ belong to the same family, the insurance effect is mute. Moreover, the payment yielded by the former security is greater than the one yielded by the latter over the entire region $\mathcal{Z}$. Hence, the difference in expected utility $\Delta_{\mathcal{Z}}EU(\theta, s_1', s_2')$ corresponds to the pure extraction effect (EE).

4.2 Aggressiveness

Because bidders can be heterogeneously risk-averse, they value the insurance embedded in each security design differently. In particular, more-risk-averse bidders value the insurance provided by steeper securities more highly—in relative terms.

Definition 4. *For any family of securities $\mathcal{S}$ and any type $\theta = (v, r)$, define*

$$\Psi(\theta, \mathcal{S}) \triangleq \frac{E\mathcal{S}(v, \sigma_{\mathcal{S}}((v, r_N))) - E\mathcal{S}(v, \sigma_{\mathcal{S}}((v, r)))}{E\mathcal{S}(v, \sigma_{\mathcal{S}}((v, r_N)))}. \tag{8}$$

We say that a bidder with type θ is more aggressive under family $\mathcal{S}'$ than under family $\mathcal{S}''$ if

$\Psi(\theta, \mathcal{S}') < \Psi(\theta, \mathcal{S}'')$.

Definition 4 states that for any bidder's type $\theta = (v, r)$, the gap between her equilibrium bid and the bid she would have submitted under risk neutrality decreases with the steepness of family $\mathcal{S}$.

Proposition 2. *For any two families of securities $\mathcal{S}'$ and $\mathcal{S}''$ such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$, we have that:*

i) any bidder with type $\theta = (v, r)$ is more aggressive under $\mathcal{S}'$ than under $\mathcal{S}''$; and

ii) for any signal v, a more-risk-averse bidder is relatively more aggressive under $\mathcal{S}'$ than under $\mathcal{S}''$—i.e.,

$$\Psi((v, \hat{r}), \mathcal{S}') - \Psi((v, r), \mathcal{S}') > \Psi((v, \hat{r}), \mathcal{S}'') - \Psi((v, r), \mathcal{S}'')$$

for all $\hat{r} > r$.

When bidders are risk-averse, their equilibrium bids are "handicapped" with respect to what their bids would have been had they been risk-neutral. Proposition 2 states that this handicap decreases when steeper securities are used, and that such a *decrement is greater* for more-risk-averse bidders. This stems from the fact that as bidders become more risk-averse, their utility functions correspond to a sequence of strict concave transformations, which implies that the insurance provided by steeper securities becomes relatively more valuable. In other words, the more risk-averse is a bidder, the relatively more aggressive is her bidding in the auction. Technically speaking, for any signal v, the function $\Psi(\theta, \mathcal{S})$ satisfies *increasing differences* in the order induced by steepness and the risk-aversion level.

4.3 Number of competitors

For any fixed security design, increasing the number of bidders always increases the seller's expected revenue when bidders are risk-neutral. Furthermore, for standard cash auctions, increasing the number of bidders by one dominates any other mechanism under the original number of bidders (Bulow and Klemperer, 1996). However, when bidders are heterogeneously risk-averse, the composition of bidders—and not only their number—matters for determining the seller's expected revenue. In particular, a seller might find it optimal to choose a security design that induces a lower number of bidders if they bid more aggressively and the seller can extract higher surplus ex-post. Hence, each seller must consider the interplay of all these effects when choosing his optimal security design.

5 Homogeneously risk-averse bidders

In this section, we present a full characterization of how bidders and sellers separate endogenously in equilibrium when bidders are homogeneously risk-averse. That is, when all bidders share the same risk-aversion parameter, which is drawn from a distribution G with support $\mathcal{R}^+$. Importantly, once the risk-aversion level is drawn, it is fixed and becomes common knowledge.

Lemma 3. *If bidders are homogeneously risk-averse, then, for any family $\mathcal{S}$, (i) the bidder's interim expected utility $\mathcal{EU}(\mathcal{S}|r,k)$ is decreasing and convex in k for any r; and (ii) the seller's interim expected payment $\mathcal{ES}(\mathcal{S}|k)$ is increasing and concave in k.*

Lemma 3 determines bidders' and sellers' expected payoffs as a function of the competitors in the selected auction. As expected, the marginal benefit (cost) of having an extra competitor for the seller (bidders) is decreasing (increasing). The next proposition establishes how the entry strategy is affected by bidders' level of risk aversion.

Proposition 3. *Suppose that bidders are homogeneously risk-averse. For any sellers' choice of families $\mathcal{S}'$ and $\mathcal{S}''$, the entry strategy $\gamma'(r|\mathcal{S}',\mathcal{S}'')$ satisfies the following properties:*

(i) Bidders use a mixed strategy $\gamma'(\mathcal{S}',\mathcal{S}'')$ that is uniquely defined by (MS).

(ii) If $\mathcal{S}'=\mathcal{S}''$, then $\gamma'(r|\mathcal{S}',\mathcal{S}'')=\frac{1}{2}$ for all $r\in\mathcal{R}^+$.

(iii) If $\mathcal{S}'$ is steeper than $\mathcal{S}''$, then $\gamma'(\hat{r}|\mathcal{S}',\mathcal{S}'')\geqslant\gamma'(r|\mathcal{S}',\mathcal{S}'')$ for all $\hat{r},r\in\mathcal{R}^+$ such that $\hat{r}>r$.

(iv) If $\mathcal{S}'$ is steeper than $\mathcal{S}''$, there exists a cutoff value $r(\mathcal{S}',\mathcal{S}'')$ such that if $r\leqslant r(\mathcal{S}',\mathcal{S}'')$, then $\gamma'(r|\mathcal{S}',\mathcal{S}'')\leqslant\gamma''(r|\mathcal{S}',\mathcal{S}'')$; and if $r>r(\mathcal{S}',\mathcal{S}'')$, then $\gamma'(r|\mathcal{S}',\mathcal{S}'')>\gamma''(r|\mathcal{S}',\mathcal{S}'')$.

Part (i) states that, in any equilibrium, bidders' strategies are totally mixed. The existence and uniqueness of an equilibrium under mixed strategies is guaranteed by the continuity of bidding strategies in signals and risk-aversion levels. Part (ii) says that when both sellers choose the same family of securities, bidders randomize equally between the two auctions, given our focus on bidders' symmetric equilibria. In this case, only the competition effect prevails, and, so, bidders' optimal strategy is to balance the expected number of bidders in each auction. In turn, part (iii) says that when sellers choose different families, the probability of joining the auction under the steeper family is increasing in bidders' risk aversion. Finally, part (iv) establishes the existence of a security-dependent threshold in the risk-aversion level that drives bidders' incentives to join each auction. Specifically, if bidders' risk aversion exceeds the given threshold, they enter the auction under the steeper family with a higher probability; otherwise, they enter the auction under the flatter family with a higher probability.

Remark 1. *If agents are homogeneously risk-averse, the expected revenue of a seller that chooses family $\mathcal{S}'$ when his opponent chooses family $\mathcal{S}''$ depends only on the expected number of bidders that join his auction. Thus, abusing notation, we let $\mathbb{ES}(\mathcal{S}'|\mathcal{S}'') = \mathbb{ES}(\mathcal{S}'|\gamma'(\mathcal{S}', \mathcal{S}''))$. Therefore, without loss of generality, we can define the expected payment for a seller who chooses family $\mathcal{S}$ in terms of the probability with which a bidder joins the $\mathcal{S}$-auction as $\mathbb{ES}(\mathcal{S}|\gamma)$.*

Using the simplification derived in the previous remark, it is possible to determine the shape of the seller's expected revenue as a function of bidders' entry probability.

Proposition 4. *Any equilibrium under homogeneous risk aversion must be symmetric for sellers.*

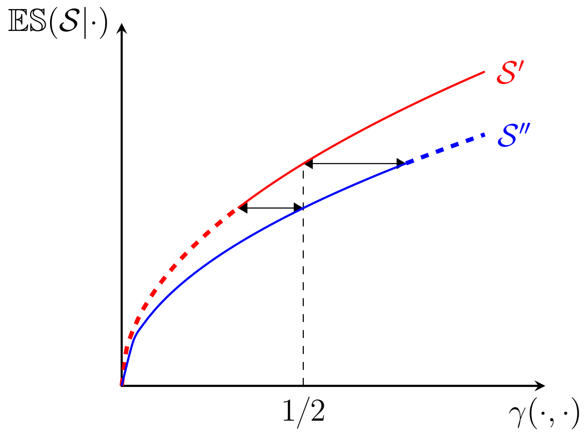


Figure 3: Nonexistence of asymmetric equilibria.

Proposition 4 establishes the symmetry of all equilibria when there are only two sellers and bidders are risk-averse; thus, showing that the focus on symmetric equilibria in duopolistic auctions is without loss of generality. Figure 3 provides a graphic intuition of this result by depicting the seller's expected revenue as a function of bidders' entry probability. For concreteness, we pick two arbitrary families $\mathcal{S}'$ and $\mathcal{S}''$, such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$, and suppose that one seller (say, Seller 1) chooses family $\mathcal{S}'$, while his opponent (Seller 2) chooses $\mathcal{S}''$.[12] For this to be an asymmetric equilibrium, neither seller must have an incentive to mimic his opponent. Seller 2 does not have an incentive to mimic Seller 1 if the probability with which bidders enter his auction is sufficiently high, such that $\mathbb{ES}(\mathcal{S}''|1-\gamma') \geqslant \mathbb{ES}(\mathcal{S}'|1/2)$. This condition is satisfied in the dashed segment of Seller 2's expected revenue curve in Figure 3. If this is the case, Seller 1's expected revenue must lie in its dashed segment as well, by virtue of the strictly increasing property between steepness and the entry probability (see Lemma A.1 in the Appendix). But now, if Seller's 1 expected revenue lies in the dashed segment, he has an incentive to mimic Seller 2 and choose family $\mathcal{S}''$. The analogous argument applies if Seller 1's expected revenue lies in the solid segment. In such a case, Seller 1 does not have an incentive to deviate, but Seller 2's expected revenue would lie in its corresponding solid segment, where a deviation is profitable.

Notably, the strictly increasing property between steepness and bidders' entry probability precludes having asymmetric equilibria, despite sellers have a large spectrum of security designs.

Proposition 5. *For any feasible family $\mathcal{S}$, if $r = r_N$ (i.e., if bidders are risk-neutral), there exists an equilibrium in which both sellers choose family $\mathcal{S}$. If $\mathcal{S}$ is flatter than a call option, there exists a cutoff $r(\mathcal{S}) \in \mathcal{R}^+$ such that if $r > r(\mathcal{S})$, the equilibrium does not survive. The only equilibrium prescribed under risk neutrality that is robust to any increment in the level of risk aversion is the one in which both sellers choose a call-option family.*

Under risk neutrality, it is always possible to define an equilibrium in which both sellers choose a family flatter than a call option (Gorbenko and Malenko, 2011). Nonetheless, when bidders become sufficiently risk-averse, the equilibrium is broken because the insurance provided by steeper securities reinforces the incentives of sellers to choose steeper families. Thus, under some caveats, Proposition 5 extends to a competitive environment the result in Fioriti and Hernandez-Chanto (2022), which shows that when bidders are sufficiently risk-averse the only family that guarantees Pareto optimality is a call option. In our environment, the existence of

[12] Because $\mathcal{S}'$ is steeper than $\mathcal{S}''$, the expected revenue curve corresponding to $\mathcal{S}'$ must lie entirely above the corresponding curve of $\mathcal{S}''$.

a symmetric equilibrium under a call option requires the level of risk aversion to be just high enough so that the insurance effect makes bidders enter such an auction with a probability higher than the one with which they would enter any other symmetric auction. This is so because when bidders are sufficiently risk-averse, the incentives of bidders and sellers are aligned in the steepness of the family utilized.

6 Heterogeneously risk-averse bidders

We show that when bidders are heterogeneously risk-averse, there exists an equilibrium in which sellers' family choices in the steepness spectrum are analogous to sellers' choices of location in a Hotelling model. To characterize such an equilibrium we rely on the theory of monotone games under private information (e.g., Athey, 2001). This allows us to determine bidders' pure strategies for each level of risk aversion, and, in particular, to characterize their switching behavior across families of securities.

6.1 Bidders' optimal strategy

For any two families $\mathcal{S}'$ and $\mathcal{S}''$, such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$, we adopt the convention of calling the "high action" the one that corresponds to choosing the $\mathcal{S}'$-auction and calling the "low action" the one that corresponds to choosing the $\mathcal{S}''$-auction.

Definition 5. *A bidder's pure strategy $\eta(\cdot|\mathcal{S}', \mathcal{S}'') : \mathcal{R}^+ \to \{\mathcal{S}', \mathcal{S}''\}$ is said to be non-decreasing if bidders (weakly) join steeper families of securities as they become more risk-averse.*

When the action set is finite, any non-decreasing strategy is a step function. Hence, if a bidder i is deciding between two auctions run under families $\mathcal{S}'$ and $\mathcal{S}''$, her strategy is determined by a cutoff value r_i^0 at which the bidder "jumps" from $\mathcal{S}''$ to $\mathcal{S}'$. That is,

$$\eta(r_i|\mathcal{S}', \mathcal{S}'') = \begin{cases} \mathcal{S}'' & \text{if } r_i < r_i^0 \\ \mathcal{S}' & \text{if } r_i \geqslant r_i^0. \end{cases} \tag{9}$$

Following Athey (2001), any strategy that follows this "jumping behavior" at r_i^0—even if there were more cutoff values—is said to be *consistent* at r_i^0. In addition, because bidders are ex ante equal, we have that $r_i^0 = r^0$ for all i.[13]

In order to guarantee the existence of a cutoff equilibrium, it is necessary that the actions of any bidder at r^0 do not affect other players' best responses. This is satisfied in our model because risk types are jointly distributed with respect to a bounded and atomless measure, and, thus, bidders' actions at any point have zero measure. Furthermore, bidders' utility functions are well defined for any entry action played by other bidders, since their interim expected utilities are bounded for all number of competitors and risk aversion parameters lie in a convex set. Therefore, if all bidders play a non-decreasing strategy consistent with r^0, the interim expected utility that a given bidder i with risk aversion r_i obtains by joining the $\mathcal{S}'$-auction corresponds to

$$\mathcal{EU}(\mathcal{S}'|r_i, r^0) = \int_{\mathcal{V}} \int_{\mathcal{L}_{\mathcal{S}'}^{I-1}(\boldsymbol{\theta}_i)} EU_{\mathcal{S}'}(\theta_i, \hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_{I-1,-i}))dH_{I-1}(r^0)(\boldsymbol{\theta}_{I-1,-i})dF(v_i), \tag{10}$$

where $H_{I-1}(r^0)$ represents the distribution of a profile of types *consistent* with the cutoff r^0.[14]

When bidders are heterogeneously risk-averse and follow a cutoff strategy, the event that a bidder has zero opponents in a given auction has zero measure. Thus, we cannot separate the conditional interim expected utility. Nonetheless, bidders' ex-ante utility is analogous to the expression in (4).

The following proposition shows the existence of a cutoff equilibrium in pure strategies, where all bidders jump at a threshold that depends on sellers' family choices.

Proposition 6. *Suppose that bidders are heterogeneously risk-averse. For any sellers' choices $\mathcal{S}'$ and $\mathcal{S}''$, with $\mathcal{S}'$ steeper than $\mathcal{S}''$, there exists a pure-strategy Nash equilibrium such that*

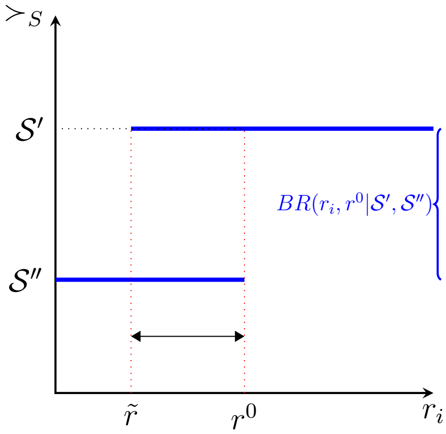
$$\eta(r_i|\mathcal{S}', \mathcal{S}'') = \begin{cases} \mathcal{S}'' & \text{if } r_i < r(\mathcal{S}', \mathcal{S}'') \\ \mathcal{S}' & \text{if } r_i \geqslant r(\mathcal{S}', \mathcal{S}'') \end{cases} \tag{11}$$

for some $r(\mathcal{S}', \mathcal{S}'') = r(\mathcal{S}'', \mathcal{S}')$.

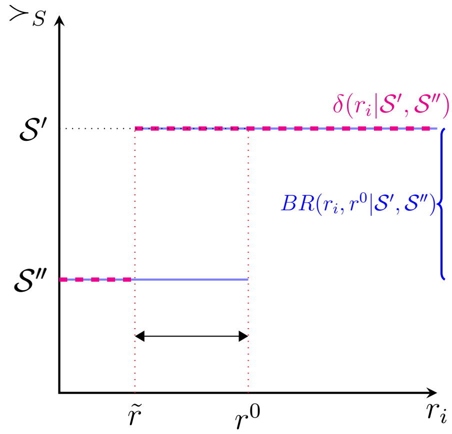
The details of the proof are presented in the Appendix, but here we give a sketch of it and the intuition behind it. First, given sellers' family choices $\langle \mathcal{S}', \mathcal{S}'' \rangle$, and assuming that all bidders

[13] When sellers choose the same family of securities, we can relabel the families using sellers' identities such that $\mathcal{S}_1$ corresponds to the family chosen by Seller 1 and $\mathcal{S}_2$ to the one chosen by Seller 2. Hence, we can redefine the set of actions such that the "higher" action corresponds to choosing Seller 2's auction and the "lower" one corresponds to choosing Seller 1's auction.

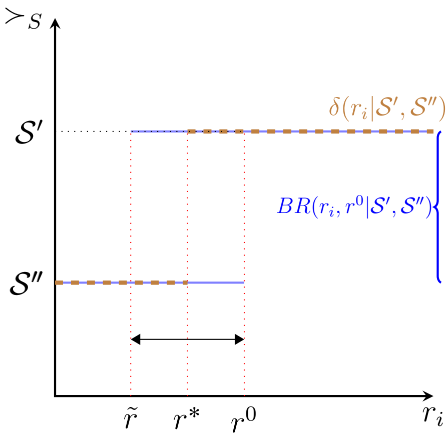
[14] The types' joint distribution depends only on the number of bidders since they are identical ex-ante.



(a) Best-response correspondence.



(b) Non-decreasing selection $\delta(r_i|\mathcal{S}', \mathcal{S}'')$ consistent with $\tilde{r}$.



(c) Non-decreasing selection $\delta(r_i|\mathcal{S}', \mathcal{S}'')$ consistent with the fixed point r^*.

Figure 4: Best-response correspondence and non-decreasing selections. Panel 4a displays bidders' best-response correspondence as a function of the risk-aversion parameter. The values $\tilde{r}$ and r^0 represent, respectively, the lowest and greatest jumping points. The arrows indicate convex combinations of such points, which are also jumping points. Panel 4b depicts a non-decreasing selection consistent with $\tilde{r}$. In turn, Panel 4c shows a non-decreasing selection consistent with the fixed point r^*.

other than i follow a cutoff strategy consistent with a risk-aversion level r^0, we determine bidder i's best-response correspondence $BR(r_i, r^0|\mathcal{S}', \mathcal{S}'')$—i.e., the set of optimal jumping points for bidder i. Then, we show that there always exists a non-decreasing selection $\delta_i(r_i|\mathcal{S}', \mathcal{S}')$ from bidder i's best-response correspondence. Finally, we show proof—via Kakutani's Fixed Point Theorem—that since bidders' interim utility functions satisfy a "single-crossing condition," the set of non-decreasing selections always have a fixed point: this corresponds to the equilibrium jumping threshold. Figure 4 depicts the steps of the procedure.

Using (11), we can compute the expected number of bidders that join each auction in equilibrium. For any $r_i \in \mathcal{R}^+$, let $\mathbb{1}_{r(\mathcal{S}',\mathcal{S}'')}(r_i)$ be equal to one if $\eta(r_i|\mathcal{S}', \mathcal{S}'') = \mathcal{S}'$ and zero if $\eta'(r_i|\mathcal{S}', \mathcal{S}'') = \mathcal{S}''$. Then, the expected number of bidders that enter each auction can be computed as

$$I'(\mathcal{S}', \mathcal{S}'') = \sum_{i=1}^{I} \int_{\mathcal{R}^+} \mathbb{1}_{r(\mathcal{S}',\mathcal{S}'')}(r_i) dG(\boldsymbol{r}) \quad \text{and} \quad I''(\mathcal{S}', \mathcal{S}'') = I - I'(\mathcal{S}', \mathcal{S}''). \tag{12}$$

The expected number of bidders is instrumental to characterize sellers' equilibrium with respect to bidders' risk-aversion types, as it can be seen later.

6.2 Sellers' optimal strategy

Given the strategy prescribed by (11), we can define the seller's expected revenue from choosing family $\mathcal{S}'$—when his opponent chooses $\mathcal{S}''$—as

$$\mathcal{ES}(\mathcal{S}'|r(\mathcal{S}', \mathcal{S}'')) = I \int_{\Theta^+} \int_{\mathcal{L}_{\mathcal{S}'}^{I-1}(\theta_i)} ES'(\theta_i, \hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_{I-1,-i})) dH_I'(r(\mathcal{S}', \mathcal{S}''))(\boldsymbol{\theta}). \tag{13}$$

Here, $H_I'(r(\mathcal{S}', \mathcal{S}''))$ represents the distribution of a profile of types that enter the $\mathcal{S}'$-auction when bidders follow a non-decreasing strategy consistent with $r(\mathcal{S}', \mathcal{S}'')$, and analogously for $H_I''(r(\mathcal{S}', \mathcal{S}''))$. The derivation in (13) is used to compute the change in payoffs as a result of deviations from a prescribed strategy.

Proposition 7. *Let $\mathcal{S}'$ and $\mathcal{S}''$ be two different families of securities. If (i) r_H is sufficiently bounded; (ii) $\mathcal{S}'$ is sufficiently steep and $\mathcal{S}''$ is sufficiently flat; (iii) the survival functions $\bar{F} \triangleq 1-F$ and $\bar{G} \triangleq 1-G$ are concave; and (iv) there exist constants $\bar{c}^\ell(I'(\mathcal{S}', \mathcal{S}''))$ and $\bar{c}^\ell(I''(\mathcal{S}', \mathcal{S}''))$,* *for $\ell \in \{1, 2\}$, such that*

$$\bar{c}^{\ell}(I'(\mathcal{S}', \mathcal{S}'')) \geqslant \lim_{I \to I'(\mathcal{S}',\mathcal{S}'')^{+}} \left| \frac{\partial \mathbb{E}[V^{(\ell:I)}]}{\partial I} \right| \geqslant \lim_{I \to I''(\mathcal{S}',\mathcal{S}'')^{-}} \left| \frac{\partial \mathbb{E}[V^{(\ell:I)}]}{\partial I} \right| \geqslant \bar{c}^{\ell}(I''(\mathcal{S}', \mathcal{S}'')), \tag{14}$$

where $\mathbb{E}[V^{(\ell:I)}]$ corresponds to the expected ℓ-highest order statistic of a sample of size I; then there exists a subgame perfect Nash equilibrium in which (i) one seller chooses $\mathcal{S}'$ and the other seller chooses $\mathcal{S}''$; and (ii) bidders separate themselves in both auctions such that those bidders with a risk-aversion parameter lower than $r(\mathcal{S}', \mathcal{S}'')$ join the $\mathcal{S}''$-auction, and those with a risk-aversion parameter greater than $r(\mathcal{S}', \mathcal{S}'')$ join the $\mathcal{S}'$-auction, for some $r(\mathcal{S}', \mathcal{S}'') \in \mathcal{R}^{+}$.

Proposition 7 shows that it is always possible to sustain an equilibrium in which sellers separate themselves in the steepness of their security design to serve the heterogeneous population of bidders, provided that the upper bound of risk aversion is sufficiently low, so that the value of insurance does not grow unbounded; the security designs are sufficiently distant in the steepness spectrum; and the distribution of the risk aversion parameter and the first two order statistics of the signals are sufficiently concave in the number of participant bidders. The concavity of the survival function $\bar{F}$ stated in condition (iii) is only necessary to assure the concavity of the expected second-highest (i.e., the $I-1$ smallest) order statistic, since the first order statistic is always concave in the sample size. Meanwhile, condition (iv) implies that the negative effect in the expected order statistics that come from a decrement in bidders' participation below $I''(\mathcal{S}', \mathcal{S}'')$—the expected number of bidders who join the auction under the steeper family—is greater in magnitude than the potential increment that comes from the increase in bidder's participation beyond the $I''(\mathcal{S}', \mathcal{S}'')$—the expected number of bidders who join the auction under the flatter family. In other words, Assumption (iv) imposes discipline in the concavity of the expected first and second order statistics on the equilibrium path.

Although the assumption of the concavity of the survival functions seems high level, it can be related to the behavior of the hazard rate as it is shown in Lemma A.4 in the Appendix.

Figure 5 depicts an equilibrium in which sellers' separate themselves in the steepness spectrum by choosing families $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}', \mathcal{S}'' \rangle$ such that $\mathcal{S}^{co} \succ \mathcal{S}' \succ \mathcal{S}'' \succ \mathcal{S}^{d}$. Bidders follow a pure-strategy cutoff equilibrium at $r(\mathcal{S}', \mathcal{S}'')$ in which bidders with risk aversion lower than $r(\mathcal{S}', \mathcal{S}'')$ joins the flatter auction, and those with risk aversion higher than than $r(\mathcal{S}', \mathcal{S}'')$ join the steeper auction. In this case, the flatter auction, features a lower mass of bidders who are

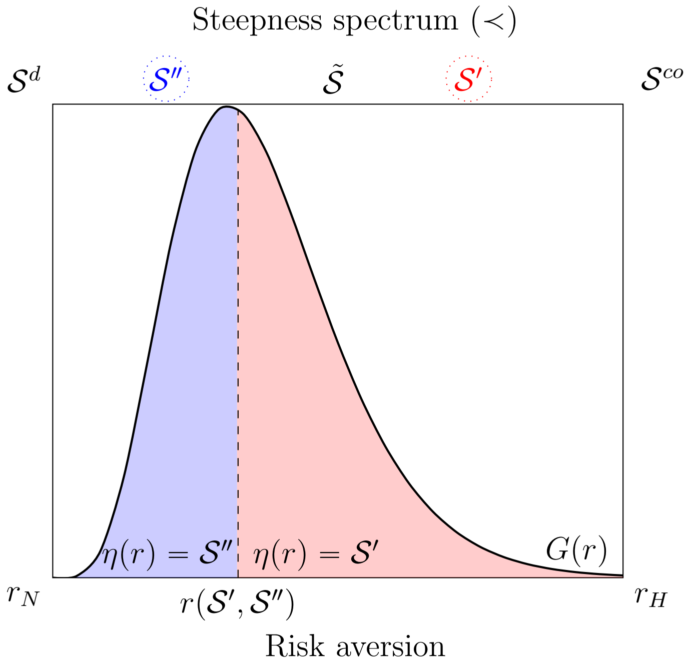


Figure 5: Example of a separating equilibrium in steepness.

more competitive than their counterparts in the steeper auction. However, the steeper auction provides enough insurance to more-risk-averse bidders to help them mitigate both the higher surplus extraction and the higher aggressiveness exhibited by their competitors in their auction. Notice that if insurance were lower, the cutoff point would be at the right, and more bidders would join the flatter auction.

7 Numerical simulation

In this section, we numerically explore how firms' risk-aversion levels affect equilibrium strategies in all stages of the game.

Assume the environment in Section 5. There are two sellers that compete for $I \geqslant 2$ bidders. The signal of each bidder V_i is represented by the net present value of the project (NPV). Signals are independently distributed across bidders from a uniform distribution with support $\mathcal{V} = [0, 2]$. The upfront investment is set at $\kappa = 0.8$. If the project is undertaken by a bidder with signal $V_i = v_i$, it yields a stochastic revenue of Z_i, which is drawn from a log-normal distribution with mean $\kappa + v_i$ and volatility $(\kappa + 1)/2 = 0.9$. To secure the payment from the winning bidder, sellers can use one of three standard families of securities to organize their auctions. Thus, $\mathscr{S} = \{\mathcal{S}^d, \mathcal{S}^e, \mathcal{S}^{co}\}$. Hence, we consider a larger set of securities, relative to the simulation in Gorbenko and Malenko (2011), which makes the characterization of equilibria more challenging.

Example 1: Homogeneously risk-averse agents Bidders are homogeneously risk-averse and seek to maximize their expected utility. Bidders' Bernoulli utility function is given by $u(m,r) = (1 - e^{-rm})/r$, where m denotes monetary prizes and r denotes bidders' realized absolute level of risk aversion. The risk-aversion parameter is fixed and commonly known. The description of the algorithm used to simulate the game is presented in the Appendix.[15]

Entry probabilities We run our algorithm for different numbers of bidders to analyze how the market size affects bidders' entry probabilities for any given sellers' family choices.

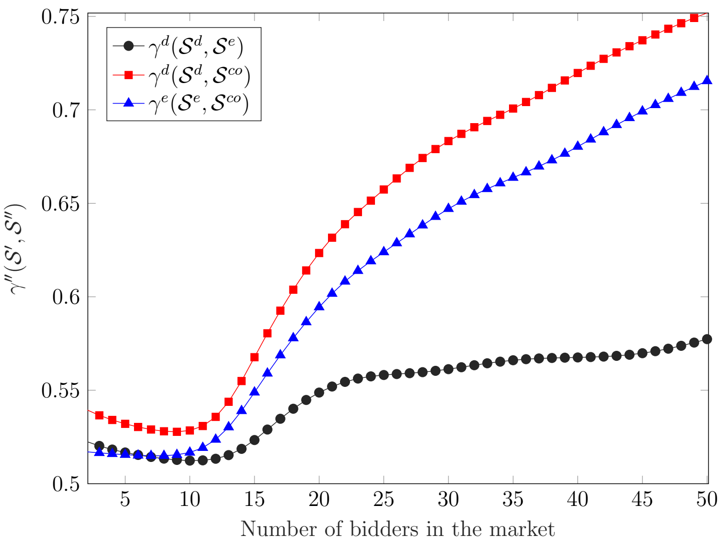


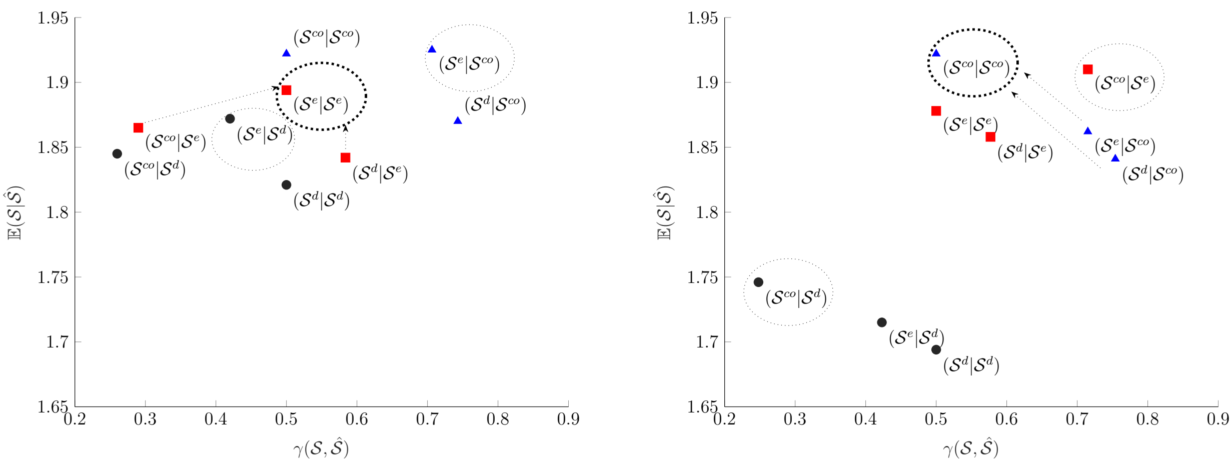
Figure 6: Bidders' entry probability as a function of the market size when they have a risk-aversion level of $r = 0.5$.

Figure 6 shows that when the number of bidders is lower than 13, for any seller's profile, the probability of entering the auction under the flatter family is always between 0.5 and 0.55. This implies that when the number of bidders is low, the competition effect dominates because bidders try to balance the opposition they face in both auctions. The probability of entering the flatter auction increases with the number of bidders, although not monotonically. This

[15]It is worth highlighting that, unlike in Gorbenko and Malenko (2011), in which any equilibrium can be characterized by the probability with which a bidder enters a given auction our algorithm must compute the interim expected utility of each bidder for each possible pair of families in $\mathscr{S}$, making the implementation more challenging.

reflects the fact that, as the competition gets sufficiently high in both auctions, bidders' entry probabilities are driven by their relative valuation of the insurance and the extraction effects. Furthermore, the gap between the entry probability to the flatter auction increases in the difference of steepness between the two families chosen by the seller. Thus, it is maximal when one seller chooses a call option and the other chooses debt.

Seller's expected revenues We now inspect the sellers' choice of security when the market is large.[16] Similar to our determination of entry probabilities, we distinguish between the cases when bidders are risk-neutral and when they are risk-averse.



(a) Risk-neutral bidders ($r = 0$).

(b) Risk-averse bidders ($r = 0.5$).

Figure 7: Seller's expected revenue and bidder's entry probability.

Figure 7 depicts the expected revenue and the entry probability corresponding to a seller that chooses the family $\mathcal{S}'$, conditional on the opposing seller choosing the family $\mathcal{S}''$. The dashed circles depict those situations in which the chosen family is a best response to the opponent's choice. In particular, the circles in thick black correspond to the Nash equilibrium. Furthermore, arrows indicate the direction of profitable deviations. When bidders are risk-neutral—i.e., Panel 7a—the equilibrium is given when both sellers choose equity. Nonetheless, as soon as bidders become risk-averse—i.e., Panel 7b—the insurance provided by steeper securities reinforces sellers' incentives to choose a steeper security (due to the greater extraction), and, thus, the equilibrium transitions towards one in which both sellers choose a call-option family, which is the steepest family.

[16]For illustration purposes, we depict the case when there are 50 bidders.

Interestingly, the equilibrium under risk-averse bidders yields the sellers greater expected revenue, although bidders shave their bids relative to what they bid when they are risk-neutral. The reason is that the insurance provided to risk-averse bidders makes them more prone to choose an auction run under a steeper family, and, as a consequence, lets sellers coordinate on choosing the steepest family. Furthermore, the presence of more aggressive bidders diminishes the bid premium when bidders are risk-neutral, mitigating the loss in surplus extraction.

Example 2: Heterogeneously risk-averse agents We now consider heterogeneously risk-averse bidders whose risk-aversion parameter follows a uniform distribution with support $[0.5, 1.5]$. We compute bidders' equilibrium in cutoff strategies following a novel algorithm presented in the Appendix.

	$\mathcal{S}^d, \mathcal{S}^d$	$\mathcal{S}^e, \mathcal{S}^e$	$\mathcal{S}^d, \mathcal{S}^c$	Median$[r]$	$\mathcal{S}^c, \mathcal{S}^c$	$\mathcal{S}^d, \mathcal{S}^e$	$\mathcal{S}^e, \mathcal{S}^c$
$r(\cdot, \cdot)$	0.63	0.76	0.78	1	1.14	1.24	1.35

Table 1: Bidders' pure-strategy cutoff equilibrium.

Cutoff values In Figure 1, we depict how bidders separate in equilibrium following a cutoff pure strategy. First, when sellers choose the same family of securities, bidders separate in an unbalanced way, with a skewness towards the right tail of the distribution—i.e., the mass of more-risk-averse bidders that join a given auction is greater. The reason is that both auctions offer the same insurance, but less-risk-averse bidders are more competitive; thus, the auction in which they participate must feature lower participation to balance the payoffs. This contrasts with the outcome in the homogeneous case, in which bidders join both auctions in an even fashion whenever both sellers choose the same family. Second, when sellers continue to choose the same family, but now steeper, the equilibrium cutoff moves to the right. This follows from the result in Proposition 2, since, as the steepness increases, more-risk-averse bidders become relatively more aggressive, and so, the mass of competitors must decrease to balance payoffs. Finally, when sellers choose different families, there is always an equilibrium in which bidders separate themselves in both auctions following the rule in Proposition 6: the more-risk-averse bidders join the steeper auction, while the less-risk-averse join the flatter auction. Furthermore, notice that, as the difference in steepness between the families chosen by sellers enlarges, the proportion of the market taken by the seller who chooses the steeper family increases.

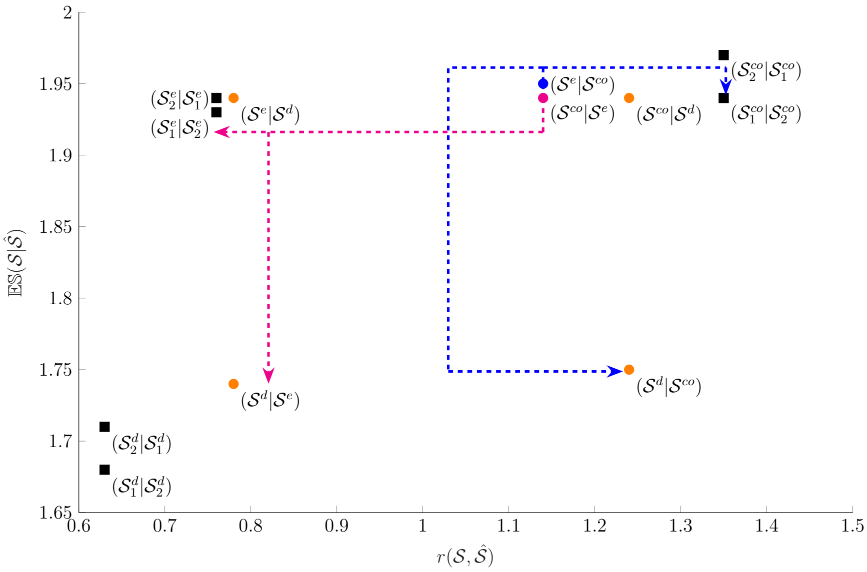


Figure 8: Sellers' expected revenue for different security-design choices when bidders are heterogeneously risk-averse.

Sellers' expected revenues We now depict sellers' expected revenue for all possible combinations of securities in $\mathscr{S}$ when the market is large.[17] In this case, the unique subgame perfect Nash equilibrium corresponds to one in which one seller chooses a call-option family and the other chooses an equity family. Meanwhile, bidders endogenously separate themselves, so, that, those with risk-aversion parameter lower than $r(\mathcal{S}^e, \mathcal{S}^{co})$ join the equity auction, while those with a risk aversion greater than $r(\mathcal{S}^e, \mathcal{S}^{co})$ join the call-option auction. The threshold $r(\mathcal{S}^e, \mathcal{S}^{co})$ is computed according to Proposition 6.

It is worth noticing that the asymmetry of bidders' risk aversion induces asymmetry in the sellers' payoffs even when both sellers choose the same family. This, in turn, affects the possible profitable deviations of each seller, as depicted in Figure 8. In the equilibrium above, we assume that if the seller who chooses equity deviates to the call option, he will obtain the lowest payoff among the ones available. This can be justified by assuming a focal equilibrium in line with the literature that compares behaviors relative to an "industry benchmark." Interestingly, in this equilibrium, the seller choosing the steepest family obtains lower expected revenue.

[17]The necessity of a large market is due only to the fact that, for tractability in the simulation, we impose a uniform distribution of the signals. The same results can be obtained in a small market under asymmetric distributions, provided that the sufficient conditions in Proposition ?? are met.

8 Discussion and extensions

We present some ways in which our model can be enriched, identifying the main changes to our framework and suggesting topics for future research.

8.1 Reserve prices

The literature that studies competing sellers in standard auctions focuses on competition via reserve prices, which arise either exogenously or as part of the design of an optimal mechanism (McAfee, 1993; Peters, 1997; Burguet and Sákovics, 1999; Hernando-Veciana, 2005; Damianov, 2012). The reason is that, in standard auctions, all payments are made upfront using cash; thus, holding constant the auction format, reserve prices become the only practical design instrument available to sellers in order to induce a different composition of participants in their auction. In contrast, security-bid auctions provide considerable flexibility in the auction's design, allowing the equilibrium constraints associated with reserve prices to be embedded within the different security designs. We characterize equilibria under reserve prices when bidders are homogeneously risk-averse.

Definition 6. *For any auction run under the family of securities $\mathcal{S} = \{S(\cdot, s) : s \in [s_L, s_H]\}$, a reserve price is a security index $\underline{s}(\mathcal{S}) \in [s_L, s_H]$ such that the seller disregard any security bid $S(\cdot, s)$ with $s \leqslant \underline{s}(\mathcal{S})$.*

A reserve price plays the role of an exclusion restriction for bidders with low valuations. As such, it entails a trade-off for the seller between the probability of allocating the project and the revenue obtained conditional on the allocation of the project. Specifically, if no bidder submits an admissible bid, the seller does not allocate the project. In contrast, if at least one bidder places an admissible bid, the seller proceeds with the project allocation, and the winning bidder is required to pay with a security that is indexed by the higher value between the reserve price, $\underline{s}(\mathcal{S})$, and the second-highest bid.

Because bidders decide which auctions to enter before learning their signals, we can define bidders' marginal type—i.e., the type who is indifferent between entering the auction and staying away—as the risk-aversion cutoff $r(\underline{s}(\mathcal{S})) \in \mathcal{R}^+$ such that

$$\int_{\mathcal{V}} EU_{\mathcal{S}}((v, r(\underline{s}(\mathcal{S}))), \underline{s}(\mathcal{S}))]dF(v) = 0. \tag{15}$$

Given that bidders with risk-aversion types $r > r(\underline{s}(\mathcal{S}))$ have utility functions that are strict concave transformations of the utility function that corresponds to $r(\underline{s}(\mathcal{S}))$, it is optimal for those bidders to stay away from the auction. Hence, for any auction under the security design $(\mathcal{S}, \underline{s}(\mathcal{S}))$, the set of risk-aversion types that induce entry can be written as $R^+(\underline{s}(\mathcal{S})) \triangleq \{r \in R^+ : r \leqslant r(\underline{s}(\mathcal{S}))\}$. Therefore, whenever a seller uses a reserve price $\underline{s}(\mathcal{S}) \in (s_L, s_H)$, we need to redefine accordingly the notion of individual rationality.

Definition 7. *For any $\mathcal{S}'$ with respective reserve price $\underline{s}(\mathcal{S}')$, the entry strategy $\gamma(\cdot|\mathcal{S}', \mathcal{S}'')$ satisfies individual rationality (IR) if, for any $\mathcal{S}''$ with reserve price $\underline{s}(\mathcal{S}'')$, $\gamma'(r|\mathcal{S}', \mathcal{S}'') = 0$ whenever $r > r(\underline{s}(\mathcal{S}'))$.*

Hence, the unconditional entry probabilities become

$$\begin{aligned} \gamma'(\mathcal{S}', \mathcal{S}'') &= \int_{R^+(\underline{s}(\mathcal{S}'))} \gamma'(r|\mathcal{S}', \mathcal{S}'') dG(r) \\ \gamma''(\mathcal{S}', \mathcal{S}'') &= \int_{R^+(\underline{s}(\mathcal{S}''))} \gamma''(r|\mathcal{S}', \mathcal{S}'') dG(r) = 1 - \gamma'(\mathcal{S}', \mathcal{S}''). \end{aligned} \tag{16}$$

Now, for any reserve price $\underline{s}(\mathcal{S})$, define $\theta(\underline{s}(\mathcal{S})) = (v(\underline{s}(\mathcal{S})), r(\underline{s}(\mathcal{S}))) \in \mathcal{V} \times R^+(\underline{s}(\mathcal{S}))$ such that

$$EU_{\mathcal{S}}(\theta(\underline{s}(\mathcal{S})), \underline{s}(\mathcal{S})) = 0.$$

That is, $\theta(\underline{s}(\mathcal{S}))$ corresponds to the type of a bidder who enters the auction and is indifferent between winning the project at the reserve price or losing the project with certainty. Notice that $v(\underline{s}(\mathcal{S}))$ corresponds to the lowest upper bound of the set of signals that induce the bidder with risk aversion $r(\underline{s}(\mathcal{S}))$ to submit an admissible bid. By the same argument as in Lemma 2, $\theta(\underline{s}(\mathcal{S}))$ can be thought as the representative of an equivalence class given by the corresponding indifference curve.

Proposition 8. *Consider an auction organized under the security design $(\mathcal{S}, \underline{s}(\mathcal{S}))$, and let $\Theta(\underline{s}(\mathcal{S})) \triangleq \{\theta \in \mathcal{V} \times R^+(\underline{s}(\mathcal{S})) : EU_{\mathcal{S}}(\theta, \underline{s}(\mathcal{S})) \geqslant 0\}$. Bidders' bidding strategy $\sigma_{\mathcal{S}}(\theta|k)$ is such that:*

(i) if $k = 1$, then $\sigma_{\mathcal{S}}(\theta|k) = \underline{s}(\mathcal{S})$ if $\theta \in \Theta(\underline{s}(\mathcal{S}))$, and $\sigma_{\mathcal{S}}(\theta|k) = 0$ otherwise; and

(ii) *if $k > 1$, then $\sigma_{\mathcal{S}}(\theta|k)$ is such that $EU_{\mathcal{S}}(\theta, \sigma_{\mathcal{S}}(\theta|k)) = 0$ if $\theta \in \Theta(\underline{s}(\mathcal{S}))$, and $\sigma_{\mathcal{S}}(\theta|k) = 0$ otherwise.*

Proof. The proof follows from immediately from Proposition 1 and the definition of $\Theta(\underline{s}(\mathcal{S}))$. ■

Proposition 9. *If bidders are homogeneously risk-averse, then when sellers' choice of security designs consist of a family of securities $\mathcal{S}$ and a reserve price $\underline{s}(\mathcal{S})$, the only possible equilibria that can arise are such that:*

(i) *both sellers choose $(\mathcal{S}, \underline{s}(\mathcal{S}))$, where $\mathcal{S} \neq \mathcal{S}^{co}$ is an equilibrium choice in the baseline model, and $\underline{s}(\mathcal{S})$ is a non-binding reserve price, i.e., $s = s_L$; or*

(ii) *both sellers choose $(\mathcal{S}^{co}, \underline{s}(\mathcal{S}^{co}))$, where $\mathcal{S}^{co}$ is an equilibrium in the baseline model and $\underline{s}(\mathcal{S}^{co})$ is a binding reserve price, i.e., $\underline{s}(\mathcal{S}^{co}) \in (s_L, s_H)$.*

That is, Proposition 9 states that a reserve price is relevant as an instrument to increase seller's revenue only when the seller is constrained by the upper bound of the steepness in the securities spectrum.

8.2 Auction format

We analyze a second-price auction because it induces a dominance-solvable game, allowing us to focus on bidders' entry strategies and sellers' choice of security design. For the first-price auction, characterizing the equilibrium is more difficult because it depends on bidders' beliefs about other competitors' multidimensional types. Nonetheless, for the case of homogeneously risk-averse bidders, Fioriti and Hernandez-Chanto (2022) obtained the equilibrium by assuming a single-crossing property in the Bernoulli utility function. The effect of the auction format when bidders are risk-averse is more noticeable, since, for any family of securities $\mathcal{S}$, the winner pays the deterministic security associated with her type in the first-price auction, whereas in the second-price auction she pays a random security determined by the bid of the second-highest bidder. Nonetheless, bidders' maximization problem in the second-price auction can be put in correspondence with the maximization problem in the first-price auction if the bidder chooses a security in the convex hull of $\mathcal{S}$.

For super-convex families (e.g., a call option), a first-price auction offers more insurance to risk-averse bidders relative to a second-price auction.[18] The reason is that the deterministic security in the first-price auction will be steeper than the random security in the second-price auction yielding the same expected payment to the seller. Hence, for super-convex families, the deterministic environment and the higher insurance provided in the first-price auction reinforces the higher ex-post extraction obtained by using a steeper security. Therefore, when sellers commit to use a second-price auction and bidders are homogeneously risk-averse, if there exists an equilibrium in which both sellers choose a family flatter than a call option, there will exist an equilibrium in which they choose a weakly steeper family if they commit to use a first-price auction.

For subconvex families (e.g., debt), the random security in the second-price auction is steeper than the corresponding deterministic security in the first-price auction. Thus, sellers face a trade-off between decreasing bidders' uncertainty (by providing a deterministic environment) and using a steeper security to extract higher surplus. As a consequence, it is not possible to guarantee the monotonicity of equilibrium in steepness across formats when the sellers choose a sub-convex family in the original equilibrium.

8.3 Risk-averse competing sellers

We analyzed risk-neutral sellers because we were interested in investigating how they compete for a population of (potentially heterogeneously) risk-averse bidders by means of their security designs, but abstracting from insurance concerns. Nonetheless, introducing risk-averse sellers would be realistic in settings in which sellers are small and not well-diversified, such as auctions to sell publishing rights or in class-action settlements. When sellers are risk-averse, the insurance goes in the opposite direction of surplus extraction in the steepness spectrum. Indeed, from the seller's perspective, debt and call options are the securities that provide, respectively, the highest and lowest insurance. Hence, when sellers are risk-averse, their incentives to "coordinate" on steeper families are weaker, as they need to balance the featured competition in the auction with the ex-post extraction and the insurance provided by the security design. A natural conjecture is that if there is an equilibrium in which sellers separate in their security

[18] A family $\mathcal{S}$ is super-convex (sub-convex) if it is steeper (flatter) than any nontrivial convex combination of the securities in $\mathcal{S}$—that is, if $\mathcal{S}$ is steeper (flatter) than its convex hull $Co(\mathcal{S})$. In turn, $\mathcal{S}$ is convex if its steepness is equal to the one of any other security in $Co(\mathcal{S})$.

design under risk neutrality, they would also do it under risk aversion. Nonetheless, this has to be formally studied as a generalization of our theoretical model and is suggested as future research.

9 Concluding remarks

Our paper provides a framework for studying how sellers compete for risk-averse bidders in security-bid auctions by means of their security-design choices. Such competition is more complex than the exhibited by sellers in standard cash auctions—in which sellers typically compete by choosing different reserve prices.

In our setup, each seller simultaneously chooses a feasible family of securities to conduct a second-price auction and commits to it. In turn, upon observing their risk-aversion type, bidders decide which auctions to join, and submit a security-bid within the family chosen by the corresponding seller.

We focus on a second-price auction, as it has the advantage that the induced game in the bidding stage is dominance-solvable. In particular, we show that each bidder's optimal strategy is to submit a security-bid that is equal to its reservation value (i.e., a security-bid such that the certainty equivalent of the induced lottery is equalized to zero), regardless of the family chosen by the seller and the number of competitors in the auction. This restriction buys us the necessary tractability to disentangle the different channels in which sellers' choices affect bidders' entry strategies; namely: (i) the number and composition of bidders in the auction; (ii) the ex-post surplus extracted and the insurance provided by the chosen security; and (iii) bidders' aggressiveness. To characterize the equilibrium we decouple these three effects and determine their feedback on seller's optimal strategies.

The interplay of these effects implies that bidders' entry strategies need to balance not only the competition effect that comes from the number of bidders in the auction (which has been largely studied in cash auctions), but also the extraction and the insurance provided by each security design. This is particularly important when bidders are heterogeneously risk-averse, since more-risk-averse agents become relatively more aggressive when a seller uses a steeper security design. In such a case, not only the number of competitors matter, but also their composition in terms of risk-aversion levels.

We show that equilibrium sellers' choice of securities depends on bidders' heterogeneity of risk aversion. In particular, when bidders are homogeneously risk-averse, the only equilibrium that survives is the one in which both sellers choose the same security design. That is, despite the fact that we do not force a symmetric equilibrium ex-ante, sellers are better off by "coordinating" on the steepness of the family chosen than by separating themselves in "steepness" to entice bidders through different combinations of insurance and surplus extraction. The reason for this result is that, under homogeneity, bidders are interested only in balancing the competition effect across auctions. Furthermore, we show that the steepness of the family in which sellers coordinate is increasing in bidders' risk aversion.

Meanwhile, when bidders are heterogeneously risk-averse, sellers separate themselves in the steepness spectrum to exploit bidders' different valuations of the insurance provided by each security design relative to its ex-post surplus extraction. Specifically, we show that one seller chooses a steeper security design to serve the more-risk-averse bidders—since they care more about the value of the insurance provided—whereas the other chooses a flatter design to serve the less-risk-averse bidders because they care more about the lower surplus extraction. To determine bidders' entry strategies when sellers totally separate we rely on the theory of monotone games under private information. This allows us to conclude that for any two families of securities, the probability of entering the auction run under the steeper family is weakly monotone in the risk-aversion level.

Finally, we discuss the role played by reserve prices in security-bid auctions and propose directions in which our analysis can be extended. In particular, we point out how our framework can be enriched by including risk-averse sellers and diverse auction formats. The further insights that might be captured in the analysis of such extensions would contribute to a more complete understanding how sellers compete in security-bid auctions.

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Appendices

A Proofs

A.1 Proof of Proposition 2

The first part of the Proposition can be found in Fioriti and Hernandez-Chanto (2022).

For the second part, it suffices to show that

$$E\mathcal{S}'(v, \sigma_{\mathcal{S}''}((v, \hat{r}))) - E\mathcal{S}''(v, \sigma_{\mathcal{S}''}((v, \hat{r}))) < E\mathcal{S}'(v, \sigma_{\mathcal{S}''}((v, r))) - E\mathcal{S}''(v, \sigma_{\mathcal{S}''}((v, r))). \quad \text{(A.1)}$$

Let z^* be the single-crossing point of $S'(\cdot, s')$ and $S''(\cdot, s'')$. Because more-risk-averse bidders suffer a greater marginal disutility when higher payoffs must be delivered under lower revenue realizations, and such marginal disutility is decreasing by the strict concavity of the utility function, we have that

$$\begin{aligned}&\mathbb{E}_Z[u(z - S'(z, s'); \hat{r})|z < z^*, v] - \mathbb{E}_Z[u(z - S''(z, s''); \hat{r})|z < z^*, v] \\ &> \mathbb{E}_Z[u(z - S'(z, s'); r)|z > z^*, v] - \mathbb{E}_Z[u(z - S''(z, s''); r)|z > z^*, v].\end{aligned}$$

Therefore, because sellers' expected revenue is continuous and increasing in the index of the security-bid—by virtue of Assumption 2—the result in (A.1) follows.

A.2 Proof of Lemma 3

For any profile of signals $\boldsymbol{v}_k = (v_i : i = 1, \ldots, k)$, denote by $v_k^{(\ell)}$ its ℓ-th order statistic.[19] We suppress the dependence on k, henceforth, to avoid cluttered notation. Because bidders are homogeneously risk-averse, for any family $\mathcal{S}$ and any $r \in \mathcal{R}^+$, we have that:

$$\begin{aligned}\mathcal{L}_{\mathcal{S}}^k((v, r)) &= \{(\boldsymbol{\theta}_k \in {\Theta^+}^k | v_j < v \ \forall j = 1, \ldots, k\}, \text{ and} \\ \hat{\sigma}_{\mathcal{S}'}(\boldsymbol{\theta}_k) &= \sigma_{\mathcal{S}}((v^{(1)}, r)|v^{(1)} < v).\end{aligned}$$

[19] When there is only one bidder, she will be the winner by default, and, so, will pay zero to the seller. This is equivalent to have the winning bidder who bids a security $S(\cdot, s)$ with $s > s_L$, and an artificial losing bidder with a type $\hat{\theta} = (\hat{v}, r_H)$ such that the induced bid is equal to s_L.

Because signals are independent, a winning bidder with signal v, who participates in an auction with $k+1$ bidders, has to pay the bid associated with the maximum signal $v^{(1)}|v^{(1)} < v$ among his k losing opponents. Thus, we can write the bidder's interim expected utility as

$$\mathcal{EU}(\mathcal{S}|r,k) = \int_{v_L}^{v_H} \left[\int_{v_L}^{v} EU_{\mathcal{S}}((v,r), \sigma_{\mathcal{S}}((v^{(1)},r)))d(F(v^{(1)})^k) \right] dF(v).$$

Differentiating the expression, we have

$$\mathcal{EU}_k(\mathcal{S}|r,k) = \int_{v_L}^{v_H} \left[\int_{v_L}^{v} EU_{\mathcal{S}}((v,r), \sigma_{\mathcal{S}}((v^{(1)},r)) \left(\frac{1}{k} + log(F(v^{(1)})) \right) d(F(v^{(1)})^k) \right] dF(v). \tag{A.2}$$

Now, since $F(v^{(1)})^k$ is the distribution of the first-order statistic of k independent signals,

$$\int_{v_L}^{v_H} d(F(v^{(1)})^k) = 1.$$

Taking its first derivative with respect to k yields:

$$\int_{v_L}^{v_H} F(v^{(1)})^{k-1} dF(v^{(1)}) + klog(F(v^{(1)}))F(v^{(1)})^{k-1} dF(v^{(1)}) = 0 \tag{A.3}$$

$$\Rightarrow \int_{v_L}^{v_H} \left(\frac{1}{k} + log(F(v^{(1)})) \right) d(F(v^{(1)})^k) = 0.$$

Now, notice that

$$\int_{v_L}^{v} \left(\frac{1}{k} + log(F(v^{(1)})) \right) d(F(v^{(1)})^k) < 0, \tag{A.4}$$

by virtue of (A.3) and the fact that $F(v^{(1)})$ is an increasing function of $v^{(1)}$. Thus, (A.2) is negative, for any v, because $\sigma_{\mathcal{S}}((v^{(1)},r))$ is also increasing in $v^{(1)}$—by Lemma 1—and $EU_{\mathcal{S}}((v,r),s)$ is decreasing in $s = \sigma_{\mathcal{S}}((v^{(1)},r))$.

Differentiating (A.2) again, $\mathcal{EU}_{kk}(\mathcal{S}|r,k)$ becomes

$$\int_{v_L}^{v_H} \left[\int_{v_L}^{v} EU_{\mathcal{S}}((v,r), \sigma_{\mathcal{S}}((v^{(1)},r)))log(F(v^{(1)})) \left(\frac{2}{k} + log(F(v^{(1)})) \right) d(F(v^{(1)})^k) \right] dF(v). \tag{A.5}$$

Because $log(F(v^{(1)})) \leqslant 0$ for any signal v, (A.5) is positive if $v^{(1)} < F^{-1}(e^{-2/k})$ and and negative if $v^{(1)} > F^{-1}(e^{-2/k})$.

Now, deriving (A.3) with respect to k yields

$$\int_{v_L}^{v_H} log(F(v^{(1)})) \left(\frac{2}{k} + log(F(v^{(1)}))\right) d(F(v^{(1)})^k) = 0.$$

This, along with the fact that (A.5) is negative if $v^{(1)} > F^{-1}(e^{-2/k})$, implies that

$$\int_{v_L}^{v} log(F(v^{(1)})) \left(\frac{2}{k} + log(F(v^{(1)}))\right) d(F(v^{(1)})^k) \geqslant 0$$

for any signal v. Hence, since $EU_{\mathcal{S}}((v,r), \sigma_{\mathcal{S}}((v^{(1)}, r)))$ is decreasing in $v^{(1)}$, the above inequality implies that $\mathcal{EU}_{kk}(\mathcal{S}|r,k) > 0$.

We can proceed analogously for the seller's expected revenue. Because bidders are homogeneously risk-averse, for any $r \in \mathcal{R}^+$, we can write

$$\mathcal{ES}(\mathcal{S}'|k) = k \int_{v_L}^{v_H} \int_{v_L}^{v} E\mathcal{S}((v,r), \sigma_{\mathcal{S}'}((v^{(1)}, r)))d(F(v^{(1)})^{k-1})\cdot$$

Deriving this expression with respect to k, $\mathcal{ES}_k(\mathcal{S}'|k)$ becomes

$$\int_{v_L}^{v_H} \int_{v_L}^{v} E\mathcal{S}((v,r), \sigma_{\mathcal{S}'}((v,r))) \left[\left(\frac{2k-1}{k-1} + klog(F(v^{(1)}))\right) d(F(v^{(1)})^{k-1})\right] dF(v) > 0, \quad \text{(A.6)}$$

since

$$\int_{v_L}^{v_H} \left(\frac{1}{k-1} + log(F(v^{(1)}))\right) d(F(v^{(1)})^{k-1}) = 0.$$

This proves that $\mathcal{ES}(\mathcal{S}'|k)$ is increasing for $k > 1$, and, in conjunction with Assumption 3, that $\mathcal{ES}(\mathcal{S}'|k)$ is increasing for all $k \geqslant 1$.

Deriving (A.6) again yields

$$\begin{aligned}\text{ES}_{kk}(\mathcal{S}'|k) = &\textstyle\int_{v_L}^{v_H} \int_{v_L}^{v} E\mathcal{S}((v,r), \sigma_{\mathcal{S}'}((v^{(1)}, r))) \left[\left(-\frac{1}{(k-1)^2} + \log F(v^{(1)})\right) d(F(v^{(1)})^{k-1})\right] dF(v) \\ &+ \textstyle\int_{v_L}^{v_H} \int_{v_L}^{v} E\mathcal{S}((v,r), \sigma_{\mathcal{S}'}((v^{(1)}, r))) \left[\left(\frac{2k-1}{k-1} + klog(F(v^{(1)}))\right)\left(\frac{1}{k-1} + \log F(v^{(1)})\right) d(F(v^{(1)})^{k-1})\right] dF(v).\end{aligned}$$

The first term is negative for any $k \geqslant 1$. Combining (A.4) and (A.6) with the fact that the seller's expected payment is increasing in $v^{(1)}$, it is possible to see that the second term is also negative. Then, $\mathcal{ES}_{kk}(\mathcal{S}'|k) < 0$.

A.3 Proof of Proposition 3

We can express the equation (MS), which gives the condition for the use of mixed strategies as

$$\sum_{k=0}^{I-1} \binom{I-1}{k} \left[\gamma'^k (1-\gamma')^{I-1-k} \mathcal{EU}(\mathcal{S}'|r,k) - (1-\gamma')^k \gamma'^{I-1-k} \mathcal{EU}(\mathcal{S}''|r,k)\right] = 0. \tag{A.7}$$

(We drop the dependence on $\mathcal{S}'$ and $\mathcal{S}''$ for simplicity.)

Denote the left-hand side of equation (A.7) by $\Phi(\gamma', \mathcal{S}', \mathcal{S}'')$ and notice that

$$\begin{aligned} \Phi(0, \mathcal{S}', \mathcal{S}'') &= \mathbb{E}_{\mathcal{V}}[EU_{\mathcal{S}'}((v,r), s_L)] - \mathcal{EU}(\mathcal{S}''|r, I-1) > 0 \\ \Phi(1, \mathcal{S}', \mathcal{S}'') &= \mathcal{EU}(\mathcal{S}'|r, I-1) - \mathbb{E}_{\mathcal{V}}[EU_{\mathcal{S}''}((v,r), s_L)] < 0. \end{aligned} \tag{A.8}$$

The first inequality in (A.8) follows immediately, while the second comes from Assumption 3. Differentiating $\Phi(\gamma', \mathcal{S}', \mathcal{S}'')$ with respect to γ', we have

$$\begin{aligned} \Phi_{\gamma'}(\gamma', \mathcal{S}', \mathcal{S}'') = &\sum_{k=0}^{I-1} \binom{I-1}{k} \left[\gamma'^k (1-\gamma')^{I-1-k} \left(\frac{k}{\gamma'} - \frac{I-1-k}{(1-\gamma')}\right) \mathcal{EU}(\mathcal{S}'|r,k)\right] \\ &+ \sum_{k=0}^{I-1} \binom{I-1}{k} \left[(1-\gamma')^k \gamma'^{I-1-k} \left(\frac{I-1-k}{\gamma'} - \frac{k}{1-\gamma'}\right) \mathcal{EU}(\mathcal{S}''|r,k)\right]. \end{aligned} \tag{A.9}$$

Notice that

$$\sum_{k=0}^{I-1} \binom{I-1}{k} \gamma'^k (1-\gamma')^{I-1-k} = \sum_{k=1}^{I} \binom{I-1}{k-1} \gamma'^{k-1} (1-\gamma')^{I-k} = 1.$$

Hence, deriving with respect to γ' on both sides, we have that

$$\sum_{k=0}^{I-1} \binom{I-1}{k} \left[\gamma'^k (1-\gamma')^{I-1-k} \left(\frac{k}{\gamma'} - \frac{I-1-k}{(1-\gamma')}\right)\right] = 0.$$

This sum is negative for all $k < \gamma'(I-1)$, and positive otherwise. Then, because $\mathcal{EU}(\mathcal{S}'|r,k)$ is a decreasing function of k, the first term in (A.9) is negative. By the same argument, the second term in (A.9) is also negative. Therefore, $\Phi_{\gamma'}(\gamma', \mathcal{S}', \mathcal{S}'')$ is a continuous and strictly decreasing function of γ', which implies that there is a unique value of γ' that solves (MS). This proves part (i).

For part (ii), observe that when $\mathcal{S}' = \mathcal{S}''$, $\mathcal{EU}(\mathcal{S}'|r,k) = \mathcal{EU}(\mathcal{S}''|r,k)$ for each k. Hence, setting $\gamma'(r|\mathcal{S}', \mathcal{S}'') = \gamma'(\mathcal{S}', \mathcal{S}'') = 1/2$ uniquely solves equation (A.7).

To prove part (iii), notice that as the risk aversion increases, the insurance embedded in steeper securities is more valuable to bidders. Thus, for any two families $\mathcal{S}'$ and $\mathcal{S}''$, such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$, we have that

$$\mathcal{EU}(\mathcal{S}'|\hat{r},k) > \mathcal{EU}(\mathcal{S}'|r,k) \quad \text{and} \quad \mathcal{EU}(\mathcal{S}''|\hat{r},k) < \mathcal{EU}(\mathcal{S}''|r,k)$$

for any $\hat{r}, r \in \mathcal{R}^+$ with $\hat{r} > r$. Hence, using the uniqueness of the solution of equation (A.7), it is straightforward to see that $\gamma'(\hat{r}|\mathcal{S}', \mathcal{S}'') > \gamma'(r|\mathcal{S}', \mathcal{S}'')$.

Finally, to prove part (iv), notice that when bidders are risk-neutral—i.e., when their risk-aversion level corresponds to the lower bound r_N—$\mathcal{EU}(\mathcal{S}'|r_N,k) < \mathcal{EU}(\mathcal{S}''|r_N,k)$ for each k. Hence, there exist $\gamma'(r_N|\mathcal{S}', \mathcal{S}'') < 1/2 < \gamma''(r_N|\mathcal{S}', \mathcal{S}'')$ that uniquely solves equation (A.7). Now, by Proposition 5 in Fioriti and Hernandez-Chanto (2022), we can guarantee the existence of a cutoff $r^L(\mathcal{S}', \mathcal{S}'') \in \mathcal{R}^+$ for any two families $\mathcal{S}'$ and $\mathcal{S}''$. This cutoff corresponds to the lowest level of risk aversion such that the insurance effect surpasses the extraction effect in equilibrium under monopoly. When there are two sellers, the cutoff $r^L(\mathcal{S}', \mathcal{S}'')$ becomes weakly lower due to the presence of competition, and, so, the same argument holds. Then, $\mathcal{EU}(\mathcal{S}'|r^L(\mathcal{S}', \mathcal{S}''),k) < \mathcal{EU}(\mathcal{S}''|r^L(\mathcal{S}', \mathcal{S}''),k)$ for each k. Thus, setting

$$\gamma'(r^L(\mathcal{S}', \mathcal{S}'')|\mathcal{S}', \mathcal{S}'') > 1/2 > \gamma''(r^L(\mathcal{S}', \mathcal{S}'')|\mathcal{S}', \mathcal{S}'')$$

uniquely solves equation (A.7).

Combining the result of (iii) with the continuity of $\gamma(\cdot|\mathcal{S}', \mathcal{S}'')$ guarantees the existence of $r_N < r(\mathcal{S}', \mathcal{S}'') < r_H$ such that $\gamma'(r(\mathcal{S}', \mathcal{S}'')|\mathcal{S}', \mathcal{S}'') = \gamma''(r(\mathcal{S}', \mathcal{S}'')|\mathcal{S}', \mathcal{S}'')$, by an application of the Intermediate Value Theorem.

A.4 Proof of Proposition 4

Towards proving the main result, it is useful to first prove the following lemma.

Lemma A.1. *For any family $\mathcal{S}$, the seller's expected revenue as a function of the expected number of bidders, $\mathbb{ES}(\mathcal{S}|\gamma)$, is always concave and strictly increasing in γ. Furthermore, it satisfies increasing differences between γ and the steepness of the security. That is, $\mathbb{ES}_\gamma(\mathcal{S}'|\gamma) - \mathbb{ES}_\gamma(\mathcal{S}''|\gamma) > 0$ for any family $\mathcal{S}'$ steeper than $\mathcal{S}''$.*[20]

Proof. Taking the derivative of the seller's expected revenue in (6) with respect to γ we obtain

$$\mathbb{ES}_\gamma(\mathcal{S}|\gamma) = \sum_{k=1}^{I} \binom{I}{k} \gamma^{k-1}(1-\gamma)^{I-k-1}(k - I\gamma)\mathcal{ES}(\mathcal{S}|k). \tag{A.10}$$

Notice that $\sum_{k=1}^{I} \binom{I}{k}\gamma^{k-1}(1-\gamma)^{I-k-1}(k - I\gamma) = 0$. Thus,

$$\begin{aligned}\mathbb{ES}_\gamma(\mathcal{S}|\gamma) = &\sum_{k=1}^{\lfloor I\gamma \rfloor} \binom{I}{k} \gamma^{k-1}(1-\gamma)^{I-k-1}(k - I\gamma)\left(\mathcal{ES}(\mathcal{S}|k) - \mathcal{ES}(\mathcal{S}|\lfloor I\gamma \rfloor)\right) \\ &+ \sum_{k=\lfloor I\gamma \rfloor+1}^{I} \binom{I}{k} \gamma^{k-1}(1-\gamma)^{I-k-1}(k - I\gamma)\left(\mathcal{ES}(\mathcal{S}|k) - \mathcal{ES}(\mathcal{S}|\lfloor I\gamma \rfloor)\right).\end{aligned} \tag{A.11}$$

Given that $\mathcal{ES}(\mathcal{S}|k)$ is increasing in k, it follows that $\mathbb{ES}_\gamma(\mathcal{S}|\gamma) > 0$.

Recall that for any $\mathcal{S}'$ steeper than $\mathcal{S}''$, $ES((v,r), \sigma_{\mathcal{S}'}((v,r))) > ES((v,r), \sigma_{\mathcal{S}''}((v,r)))$. Hence, using (A.6), $\mathcal{ES}_k(\mathcal{S}'|k) > \mathcal{ES}_k(\mathcal{S}''|k)$, which implies that

$$\left|\mathcal{ES}(\mathcal{S}'|k) - \mathcal{ES}(\mathcal{S}'|[I\gamma])\right| > \left|\mathcal{ES}(\mathcal{S}''|k) - \mathcal{ES}(\mathcal{S}''|[I\gamma])\right|.$$

As a consequence, using (A.11), we can directly verify that $\mathbb{ES}_\gamma(\mathcal{S}'|\gamma) - \mathbb{ES}_\gamma(\mathcal{S}''|\gamma) > 0$.

Deriving again with respect to γ, we obtain

$$\mathbb{ES}_{\gamma\gamma}(\mathcal{S}|\gamma) = \sum_{k=1}^{I} \lambda(\gamma, I, k)\mathcal{ES}(\mathcal{S}|k), \tag{A.12}$$

where

$$\lambda(\gamma, I, k) = \binom{I}{k}\gamma^k(1-\gamma)^{I-k}\left(\left(\frac{k}{\gamma} - \frac{I-k}{1-\gamma}\right)^2 - \frac{k}{\gamma^2} - \frac{I-k}{(1-\gamma)^2}\right).$$

[20]The differentiability of $\mathbb{ES}(\mathcal{S}|\gamma)$ with respect to γ is satisfied by the assumptions on the distribution H.

Now, consider the following probability distribution functions:

$$\mathbb{P}_{M_1}(K = k) = \begin{cases} \frac{\lambda(\gamma, I, k)}{\sum_{k<k_1, k>k_2} \lambda(\gamma, I, k)} & \text{if } k < k_1 \text{ and } k > k_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}_{M_2}(K = k) = \begin{cases} \frac{-\lambda(\gamma, I, k)}{\sum_{k<k_1, k>k_2} \lambda(\gamma, I, k)} & \text{if } k \in (k_1, k_2). \\ 0 & \text{otherwise} \end{cases}$$

Following the techniques introduced by Gorbenko and Malenko (2011, p. 1837), it is straightforward to show that $M_2(k)$ second-order stochastically dominates $M_1(k)$. Thus, for any concave transformation $\varphi(\cdot)$,

$$\sum_{k=0}^{I} \lambda(\gamma, I, k)\varphi(k) \leqslant 0. \tag{A.13}$$

By Lemma 3, $\mathcal{ES}(\mathcal{S}|k)$ is concave in k. Hence, using (A.12) and (A.13) it follows that $\mathbb{ES}_{\gamma\gamma}(\mathcal{S}|\gamma) \leqslant 0$. ■

Uniqueness We prove the uniqueness of the symmetric equilibrium by contradiction. Suppose, without loss of generality, that Seller 1 and Seller 2 choose, respectively, $\mathcal{S}'$ and $\mathcal{S}''$ with $\mathcal{S}'$ steeper than $\mathcal{S}''$. Let $\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')$ be the minimum entry probability such that Seller 1 does not have an incentive to deviate to choose family $\mathcal{S}''$. If Seller 1 were to deviate, then, by Proposition 3, bidders' optimal strategy would be to enter both auctions with probability 1/2. Hence, $\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')$ satisfies

$$\mathbb{ES}(\mathcal{S}'|\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')) = \mathbb{ES}(\mathcal{S}''|1/2). \tag{A.14}$$

Similarly, $\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')$ corresponds to the minimum entry probability such that Seller 2 does not have an incentive to deviate to choose $\mathcal{S}'$. Thus,

$$\mathbb{ES}(\mathcal{S}''|\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')) = \mathbb{ES}(\mathcal{S}'|1/2). \tag{A.15}$$

As a consequence, sellers' choices constitute an equilibrium if and only if

$$\gamma'(\mathcal{S}', \mathcal{S}'') \geqslant \tilde{\gamma}'(\mathcal{S}', \mathcal{S}'') \text{ and } \gamma''(\mathcal{S}', \mathcal{S}'') \geqslant \tilde{\gamma}''(\mathcal{S}', \mathcal{S}''). \tag{A.16}$$

Now, by Lemma A.1, $\mathbb{ES}(\mathcal{S}|\gamma)$ satisfies increasing differences between γ and the steepness of the security. Hence, we have that

$$\mathbb{ES}(\mathcal{S}'|1/2) - \mathbb{ES}(\mathcal{S}'|\gamma'(\mathcal{S}', \mathcal{S}'')) \geqslant \mathbb{ES}(\mathcal{S}''|1/2) - \mathbb{ES}(\mathcal{S}''|\gamma'(\mathcal{S}', \mathcal{S}'')).$$

Using the equivalence in (A.14), this implies that

$$\mathbb{ES}(\mathcal{S}''|\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|1/2) \geqslant \mathbb{ES}(\mathcal{S}'|\gamma'(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|\gamma'(\mathcal{S}', \mathcal{S}'')). \tag{A.17}$$

Proceeding analogously for $\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')$ we obtain that

$$\mathbb{ES}(\mathcal{S}'|1/2) - \mathbb{ES}(\mathcal{S}'|\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')) \leqslant \mathbb{ES}(\mathcal{S}'|\gamma''(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|\gamma''(\mathcal{S}', \mathcal{S}'')). \tag{A.18}$$

Subtracting (A.18) from (A.17), we have

$$\begin{aligned}
&[\mathbb{ES}(\mathcal{S}''|\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|1/2)] - [\mathbb{ES}(\mathcal{S}'|1/2) - \mathbb{ES}(\mathcal{S}'|\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')] \\
\geqslant &[\mathbb{ES}(\mathcal{S}'|\gamma'(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|\gamma'(\mathcal{S}', \mathcal{S}''))] - [\mathbb{ES}(\mathcal{S}'|\gamma''(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|\gamma''(\mathcal{S}', \mathcal{S}'')] .
\end{aligned}$$

The second line in the inequality above is positive by the increasing-differences property of $\mathbb{ES}(\mathcal{S}|\gamma)$ between the steepness of $\mathcal{S}$ and the entry probability γ. Thus,

$$\mathbb{ES}(\mathcal{S}''|\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}''|1/2) \geqslant \mathbb{ES}(\mathcal{S}'|1/2) - \mathbb{ES}(\mathcal{S}'|\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'').$$

Therefore, because $\mathcal{S}'$ is steeper than $\mathcal{S}''$ and $\mathbb{ES}(\mathcal{S}|\gamma)$ is concave in γ, we have that

$$\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'') - 1/2 \geqslant \tilde{\gamma}'(\mathcal{S}', \mathcal{S}'') - 1/2.$$

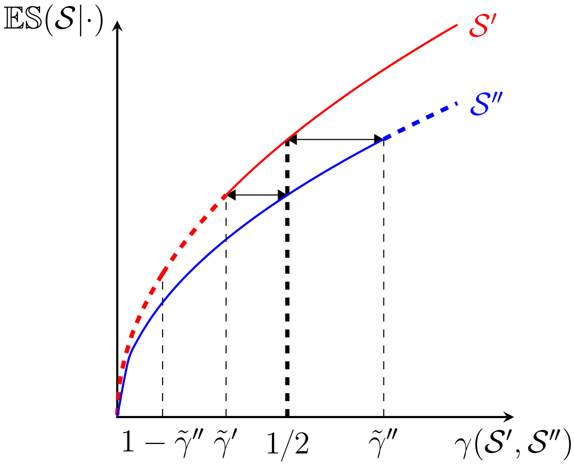

Figure A.1: **Increasing differences between steepness and bidders' entry probability**. By the increasing difference argument, the distance between $\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')$ and 1/2 is greater than the distance between $\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'')$ and 1/2.

Therefore,

$$\tilde{\gamma}''(\mathcal{S}', \mathcal{S}'') > 1 - \tilde{\gamma}'(\mathcal{S}', \mathcal{S}'').$$

Consequently, if $\gamma''(\mathcal{S}', \mathcal{S}'') \geqslant \tilde{\gamma}''(\mathcal{S}', \mathcal{S}'')$, it follows that:

$$\gamma''(\mathcal{S}', \mathcal{S}'') > 1 - \tilde{\gamma}'(\mathcal{S}', \mathcal{S}'').$$

By definition, $\gamma''(\mathcal{S}', \mathcal{S}'') = 1 - \gamma'(\mathcal{S}', \mathcal{S}'')$, which implies that

$$1 - \gamma'(\mathcal{S}', \mathcal{S}'') > 1 - \tilde{\gamma}'(\mathcal{S}', \mathcal{S}'') \text{ and}$$
$$\tilde{\gamma}'(\mathcal{S}', \mathcal{S}'') > \gamma'(\mathcal{S}', \mathcal{S}''),$$

which contradicts the equilibrium condition in (A.16). The argument of the proof is illustrated in Figure A.1.

A.5 Proof of Proposition 5

Let us consider a strategy profile in which both sellers choose the same family $\mathcal{S}$. Hence, the net payoff of deviating to a (weakly) steeper family $\mathcal{S}'$ can be expressed as follows:

$$\bar{\xi}(\mathcal{S}'|\mathcal{S}) = \mathbb{ES}(\mathcal{S}'|\gamma'(\mathcal{S}, \mathcal{S}')) - \mathbb{ES}(\mathcal{S}|1/2).$$

Similarly, the net payoff of deviating to a (weakly) flatter family $\mathcal{S}''$ can expressed as:

$$\underline{\xi}(\mathcal{S}''|\mathcal{S}) \triangleq \mathbb{ES}(\mathcal{S}'|\gamma''(\mathcal{S}, \mathcal{S}'')) - \mathbb{ES}(\mathcal{S}|1/2).$$

Following Proposition 2 in Gorbenko and Malenko (2011), there always exists an equilibrium in which all sellers choose the same family of securities $\mathcal{S}$ which can be either interior (i.e., stepper than a debt family or steeper than call option family) or lying on the boundary. Hence, by definition, $\bar{\xi}(\mathcal{S}'|\mathcal{S}) \leqslant 0$ for all $\mathcal{S}' \succ \mathcal{S}$ and $\underline{\xi}(\mathcal{S}''|\mathcal{S}) \leqslant 0$ for all $\mathcal{S}'' \succ \mathcal{S}$.

We first consider an interior equilibrium under risk neutrality. By Proposition 3, we have that, as bidders' risk aversion increases starting from r_N, the induced entry probability, $\gamma(r|\mathcal{S}, \mathcal{S}')$, increases for all $\mathcal{S}'$ steeper than $\mathcal{S}$. Analogously, $\gamma(r|\mathcal{S}, \mathcal{S}'')$ decreases for all $\mathcal{S}''$ flatter than $\mathcal{S}$. In addition, the seller's expected payment decreases relatively less for steeper families because bidders behave relatively more aggressively as they become more risk averse (cf. Proposition 2). This implies that, for any $\mathcal{S}' \succ \mathcal{S}''$, the distance between $\mathbb{ES}(\mathcal{S}'|\gamma)$ and $\mathbb{ES}(\mathcal{S}''|\gamma)$ increases in the entry probability γ, and consequently, $\underline{\xi}(\mathcal{S}''|\mathcal{S})$ decreases, while $\bar{\xi}(\mathcal{S}'|\mathcal{S})$ increases. Therefore, if any interior symmetric equilibrium is no longer sustainable due an increase in bidders' risk aversion, a new equilibrium under a steeper family would emerge.

We will now show that any interior symmetric equilibrium can fail for some sufficiently high $r \in \mathcal{R}^+$. Fix a symmetric equilibrium under a family $\mathcal{S}$ that is steeper than debt and flatter than a call option. Following Proposition 3, there exists an $r \in \mathcal{R}^+$ sufficiently high, such that $\gamma(r|\mathcal{S}, \mathcal{S}') < 1/2$, which by definition implies that $\bar{\xi}(\mathcal{S}'|\mathcal{S}) > 0$ and, in turn, that the equilibrium can be no longer sustained. Importantly, such level of risk aversion corresponds to an upper bound, since the "destruction of the equilibrium" can happen at lower levels of risk aversion because the distance between $\mathbb{ES}(\mathcal{S}'|\cdot)$ and $\mathbb{ES}(\mathcal{S}|\cdot)$ increases in r. Indeed, if this distance becomes high enough, it could be the case that, despite having $\gamma(r|\mathcal{S}, \mathcal{S}') > 1/2$, $\bar{\xi}(\mathcal{S}'|\mathcal{S}) > 0$, giving bidders an incentive to deviate to $\mathcal{S}'$.

Now, if the symmetric equilibrium under risk neutrality occurs when both sellers select a debt family, we can easily replicate the previous analysis by disregarding $\underline{\xi}$. Finally, if the equilibrium occurs when both sellers choose a call-option family, it can survive when the level of risk aversion increases due to two effects: (i) the entry probability increases by virtue of the higher insurance provided to bidders; and (ii) the seller's expected payment is less affected by the loss of bidders' competitiveness due to their risk aversion, since they become more aggressively under steeper families.

A.6 Proof of Proposition 6

A key concept in our proof is the notion of the Single-Crossing Condition (SCC) in the auction-entry game, which, in turn, depends on the Single-Crossing Property (SCP) on bidders' unconditional interim expected utility. We define both below.

Definition A.1. *Bidders' unconditional interim expected utility satisfies the Single-Crossing Property (SCP) if for all $r' > r''$, $\mathbb{EU}(\mathcal{S}'|r'', \mathcal{S}'') - \mathbb{EU}(\mathcal{S}''|r'', \mathcal{S}') \geqslant 0$ implies $\mathbb{EU}(\mathcal{S}'|r', \mathcal{S}'') - \mathbb{EU}(\mathcal{S}''|r', \mathcal{S}') \geqslant 0$.*

Definition A.2. *The auction-entry game is said to satisfy the Single-Crossing Condition (SCC) if for each bidder i, whenever every opponent j uses a strategy η_j that is non-decreasing, bidder i's unconditional interim expected utility in (10) satisfies the Single-Crossing Property (SCP).*

That is, the unconditional interim expected utility satisfies the SCP if SCC holds whenever the bidders' follow a threshold strategy in their risk-aversion type.

Lemma A.2. *For any two families $\mathcal{S}'$ and $\mathcal{S}''$, such that $\mathcal{S}'$ is steeper than $\mathcal{S}''$, the auction-entry game satisfies the SCC.*

Proof. Suppose that all bidders $j \neq i$ play non-decreasing strategies described by the vector of cutoff values $\boldsymbol{r}_{-i}$. We need to analyze two cases. The first one is trivial and occurs when either $\mathcal{EU}(\mathcal{S}'|r_i, \boldsymbol{r}_{-i}) > \mathcal{EU}(\mathcal{S}''|r_i, \boldsymbol{r}_{-i})$ or $\mathcal{EU}(\mathcal{S}'|r_i, \boldsymbol{r}_{-i}) < \mathcal{EU}(\mathcal{S}''|r_i, \boldsymbol{r}_{-i})$ for all $r_i \in \mathcal{R}^+$. In such a case, it is easy to verify that the SCP is satisfied since the preference of one auction over the other is preserved as r increases. The second case is more interesting. Here, depending on the value of r_i, $\mathcal{EU}(\mathcal{S}'|r_i, \boldsymbol{r}_{-i})$ can be either higher or lower than $\mathcal{EU}(\mathcal{S}''|r_i, \boldsymbol{r}_{-i})$. Nonetheless, by continuity, there exists a value $\tilde{r}_i$ such that $\mathcal{EU}(\mathcal{S}'|\tilde{r}_i, \boldsymbol{r}_{-i}) = \mathcal{EU}(\mathcal{S}''|\tilde{r}_i, \boldsymbol{r}_{-i})$. Additionally, due to the presence of the insurance effect, there exists a value $\epsilon > 0$ sufficiently small such that $\mathcal{EU}(\mathcal{S}'|\tilde{r}_i + \epsilon, \boldsymbol{r}_{-i}) > \mathcal{EU}(\mathcal{S}''|\tilde{r}_i + \epsilon, \boldsymbol{r}_{-i})$ and $\mathcal{EU}(\mathcal{S}'|\tilde{r}_i - \epsilon, \boldsymbol{r}_{-i}) < \mathcal{EU}(\mathcal{S}''|\tilde{r}_i - \epsilon, \boldsymbol{r}_{-i})$. Now, by Proposition 2, the handicap in competitiveness of any risk-averse bidder decreases when steeper securities are used; and the absolute value of such a decrement is increasing in the bidder's risk aversion. Consequently, $\mathcal{EU}(\mathcal{S}'|\tilde{r}_i, \boldsymbol{r}_{-i})$ decreases less (more) than $\mathcal{EU}(\mathcal{S}''|\tilde{r}_i, \boldsymbol{r}_{-i})$ as the risk aversion increases (decreases). Therefore, our previous argument can be extended—via continuity—to show that $\mathcal{EU}(\mathcal{S}'|r_i, \boldsymbol{r}_{-i}) > \mathcal{EU}(\mathcal{S}''|r_i, \boldsymbol{r}_{-i})$ for all $r_i > \tilde{r}_i$ and $\mathcal{EU}(\mathcal{S}'|r_i, \boldsymbol{r}_{-i}) < \mathcal{EU}(\mathcal{S}''|r_i, \boldsymbol{r}_{-i})$ for all $r_i < \tilde{r}_i$. ■

The intuition behind Lemma A.2 hinges on the interplay among the extraction, competition, and insurance effects. Indeed, bidders follow a non-decreasing strategy if there exists a risk-aversion threshold such that for bidders with risk-aversion types greater than the threshold, the insurance effect surpasses the other two effects. If this holds, it is immediate to notice that the unconditional interim expected utility satisfies the SCP.

Now, for any two families $\mathcal{S}'$ and $\mathcal{S}''$, we define

$$BR(r_i, r^0|\mathcal{S}', \mathcal{S}'') \triangleq \underset{\mathcal{S}\in\{\mathcal{S}', \mathcal{S}''\}}{\operatorname{argmax}} \mathcal{EU}(\mathcal{S}|r_i, r^0) \tag{A.19}$$

as the best-response correspondence of a bidder with risk aversion r_i, conditional on her opponents following a cutoff strategy consistent with r^0. By Lemma 1 in Milgrom and Shannon (1994), $BR(r_i, r^0|\mathcal{S}', \mathcal{S}'')$ is non-decreasing in the strong set order, which implies that there is a non-decreasing selection

$$\delta(r_i|\mathcal{S}', \mathcal{S}'') \in BR(r_i, r^0|\mathcal{S}', \mathcal{S}'')$$

for all $r_i \in \mathcal{R}^+$.[21] This selection can be represented by a cutoff $\tilde{r}$ such that bidders follow a non-decreasing strategy consistent with $\tilde{r}$. Therefore, the set of of all cutoff values that represent best-response non-decreasing strategies can be defined as follows:

$$\mathcal{H}(r^0|\mathcal{S}', \mathcal{S}'') \triangleq \{\tilde{r} \in \mathcal{R}^+ \mid \exists \eta(\cdot|\mathcal{S}', \mathcal{S}'') \text{ consistent with } \tilde{r} : \forall r_i \in \mathcal{R}^+, \eta(r_i|\mathcal{S}', \mathcal{S}'') \in BR(r_i, r^0|\mathcal{S}', \mathcal{S}'')\}.$$

[21]For two sets of real numbers A and B, we say that $A \geqslant_{sso} B$, which is read as "A is greater than or equal to B in the Strong Set Order (SSO)" if for any $a \in A$ and $b \in B$, $\min\{a, b\} \in B$ and $\max\{a, b\} \in A$.

Following Athey (2001), for any two families $\mathcal{S}', \mathcal{S}''$, with $\mathcal{S}'$ steeper than $\mathcal{S}''$, $\mathcal{H}(\cdot|\mathcal{S}', \mathcal{S}'') : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ is convex if $BR(\cdot, \cdot|\mathcal{S}', \mathcal{S}'')$ is non-decreasing in the strong set order. Furthermore, given that $\mathcal{R}^+$ is a compact and convex set, Kakutani's fixed point theorem can be applied. Hence, because $\mathcal{H}$ is non empty and convex, it is necessary to only show that $\mathcal{H}$ has a closed graph to apply the theorem. By definition, $\mathcal{EU}(\mathcal{S}|r_i, r)$ is continuous in r, and, thus, SSC holds in our environment in virtue of Lemma A.2. Then, consider a sequence $\{r_1^k, r_2^k\}$ that converges to (r_1, r_2) such that $r_2^k \in \mathcal{H}(r_1^k)$ for all k. Since r_2^k converges to r_2, there exists a K such that, for all $k > K$, $\mathcal{EU}(\mathcal{S}'|r_i, r_1^k) \geqslant \mathcal{EU}(\mathcal{S}''|r_i, r_1^k)$ for some $r_i > r_2^K$. Therefore, by continuity, $\mathcal{EU}(\mathcal{S}'|r_i, r_1) \geqslant \mathcal{EU}(\mathcal{S}''|r_i, r_1)$ for any $r_i > r_2$. By the same argument, for all $k > K$, $\mathcal{EU}(\mathcal{S}'|r_i, r_1^k) \leqslant \mathcal{EU}(\mathcal{S}''|r_i, r_1^k)$ for any $r_i < r_2^K$, which implies that $\mathcal{EU}(\mathcal{S}'|r_i, r_1) \leqslant \mathcal{EU}(\mathcal{S}''|r_i, r_1)$ if $r_i < r_2$. It follows that $r_2 \in \mathcal{H}(r_1)$, and, thus, $\mathcal{H}$ has a closed graph. As a consequence, there exists a fixed point $r(\mathcal{S}', \mathcal{S}'')$ such that for all bidder i, $BR(r_i, r(\mathcal{S}', \mathcal{S})|\mathcal{S}', \mathcal{S}'') = r(\mathcal{S}', \mathcal{S}'')$.

A.7 Proof of Proposition 7

We first state and prove two lemmas regarding the behavior of (i) survival functions, and (ii) the first- and second-order statistics of random vectors.

Concavity (convexity) of the survival function

Lemma A.3. *For any distribution Λ of a random variable $x \in [x_L, x_H]$, that is absolutely continuous with density λ, we have that:*

(i) if the hazard rate $\psi(x) \triangleq \frac{\lambda(x)}{1-\Lambda(x)}$ is decreasing, the survival function $\bar{\Lambda}$ is both convex and log convex; and

(ii) if the hazard rate is increasing, the survival function $\bar{\Lambda}$ is log concave; if, additionally, there exists a value $x^* \in (x_L, x_H) : \frac{d}{dx}\psi(x) > \psi(x)^2$, then $\bar{\Lambda}$ is concave for all $x > x^*$, and otherwise convex.

Proof.

Notice that the distribution Λ can be written in terms of its hazard rate ψ as follows:

$$\Lambda(x) = 1 - exp\left(-\int_{x_L}^{x} \psi(x)dx\right).$$

This implies that

$$log(\bar{\Lambda}(x)) = -\int_{x_L}^{x} \psi(x).$$

Applying Leibinz derivation rule, we obtain that

$$\begin{aligned} \frac{d}{dx} log(\bar{\Lambda}(x)) &= -\psi(x) \\ \frac{d^2}{dx^2} log(\bar{\Lambda}(x)) &= -\frac{d}{dx}\psi(x). \end{aligned} \tag{A.20}$$

From the last expression, we can conclude that if the hazard rate is decreasing (increasing), the survival function $\bar{\Lambda}$ is log-convex (log-concave).

Now, we can express the second derivative of $\bar{\Lambda}(x)$ as follows:

$$\begin{aligned} \frac{d^2}{dx^2}\bar{\Lambda}(x) &= \bar{\Lambda}(x)\frac{d^2}{dx^2} \\ &= log(\bar{\Lambda}(x)) + \frac{1}{\bar{\Lambda}(x)}\left(\frac{d}{dx}\bar{\Lambda}(x)\right)^2 \\ &= \bar{\Lambda}(x)\left[\frac{d^2}{dx^2} log(\bar{\Lambda}(x)) + \left(\frac{1}{\bar{\Lambda}(x)}\frac{d}{dx}\bar{\Lambda}(x)\right)^2\right] \\ &= \bar{\Lambda}(x)\left[\frac{d}{dx^2} log(\bar{\Lambda}(x)) + \left(\frac{d}{dx} log(\bar{\Lambda}(x))\right)^2\right] \\ &= -\bar{\Lambda}(x)\left[\frac{d}{dx}\psi(x) - \psi(x)^2\right]. \end{aligned} \tag{A.21}$$

Then, if the hazard rate $\psi(x)$ is increasing, we can define

$$x^* = \inf\left\{x \in (x_L, x_H) : \frac{d}{dx}\psi(x) > \psi(x)^2\right\}. \tag{A.22}$$

Therefore, the survival function is concave for all values $x > x^*$, and, otherwise, convex.

■

We can use this result to show how the expected number of participant bidders change when sellers change their security design.

Lemma A.4. *Let $X^{(\ell:I)}$ be the ℓ-highest order statistic of I random variables $X_1, X_2, \ldots, X_I$ drawn independently and identically from a distribution Λ. Hence, the highest-order statistic is* *concave in I. Furthermore, if the survival function $\bar{\Lambda}$ is sufficiently concave, or the sample I is sufficiently large, the second-highest order statistic is also concave in I.*

Proof. Let $\mathbb{E}[X^{(\ell:I)}]$ be the expected value of the ℓ-smallest order statistic (conversely, the $I-\ell+1$-largest statistic) for $\ell = 1, \dots I$. Hence, following Lemma 2 in David (1997), we have that:

$$\mathbb{E}[X^{(\ell+1:I)}] - \mathbb{E}[X^{(\ell:I-1)}] = \binom{I-1}{\ell} \int_{x_L}^{x_H} \Lambda^{\ell}(x)(1-\Lambda(x))^{I-\ell} dx. \tag{A.23}$$

Hence, setting $\ell = I-1$ we have that:

$$\mathbb{E}[X^{(I:I)}] - \mathbb{E}[X^{(I-1:I-1)}] = \binom{I-1}{I-1} \int_{x_L}^{x_H} \Lambda^{I-1}(x)(1-\Lambda(x)) dx.$$

Applying the same recursion with a sample size of $I+1$ we have that:

$$\mathbb{E}[X^{(I+1:I+1)}] - \mathbb{E}[X^{(I:I)}] = \binom{I}{I} \int_{x_L}^{x_H} \Lambda^{I}(x)(1-\Lambda(x)) dx.$$

Therefore, since Λ is a distribution,

$$\int_{x_L}^{x_H} \Lambda^{I}(x)(1-\Lambda(x)) dx < \int_{x_L}^{x_H} \Lambda^{I-1}(x)(1-\Lambda(x)) dx,$$

and, thus, $\mathbb{E}[X^{(I:I)}]$ is increasingly concave in the sample size I.

Repeating the process by replacing $\ell = I-2$ in (A.23), and then, augmenting the sample size by one observation, we get that

$$\mathbb{E}[X^{(I-1:I)}] - \mathbb{E}[X^{(I-2:I-1)}] = \binom{I-1}{I-2} \int_{x_L}^{x_H} \Lambda^{I-2}(x)(1-\Lambda(x))^2 dx$$
$$\mathbb{E}[X^{(I-1:I+1)}] - \mathbb{E}[X^{(I-1:I)}] = \binom{I}{I-1} \int_{x_L}^{x_H} \Lambda^{I-1}(x)(1-\Lambda(x))^2 dx.$$

Hence, $\mathbb{E}[X^{(I-1:I)}]$ is concave in I if and only if

$$I \int_{x_L}^{x_H} \Lambda^{I-1}(x)(1-\Lambda(x))^2 < (I-1) \int_{x_L}^{x_H} \Lambda^{I-2}(x)(1-\Lambda(x))^2 dx. \tag{A.24}$$

Condition (A.24) is satisfied if the survival function $\bar{\Lambda}(x)$ is sufficiently concave (equivalently, if the value x^* in (A.22) is sufficiently low) or if the sample size, I, is sufficiently large. ■

Lemma A.3 shows the sufficient conditions for the concavity (convexity) of an arbitrary survival function. Similarly, A.4 shows the conditions for the concavity of the first- and second-order statistics of a random vector. Proposition 7 provides those conditions, and, thus, guarantees the concavity of $\bar{G}$ and $\bar{F}$, and also the concavity of $V^{(1:I)}$ and $V^{(2:I)}$. Additionally, the statement of the proposition requires that for $\ell \in \{1, 2\}$

$$\bar{c}^{\ell}(I'(\mathcal{S}', \mathcal{S}'')) \geqslant \lim_{I \to I'(\mathcal{S}', \mathcal{S}'')^{+}} \left| \frac{\partial \mathbb{E}[V^{(I-\ell+1:I)}]}{\partial I} \right| \geqslant \lim_{I \to I''(\mathcal{S}', \mathcal{S}'')^{-}} \left| \frac{\partial \mathbb{E}[V^{(I-\ell+1:I)}]}{\partial I} \right| \geqslant \bar{c}^{\ell}(I''(\mathcal{S}', \mathcal{S}'')). \tag{A.25}$$

Condition (A.25) indicates that the expected first- and second-order statistics are bounded within certain limits when the expected number of bidders decreases below $I''(\mathcal{S}', \mathcal{S}'')$—i.e., the expected number of entrants attracted by the $\mathcal{S}''$-auction—but remains within the range of the number of bidders attracted by the $\mathcal{S}'$-auction, $I'(\mathcal{S}', \mathcal{S}'')$. These bounds are defined by the constants $\bar{c}(I(\mathcal{S}', \mathcal{S}''))$ and $\bar{c}(I(\mathcal{S}', \mathcal{S}''))$. Consequently, the competition effect is more sensitive in this region.

Seller's competition and extraction effects As we decomposed bidders' expected payoffs between the extraction and insurance effects, it is also useful to decompose sellers' expected payoffs between the extraction and competition effects.

Definition A.3. *Suppose that sellers are playing the initial profile of strategies $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$. The extraction effect that Seller $j \in \{1, 2\}$ obtains by deviating and choosing the family $\hat{\mathcal{S}}_j$ is defined as*

$$\Delta \mathcal{ES}^{e}(\mathcal{S}_j, \hat{\mathcal{S}}_j | \langle \mathcal{S}_1, \mathcal{S}_2 \rangle) \triangleq \mathcal{ES}(\hat{\mathcal{S}}_j | r(\mathcal{S}_1, \mathcal{S}_2)) - \mathcal{ES}(\mathcal{S}_j | r(\mathcal{S}_1, \mathcal{S}_2)), \tag{EE_S}$$

and the competition effect is defined as

$$\Delta \mathcal{ES}^{c}(\mathcal{S}_j, \hat{\mathcal{S}}_j | \langle \mathcal{S}_1, \mathcal{S}_2 \rangle) \triangleq \mathcal{ES}(\hat{\mathcal{S}}_j | r(\hat{\mathcal{S}}_j, \mathcal{S}_{-j})) - \mathcal{ES}(\hat{\mathcal{S}}_j | r(\mathcal{S}_j, \mathcal{S}_{-j})). \tag{CE}$$

The extraction effect fixes the "population" of bidders corresponding to the original profile of strategies $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ and computes the difference in extraction for seller $j \in \{1, 2\}$ when he switches from the family $\mathcal{S}_j$ to the family $\hat{\mathcal{S}}_j$. Meanwhile, the competition effect computes the difference in extraction under the new family due to the change in the population of bidders.

To determine sellers' equilibrium strategies, suppose that they play the strategy profile $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}', \mathcal{S}'' \rangle$ where $\mathcal{S}'$ is steeper than $\mathcal{S}''$. By Proposition 6, there exists a threshold $r(\mathcal{S}', \mathcal{S}'')$ such that bidders with risk aversion lower than $r(\mathcal{S}', \mathcal{S}'')$ join the $\mathcal{S}''$-auction, and those with risk aversion greater than $r(\mathcal{S}', \mathcal{S}'')$ join the $\mathcal{S}'$-auction.

Notice that if r_H is sufficiently bounded, the effects on bidder's expected utility that come from the competition and extraction effects are stronger than the effect that comes from insurance. This is important to take into account at the time to analyze deviations, since we need to consider how the extraction, insurance, and competition effects must change to make the *marginal bidder* indifferent in the new alleged equilibrium.

Seller 1's possible deviations If Seller 1 (the one choosing the steeper family) decides to deviate from the prescribed strategy, there are three possible cases we must analyze.

1.i) Deviation to a family $\hat{\mathcal{S}}'$ that is steeper than $\mathcal{S}'$. In this case, the new cutoff value $r(\hat{\mathcal{S}}', \mathcal{S}'')$ requires to lower the competition coming from the most risk-averse bidders to balance the higher surplus extraction—net of the greater insurance provided. This leads Seller 1 to lose all bidders with risk aversion types $r \in [r(\mathcal{S}', \mathcal{S}''), r(\hat{\mathcal{S}}', \mathcal{S}'')]$. That is, Seller 1 would lose, in expectation, $\Delta I'(\hat{\mathcal{S}}', \mathcal{S}') = G(r(\hat{\mathcal{S}}', \mathcal{S}'')) - G(r(\mathcal{S}', \mathcal{S}''))$ bidders.

1.ii) Deviating to a family $\breve{\mathcal{S}}'$ that is flatter than $\mathcal{S}'$ but steeper than $\mathcal{S}''$. In this case, the new cutoff value, $r(\breve{\mathcal{S}}', \mathcal{S}'')$, must increase the competition from the most risk-averse bidders to balance the lower surplus extraction—net of the lower insurance provided. This leads Seller 1 to gain all bidders with risk aversion $r \in [r(\breve{\mathcal{S}}, \mathcal{S}''), r(\mathcal{S}', \mathcal{S}'')]$, and, thus, to gain, in expectation, $\Delta I'(\breve{\mathcal{S}}', \mathcal{S}') = G(r(\mathcal{S}', \mathcal{S}'')) - G(r(\breve{\mathcal{S}}', \mathcal{S}''))$ bidders.

1.iii) Deviating to a family $\tilde{\mathcal{S}}'$ that is flatter than $\mathcal{S}''$. Here, the cutoff value $r(\tilde{\mathcal{S}}, \mathcal{S}'')$ requires to balance the greater competition from the least-risk averse bidders (i.e., the most competitive bidders) and the lower insurance provided, with the lower surplus extraction. This leads Seller 1 to gain all bidders with risk aversion $r \in [r_N, r(\tilde{\mathcal{S}}', \mathcal{S}'')]$ and to lose all bidders with with risk aversion $r \in [r(\mathcal{S}', \mathcal{S}''), r_H]$. Hence, in expectation, the change in the number of bidders that join Seller 1's auction is given by $\Delta I'(\tilde{\mathcal{S}}', \mathcal{S}'') = G(r(\tilde{\mathcal{S}}', \mathcal{S}'')) - [1 - G(r(\mathcal{S}', \mathcal{S}''))]$.

Deviation 1.i) is not profitable for Seller 1 because the expected number of bidders decrease, the base of bidders become less competitive, and the original security choice is sufficiently steep. Indeed, in the profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$, Seller 1 has already a small expected number of participants, which becomes even smaller by the magnitude $\Delta I(\hat{\mathcal{S}}', \mathcal{S}')$. Hence, by the concavity of the expected first- and second-largest statistic in the sample size, the effect on the competition effect $\Delta \mathcal{ES}^c(\mathcal{S}', \hat{\mathcal{S}}' | \langle \mathcal{S}', \mathcal{S}'' \rangle)$ is large.[22] Furthermore, the base of bidders will consist of the most risk-averse bidders (i.e., the least competitive), which will not be sufficiently aggressive given that the risk-aversion parameter is sufficiently bounded. Therefore, because the family $\mathcal{S}'$ is sufficiently steep, any additional gain that comes from the higher extraction effect, $\Delta \mathcal{ES}^e(\mathcal{S}', \hat{\mathcal{S}}' | \langle \mathcal{S}', \mathcal{S}'' \rangle)$, will not be enough to compensate the loss that comes from the competition effect.

Meanwhile, deviation 1.ii) is not profitable since the expected first- and second-highest order statistics of the valuation distribution are sufficiently concave in the sample size—in virtue of condition (A.25)—and the survival function of the risk-aversion parameter is also sufficiently concave. The former feature of the valuations distribution, in combination with the fact that the deviation is toward a flatter security, imply that the effect of the greater expected number of bidders on the expected revenue, captured by the competition effect $\Delta \mathcal{ES}^c(\mathcal{S}', \breve{\mathcal{S}}' | \langle \mathcal{S}', \mathcal{S}'' \rangle)$, is moderate. Additionally, the concavity of the survival function implies that the increase in bidders' competitiveness is also moderate. This is so because the cutoff value $r(\breve{\mathcal{S}}, \mathcal{S}'')$ that induces the new expected number of bidders $I'(\breve{\mathcal{S}}', \mathcal{S}'')$ is sufficiently close to the original cutoff $r(\mathcal{S}', \mathcal{S}'')$. Additionally, as bidders become less risk-averse, they become relatively less aggressive, making the lower extraction effect strong enough to offset the positive competition effect.

Finally, deviation 1.iii) is not profitable due to two reasons. First, the deviation would be towards a significantly flatter security, taking into account the opponent's chosen security. Second, the expected number of bidders attracted would be such that $I'(\tilde{\mathcal{S}}', \mathcal{S}'') \leqslant I''(\mathcal{S}', \mathcal{S}'')$. This can be observed by comparing the profiles in which sellers choose, respectively, $\langle \mathcal{S}', \mathcal{S}'' \rangle$ and $\langle \tilde{\mathcal{S}}', \mathcal{S}'' \rangle$. In both cases, selecting the flatter security attracts more bidders, in expectation, because the risk-aversion parameter is bounded, and, so is the insurance benefit. However, when the profile played is $\langle \tilde{\mathcal{S}}', \mathcal{S}'' \rangle$, Seller 2 (the one choosing the steeper security in this

[22]When discussing the relevant effects of the expected first- and second-order statistics, we do it relative to the lower bound implied by the Jensen's inequality. This is possible since our interest is on providing sufficient conditions to determine the overall effect of the competition, extraction, and insurance channels.

profile) attracts a greater expected number of bidders compared to Seller 1 when the profile played is $\langle \mathcal{S}', \mathcal{S}'' \rangle$. This is so because the marginal value of insurance when seller choose $\langle \tilde{\mathcal{S}}', \mathcal{S}'' \rangle$ is greater than when they choose $\langle \mathcal{S}', \mathcal{S}'' \rangle$, making the steeper family relatively more appealing in the former case. This feature, in combination with condition (A.25), implies that the lower extraction effect $\mathcal{ES}^c(\mathcal{S}', \breve{\mathcal{S}}' | \langle \mathcal{S}', \mathcal{S}'' \rangle)$ more than offsets the greater competition effect.

Seller 2's possible deviations Similarly, if Seller 2 (the one choosing the flatter family) decides to deviate from the prescribed strategy, there are also three possible cases to analyze, which have different implications for the set of bidders that join his auction.

2.i) Deviation to a family $\hat{\mathcal{S}}''$ that is flatter than $\mathcal{S}''$. In this case, the new cutoff $r(\mathcal{S}', \hat{\mathcal{S}}'')$ requires to balance the lower extraction—net of the lower insurance—with a higher competition from the least risk-averse bidders, which are the most competitive. Then, Seller 2 will gain all bidders with risk aversion $r \in [r(\mathcal{S}', \mathcal{S}''), r(\mathcal{S}', \hat{\mathcal{S}}'')]$, and, thus, in expectation, will attract $\Delta I''(\hat{\mathcal{S}}'', \mathcal{S}'') = G(r(\mathcal{S}', \hat{\mathcal{S}}'')) - G(r(\mathcal{S}', \mathcal{S}''))$ bidders.

2.ii) Deviating to a family $\breve{\mathcal{S}}''$ that is steeper than $\mathcal{S}''$ but flatter than $\mathcal{S}'$. In this case, the new cutoff value $r(\mathcal{S}', \breve{\mathcal{S}}'')$, decreases the competition from the moderate risk-averse bidders to compensate for the higher surplus extraction—net of the higher insurance provided. This leads Seller 2 to lose all bidders with risk aversion $r \in [r(\breve{\mathcal{S}}, \mathcal{S}''), r(\mathcal{S}', \mathcal{S}'')]$. Thus, in expectation, he loses $\Delta I''(\breve{\mathcal{S}}'', \mathcal{S}'') = G(r(\mathcal{S}', \mathcal{S}'')) - G(r(\breve{\mathcal{S}}', \mathcal{S}''))$ bidders.

2.iii) Deviating to a family $\tilde{\mathcal{S}}''$ that is steeper than $\mathcal{S}'$. Here, the cutoff value $r(\mathcal{S}', \tilde{\mathcal{S}}'')$ requires to balance the greater surplus extraction, net of the higher extraction, with a lower competition from the most risk-averse bidders—i.e., the least-competitive bidders. This leads Seller 2 to lose all bidders with risk aversion $r \in [r_N, r(\mathcal{S}', \mathcal{S}'')]$ and to gain all bidders with with risk aversion $r \in [r(\mathcal{S}', \tilde{\mathcal{S}}''), r_H]$. Hence, in expectation, the change in the number of bidders that join Seller 2's auction is given by $\Delta I''(\tilde{\mathcal{S}}'', \mathcal{S}'') = [1 - G(r(\mathcal{S}', \tilde{\mathcal{S}}'')] - G(r(\mathcal{S}', \mathcal{S}''))$.

Deviation 2.i) is not profitable since Seller 2's original security choice is sufficiently flatter and, by condition (A.25), the expected first- and second-largest order statistics are sufficiently concave for any expected number of bidders greater than $I''(\mathcal{S}', \mathcal{S}'')$. Then, the positive competition effect $\mathcal{ES}^c(\mathcal{S}'', \hat{\mathcal{S}}''|\langle\mathcal{S}', \mathcal{S}''\rangle)$ is not strong enough to offset the negative extraction effect $\mathcal{ES}^e(\mathcal{S}'', \hat{\mathcal{S}}''|\langle\mathcal{S}', \mathcal{S}''\rangle)$.

Similarly, deviation 2.ii) is not profitable, since condition (A.25) implies that the loss coming from the competition effect is higher in magnitude, starting from a expected number of bidders of $I''(\mathcal{S}', \mathcal{S}'')$, and, thus, it is strong enough to offset the greater extraction effect.

Finally, deviation 2.iii) is not profitable since by the same argument as before $I''(\mathcal{S}', \tilde{\mathcal{S}}'') \leqslant I'(\mathcal{S}', \mathcal{S}'')$. Hence, by condition (A.25), the decrease in the expected first- and second-order statistic is large. Furthermore, the base of bidders become less competitive and are moderately aggressive, which makes the greater extraction effect insufficient to offset the losses that come from the competition effect.

A.8 Proof of Proposition 9

We use Proposition 1 in Sogo et al. (2016), and a similar argument in Proposition 4, to show that any equilibrium must exhibit symmetry, even sellers can use reserve prices as part of their security design. This implies that $(\mathcal{S}', \underline{s}(\mathcal{S}')) = (\mathcal{S}'', \underline{s}(\mathcal{S}''))$, as long as the increasing-differences condition between the entry probability and the steepness of the selected family is satisfied.

Now, suppose by contradiction that both sellers choose a security design $(\mathcal{S}', \underline{s}(\mathcal{S}'))$, where $\mathcal{S}' \neq \mathcal{S}^{co}$ and the "reserve price" $\underline{s}(\mathcal{S}') \in (s_L, s_H)$. Then, because a binding reserve price excludes bidders that otherwise were willing to enter the auction in its absence, it is always possible to find a security $\tilde{\mathcal{S}}$ steeper than $\mathcal{S}'$, and a reserve price $\underline{s}, (\tilde{\mathcal{S}}) < \underline{s}(\mathcal{S}')$ such that $\tilde{\gamma}(r_N|\tilde{\mathcal{S}}, \mathcal{S}') = \gamma'(r_N|\tilde{\mathcal{S}}, \mathcal{S}')$. Indeed, if $\mathcal{S}'$ belongs to a sub-convex family of securities, $\tilde{\mathcal{S}}$ can be selected from the convex hull of $\tilde{\mathcal{S}}$. Otherwise, $\tilde{\mathcal{S}}$ can be constructed as a convex combination between debt and a call option. The entry probabilities can be equalized since—from the bidder's perspective—the auction under the steeper security induces a higher probability of allocating the project, but extract a greater surplus conditional on allocation; whereas the flatter security offers a lower probability of trading but extract a lower surplus ex-post.

Now, since steeper securities provide more insurance to risk-averse bidders, we have that $\tilde{\gamma}(r|\tilde{\mathcal{S}}, \mathcal{S}') > \gamma'(r|\tilde{\mathcal{S}}, \mathcal{S}')$ for all $r > r_N$. Therefore, because steeper securities attract more entry and extract higher surplus ex-post, sellers that choose a binding reserve price and a family of securities flatter than a call option would have a profitable deviation. The same logic can be applied up to the point where both sellers choose a call option. At that point, there could be a symmetric equilibrium with binding reserve prices. In particular, we could find the highest binding reserve price that can be part of a symmetric equilibrium. Such a price is the one that makes sellers indifferent between choosing a call option under that price and deviating to the closest flatter security under a no-binding reserve price.

B Numerical Simulation

B.1 Homogeneously risk-averse bidders

We describe the simulation algorithm using backward induction.

- **Step 3: Interim expected utility in the auction stage**. For each number of bidders I and each number $1 \leqslant k \leqslant I-1$, we simulate the empirical distribution of the first-order statistic with respect to bidders' signals for each family $\mathcal{S}$, by making one million draws from the uniform distribution. This empirical distribution is necessary to compute the maximum highest bid from a group of k participants. With this empirical distribution on hand, we compute the winning probability and the expected utility for any bidder i with risk aversion r. (See equation (3)).

- **Step 2: Bidders' entry strategies**. For any seller's strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$, we compute bidders' entry probability strategies $\gamma'(r|\mathcal{S}', \mathcal{S}'')$ using the interim expected utility obtained in Step 1. Specifically, we solve the non-linear equation defined in (MS).

- **Step 1: Sellers' expected revenue**. Having computed $\gamma'(r|\mathcal{S}', \mathcal{S}'')$ in the previous step, we construct an *auxiliary* random index ω to implement the entry strategy prescribed by $\gamma'(r|\mathcal{S}', \mathcal{S}'')$. The random index ω is drawn from a uniform distribution with support $[0, 1]$, and is used to randomize bidders across auctions. Specifically, we draw an index ω_i for each bidder i and then follow a threshold rule: if $\omega_i \leqslant \gamma'(r|\mathcal{S}', \mathcal{S}'')$, then bidder i goes to the auction with the steeper family; otherwise, she goes to the auction with the flatter family. Once we distribute bidders across auctions in a random fashion, we compute the sellers' revenue using equation (5). We repeat this process one million times, and compute sellers' expected revenue in (6) as the average revenue over all iterations.

- **Step 0: Sellers' choice of design**. We repeat Steps 2-3 for each possible strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$. Then, we obtain the Nash equilibrium relative to the computed matrix of payoffs.

B.2 Heterogeneously risk-averse bidders

- **Step 1: Jumping strategies.** Fix a sellers' strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$ and an arbitrary bidder i with a fixed risk-aversion level $r_i \in \mathcal{R}^+$. Then, for each I and k such that $1 \leqslant k \leqslant I - 1$, we assume that the strategy of bidder i's opponents is determined by the rule of Proposition 6 relative to a cutoff determined, precisely, by bidder's i risk aversion, r_i.[23] Then, we make one thousand draws of the signals and risk-aversion levels. For any realization of bidder's i signal v_i, we compute the distribution of the losing bids against the type $\theta_i = (v_i, r_i)$. With this distribution in hand, we compute the ex-ante expected utility of joining each auction, i.e., $\mathcal{EU}(\mathcal{S}', r_i, r_i)$ and $\mathcal{EU}(\mathcal{S}'', r_i, r_i)$ (cf equation 10). These correspond to the expected utilities of a bidder i with risk aversion r_i of joining each auction when all other bidders "jump" exactly at the value r_i.

- **Step 2: Candidate equilibrium cutoff.** We repeat Step 1 for every $r_i \in \mathcal{R}^+$ until finding a type $\hat{r}$ of indifference, which is a type such that $\mathcal{EU}(\mathcal{S}', \hat{r}, \hat{r}) = \mathcal{EU}(\mathcal{S}'', \hat{r}, \hat{r})$. The value $\hat{r}$ constitutes the candidate for the equilibrium cutoff value—i.e., $\hat{r} = r(\mathcal{S}', \mathcal{S}'')$.

- **Step 3: Verification of the proposed equilibrium.** To check the validity of $\hat{r}$ as an equilibrium candidate, we first compute Step 1 relative to the cutoff value $\hat{r}$. Then, we verify that the rule embedded in Proposition 6 is satisfied in an ϵ-neighborhood, for $\epsilon > 0$ sufficiently small. That is, we we check that for any $r_i \neq \hat{r}$, $|\mathcal{EU}(\mathcal{S}', r_i, \hat{r}) - \mathcal{EU}(\mathcal{S}'', r_i, \hat{r})| < \epsilon$.

- **Step 4 Seller's expected revenue.** For a fixed seller's strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$, we distribute bidders across auctions following the rule of Proposition 6, relative to the cutoff $r(\mathcal{S}', \mathcal{S}'')$ found in Step 2. We make one million draws of signals and risk-aversion levels, and compute the seller's expected revenue for each family in the sellers' profile. Sellers' expected revenue is computed as the average revenue over all iterations.

[23] In the case of a strategy profile where both sellers choose the same family—i.e., $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}, \mathcal{S} \rangle$ for some $\mathcal{S}$—we define a focal point so that all bidders $j \neq i$ with risk aversion $r_j \leqslant r_i$ join the $\mathcal{S}_1$-auction and all bidders with risk aversion $r_j > r_i$ join the $\mathcal{S}_2$-auction.

- **Step 5: Sellers' Nash equilibrium.** We repeat step 4 for each possible strategy profile $\langle \mathcal{S}', \mathcal{S}'' \rangle$. Then, we obtain the Nash equilibrium relative to the computed matrix of payoffs.